Acta Physica Hungarica 64 (4), pp. 353-360 (1988)

A KERR-LIKE RADIATING METRIC IN THE EXPANDING UNIVERSE

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(Received 3 June 1986)

Using the method of differential forms a Kerr-like metric is derived as an exact solution of Einstein's field equations corresponding to a perfect fluid distribution plus a pure radiation field. The solution is interpreted as a Kerr-like radiating metric in the cosmological background of an expanding universe. The radiating Kerr metric and the radiating associated Kerr metric are derived as particular cases. The details of the solution are also discussed.

1. Introduction

Vaidya [1] has discussed a metric which reduces to the metric of an expanding universe in the absence of the source and becomes the well known Kerr metric (Kerr [2]) in the absence of the expansion of the universe. The source for Vaidya's solution is an imperfect fluid (i.e. the pressures in three spatial directions are not equal). Guha Thakurta [3] has also discussed the Kerr metric in the background of expanding universe. He has used the field equations corresponding to a perfect fluid with heat flow for his discussion.

Kerr metric describes the exterior gravitational field of a rotating body. Many people have tried to construct the non-static generalization of Kerr metric. Vaidya and Patel [4] have obtained a radiating Kerr solution in terms of a Kerr–Schild (Kerr and Schild [5]) metric. Their solution describes the exterior gravitational field of a radiating rotating body. Vaidya, Patel and Bhatt [6] have also constructed Kerr-like radiating solutions of Einstein's equations. These solutions are simpler than the one reported earlier by Vaidya amd Patel [4]. The main purpose of the present investigation is to obtain an exact explicit solution of Einstein's equations which describes the radiating Kerr and the radiating associated Kerr solution, discussed by Vaidya, Patel and Bhatt [6], in the cosmological background of an expanding universe.

For the derivation of our solution we shall use Cartan's exterior calculus of differential forms. This method of differential forms is standard now. Therefore we shall not enter into the details here.

2. Equations of structure

For calculations leading to the equations of structure, we take the line element in the form

$$ds^2 = e^{2F}[2(du + g\sin\alpha d\beta)dt - M^2(d\alpha^2 + \sin^2\alpha d\beta^2) - 2L(du + g\sin\alpha d\beta^2], \quad (1)$$

were $F = F(t), M = M(u,\alpha,t), g = g(\alpha)$ and $L = L(u,\alpha,t)$. The metric (1) is conformal to the Kerr-NUT metric discussed by Vaidya, Patel and Bhatt [6], the conformal factor being a function of time t.

In the space-time manifold defined by the metric (1), let us introduce the following basic 1 - forms

$$\begin{aligned} \theta^1 &= e^F(du + g\sin\alpha d\beta), & \theta^2 &= e^F d\alpha, \\ \theta^4 &= e^F dt - L\theta^1, & \theta^3 &= e^F M\sin\alpha d\beta. \end{aligned} \quad (2)$$

Therefore the metric (1) becomes

$$ds^2 = 2\theta^1\theta^4 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)}\theta^a\theta^b. \quad (3)$$

Here and in what follows the bracketed indices indicate tetrad components with respect to the tetrad (2). Using (2) it is easy to compute the exterior derivatives $d\theta^a$, and Cartan's first structure equation $d\theta^a = -W^a_b\Lambda\theta^b$ will the give the connection 1 - forms w^a_b. A straightforward calculation gives

$$\begin{aligned}
e^F w^1_1 &= -e^F w^4_4 = (L_t + LF_t)\theta^1 - F_t\theta^4, \\
e^F w^1_2 &= e^F w^2_4 = \left(\frac{f}{m^2}\right)\theta^3 + \left(F_t + \frac{M_t}{M}\right)\theta^2, \\
e^F w^1_3 &= e^F w^3_4 = -\left(\frac{f}{M^2}\right)\theta^2 + \left(F_t + \frac{M_t}{M}\right)\theta^3, \\
e^F w^2_1 &= e^F w^4_2 = \frac{-L_\alpha}{M}\theta^1 + \left[\frac{M_u}{M} + L(F_t + \frac{M_t}{M})\right]\theta^2 - \frac{fL}{M^2}\theta^3, \\
e^F w^3_1 &= e^F w^4_3 = \frac{gL_u}{M}\theta^1 + \frac{fL}{M^2}\theta^2 + \left[\frac{M_u}{M} + L\left(F_t + \frac{M_t}{M}\right)\right]\theta^3, \\
e^F w^2_3 &= -e^F w^3_2 = -\frac{fL}{M^2}\theta^1 - g\frac{M_u}{M}\theta^2 - \frac{(M_\alpha + M_{cot}\alpha)}{M^2}\theta^3 + \frac{f}{M^2}\theta^4
\end{aligned}$$

with $w_{ab} + w_{ba} = 0, f = \frac{1}{2}(g_\alpha + g\cot\alpha)$, and a suffix denoting partial derivatives (e.g. $g_\alpha = \frac{\partial g}{\partial\alpha}, F_t = \frac{\partial F}{\partial t}$ etc.) From the second equation of structure

$$dw^a_b + w^a_c\Lambda w^c_b = R^a_{bcd}\theta^c\Lambda\theta^d.$$

We can now compute the curvature components R^a_{bcd}. For the sake of brevity they are not listed here. If $R_{(ab)} = R^c abc$ denote the tetrad components of the Ricci tensor, we find that

$$
\begin{aligned}
R_{(23)} =& 0, \\
e^{2F} R_{(44)} =& 2\left(\frac{M_{tt}}{M} - \frac{f^2}{M^4}\right) + 2(F_{tt} - F_t^2), \\
e^{2F} R_{(24)} =& \frac{g}{M}\left[\left(\frac{M_t}{M}\right)_y - (f/M^2)_u\right], \\
e^{2F} R_{(34)} =& \frac{-g}{M}\left[\left(\frac{M_t}{M}\right)_u + (f/M^2)_y\right], \\
e^{2F} R_{(12)} =& Le^{2F} R_{(24)} + \frac{2g}{M} L_y F_t + \frac{g}{M}\left[\left(L_t + \frac{M_u}{M}\right)_y + \left(\frac{2fL}{M^2}\right)_u\right], \\
e^{2F} R_{(13)} =& Le^{2F} R_{(34)} - \frac{2g}{M} L_u F_t + \frac{g}{M}\left[-\left(L_t + \frac{M_u}{M}\right)_u + \left(\frac{2fL}{M^2}\right)_y\right], \\
e^{2F} R_{(14)} =& L_{tt} + \frac{2}{M}\left[M_{ut} + (LM_t)_t + \frac{Lf^2}{M^3}\right] + \\
& + 4LF_{tt} + 2LF_t\left(F_t + \frac{2M_t}{M}\right) + 2F_t\left(2L_t + \frac{M_u}{M}\right). \\
e^{2F} R_{(22)} =& e^{2F} R_{(33)} = \frac{1}{M_2}\left[g^2\left(\frac{M_u}{M}\right)_u + g^2\left(\frac{M_y}{M}\right)_y + \right. \\
& \left. + \frac{2fM_y}{M} + \frac{4f^2L^2}{M} - 1 - (M^2)_{ut} + \{L(M^2)_t\}_t\right] - \\
& - 2F_t\left(L_t + \frac{2M_u}{M}\right) - 4LF_t\left(F_t + \frac{2M_t}{M}\right) - 2LF_{tt}, \\
e^2 F R_{(11)} =& L^2 e^{2F} R_{(44)} + \frac{1}{M^2}\left[g^2(L_{uu} + L_{yy}) + \right. \\
& \left. + 2fL_y + 2L_u M M_t + 4LMM_{tu} - 2L_t M M_u + 2MM_{uu}\right] + 2L_u F_t.
\end{aligned}
\tag{5}
$$

In the above the variable y replaces the variable α in differentiation, the defining relation being

$$g d\alpha = dy. \tag{6}$$

3. The field equations

We shall try to solve the following field equations

$$R_{ik} - \frac{1}{2} g_{ik} R = -8\pi[(p+\rho)v_i v_k - p g_{ik} + \sigma w_i w_k] - \Lambda g_{ik}, \tag{7}$$

with

$$g^{ik}v_iv_k = 1, \quad g^{ik}w_iw_k = 0, \quad g^{ik}v_iw_k = 1. \tag{8}$$

The last condition in (8) is the normalizing condition.Here $\sigma w_i w_k$ is the tensor arising out of the flowing null radiation, v^i is the flow vector of the perfect fluid and Λ is the cosmological constant. The other symbols occurring in (7) have their usual meanings.

It is painless to see that the field equations (7) can be expressed in the tetrad form as

$$R_{(ab)} = \Lambda g_{(ab)} - 8\pi\sigma^w(a)^w(b) - 8\pi\left[(p+\rho)v_{(a)}v_{(b)} - \frac{1}{2}(\rho - p)g_{(ab)}\right]. \tag{9}$$

For the metric (1) and the tetrad (2) we take the tetrad components of the vectors v_i and w_i as

$$v_{(a)} = \left(\frac{1}{2\lambda}, 0, 0, \lambda\right), w_{(a)} = \left(\frac{1}{\lambda}, 0, 0, 0\right), \tag{10}$$

where λ is a function of the co-ordinates to be determined from the field equations. It is easy to see that v_i and w_i given by (10) satisfy the conditions (8). The equations (9) and (10) imply the following relations:

$$R_{(24)} = 0, \qquad R_{(34)} = 0, \tag{11}$$

$$R_{(12)} = 0, \qquad R_{(13)} = 0, \tag{12}$$

$$8\pi p = -R_{(14)} - \Lambda, \tag{13}$$

$$8\pi\rho = -2R_{(22)} - R_{(14)} - \Lambda, \tag{14}$$

$$2\lambda^2 = \frac{R_{(44)}}{R_{(22)} + R_{(14)}}, \tag{15}$$

$$16\pi\sigma = R_{(22)} + R_{(14)} - \frac{R_{(44)}R_{(11)}}{R_{(22)} + R_{(14)}}. \tag{16}$$

Here the tetrad components $R_{(ab)}$ of the Ricci tensor are given by (5).

4. The solution of the field equations

Take the two equations (11). These involve only one unknown function, M. One can easily verify that these equations admit the solution

$$M^2 = (f/Y)(X^2 + Y^2), \tag{17}$$

with

$$X_u = -Y_y, \qquad X_y = Y_u, \qquad X_t = -1, \qquad Y_t = 0. \tag{18}$$

For our purpose we shall take a particular case of the above solution. We assume $f = Y$. Therefore, the above solution becomes

$$M^2 = X^2 + Y^2, \tag{19}$$

with

$$Y = -ay + B, \quad X = au - t. \tag{20}$$

Here a and B are undetermined constants and no additive constant is shown explicitly in x, because such a constant can always be incorporated in the t co-ordinate. The two equations (12) can be explicitly written as.

$$-L_{tu} - \left(\frac{M_u}{M}\right)_u + \left(\frac{2fL}{M^2}\right)_y - 2L_u F_t = 0$$

and

$$L_{ty} + \left(\frac{M_u}{M}\right)_y + \left(\frac{2fL}{M^2}\right)_u + 2L_y F_t = 0.$$

Using the results (19) and (20) a solution of the above two differential equations can be expressed as

$$2L = a + \left[a - 1 + 2\frac{(F^* X + E^* Y)}{x^2 + y^2}\right] e^{-2F}, \tag{21}$$

with

$$E^* = -(a-1)Y, \quad F^* = -a(a-1)u - m, \tag{22}$$

where m is a constant of integration. The pressure p, the density ρ, λ^2 and the radiation density σ can be determined from (13), (14), (15) and (16). They are given by

$$8\pi p = \Lambda - e^{-2F}[a(2F_{tt} + F_t^2) + e^{-2F}(2L_0 - a)(F_{tt} - F_t^2)], \tag{23}$$

$$\begin{aligned} 8\pi(p+\rho) = & -2e^{-2F}\left[\frac{a-1}{X^2+Y^2} + a(F_{tt} - F_t^2)\right] - \\ & - 2e^{-4F}\left[\frac{1-a}{X^2+Y^2} - (2L_o - a)F_t^2 - \frac{2F_t\{(a-1)t + m\}}{X^2+Y^2}\right], \end{aligned} \tag{24}$$

$$\lambda^2 = \frac{2(F_{tt} - F_t^2)e^{-2F}}{-8\pi(p+\rho)}, \tag{25}$$

$$\begin{aligned} 16\pi\sigma = & -4\pi(p+\rho) + \frac{1}{4\pi(p+\rho)}\left[\frac{4ae^{-6F}}{X^2+Y^2}(F_{tt} - F_t^2)\{(a-1) + F^* F_t\} + \right. \\ & \left. + (F_{tt} - F_t^2)e^{-4F}\{a + (2L_0 - a)e^{-2F}\}^2\right], \end{aligned} \tag{26}$$

where $2L_0$ is given by

$$2L_0 = 2a - 1 + \frac{2(E^*Y + F^*X)}{X^2 + Y^2} \tag{27}$$

and F^* and E^* are given by (22).

With $2f = g_\alpha + g\cot\alpha, gd\alpha = dy$ and $f = Y$, the result (20) shows that the function Y satisfies the differential equation

$$Y_{zz}(1 - z^2) - 2zY_z + 2aY = 0, \quad z = \cos\alpha, \tag{28}$$

This differential equation admits a power series solution if $a \geq -\frac{1}{8}$. Given a satisfying this condition, one can find a real number n such that $2a = n(n+1)$, and so (28) becomes the Legendre equation

$$Y_{zz}(1 - z^2) - 2zY_z + n(n+1)y = 0. \tag{29}$$

The solution of (29) can be expressed as

$$Y = KP_n(\cos\alpha) + NQ_n(\cos\alpha), \tag{30}$$

where K and N are arbitrary constants and P and Q are respectively the Legendre polynomial and the associated Legendre polynomial of order n. It is easy to see that

$$g\sin\alpha = \frac{K}{a}\frac{dP_n}{dz}\sin^2\alpha + \frac{N}{a}\frac{dQ_n}{dz}\sin^2\alpha. \tag{31}$$

Thus the final form of the metric of our solution can be explicitly expressed as

$$\begin{aligned} ds^2 = e^{2F}\Bigg[& 2\{du + \left(\frac{K}{a}\frac{dP_n}{dz} + \frac{N}{a}\frac{dQ_n}{dz}\right)\sin^2\alpha d\beta\}dt - \\ & - 2L\left\{du + \left(\frac{K}{a}\frac{dP_n}{dz} + \frac{N}{a}\frac{NQ_n}{dz}\right)\sin^2\alpha d\beta\right\}^2 - \\ & - (X^2 + Y^2)(d\alpha^2 + \sin^2\alpha d\beta^2)\Bigg], \end{aligned} \tag{32}$$

where

$$X = au - t, \quad Y = KP_n(\cos\alpha) + NQ_n(\cos\alpha),$$
$$2L = a + \left[(1 - a) + \frac{2X\{(1-a)t - m\}}{X^2 + Y^2}\right]e^{-2F}.$$

When $F = 0, \Lambda = 0$, we have verified that $p = 0, \rho = 0$ but $\sigma \neq 0$. In this case the metric (32) reduces to the radiating Kerr-like metric discussed by Vaidya, Patel and Bhatt [6].

Let us study the situation when $a = 1$. In this case $n = 1$ and consequently Y is given by

$$Y = K\cos\alpha + N(\cos\alpha \log\tan\frac{\alpha}{2} + 1). \tag{33}$$

In this case the parameters p, ρ, λ^2 and σ are given by

$$\begin{aligned}
8\pi p =& \Lambda - e^{-2F}(2F_{tt} + F_t^2) + \frac{2mre^{-2F}}{X^2+Y^2}(F_{tt} - F_t^2),\\
8\pi(p+\rho) =& -2e^{2F}(F_{tt} - F_t^2) + \frac{4me^{-4F}(rF_t^2 + F_t)}{r^2+Y^2},\\
\lambda^2 =& \frac{2(F_{tt} - F_t^2)e^{-2F}}{-8\pi(p+\rho)},\\
16\pi\sigma =& \frac{1}{4\pi(p+\rho)}\left[-\frac{4mF_t}{r^2+Y^2}(F_{tt} - F_t^2)e^{-6F} + \right.\\
&\left. +(F_{tt} - F_t^2)e^{-4F}\left\{1 + \frac{2mr}{r^2+Y^2}e^{-2F}\right\}^2\right] - 4\pi(p+\rho),
\end{aligned} \tag{34}$$

where Y is given by (33) and $x = u - t = -r$. From (34) it is obvious that when $F = 0, \Lambda = 0$ we get $p = \rho = \sigma = 0$. Thus, we get an empty space time described by the metric

$$\begin{aligned}
ds^2 =& 2[du + \{K\sin^2\alpha + N(\sin^2\alpha\log\tan\frac{\alpha}{2} - \cos\alpha)\}d\beta]dt-\\
&-(r^2+Y^2)(d\alpha^2 + \sin^2\alpha d\beta^2) - \left[1 + \frac{2mr}{r^2+Y^2}\right]\times\\
&\times[du + \{K\sin^2\alpha + N(\sin^2\alpha\log\tan\frac{\alpha}{2} - \cos\alpha)d\beta\}]^2,
\end{aligned} \tag{35}$$

where Y is given by (33).

The metric (35) is the particular case of Kerr-like vacuum metric discussed by Demianski [7] with slight changes of notation. When $N = 0$, (35) reduces to the well-known Kerr metric. When $K = 0$, (35) reduces to the associated Kerr metric discussed by Vaidya [8]. Here it should be noted that when $m = 0$ the metric (35) reduces to the flat metric. The explicit transformations of coordinates for this purpose are given by Demianski [7]. Therefore when $a = 1$ and $m = 0$ the metric (32) reduces to the metric of an expanding universe.

Thus the metric (32) with $a = 1$ and $N = 0$ gives us Kerr metric in the background of an expanding universe. Also the metric (32) with $a = 1$ and $K = 0$ represents the field of the associated Kerr source embedded in an expanding universe.

We can interpret the metric (32) when $N = 0$ as the radiating Kerr metric in the cosmological background of an expanding universe. Similarly when $K = 0$, the metric (32) represents the radiating associated Kerr metric, in the background of an expanding universe.

When the background universe is pressure-free, then we have

$$2F_{tt} + F_t^2 = 0. \tag{36}$$

The Eq. (36) can be easily integrated to have

$$e^{2F} = (lt + q)^4, \tag{37}$$

where l and q are constants of integration. The metric (32) with e^{2F} given by (37) represents a radiating Kerr-like metric in the cosmological background of Einstein–de Sitter universe.

Acknowledgement

One of the authors (S.S.K) is highly indebted to the University Grants Commission, New Delhi, for the award of a Junior Research Fellowship.

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