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CONVERSION OF GRAVITATIONAL WAVES INTO ELECTROMAGNETIC WAVES IN A BIANCHI TYPE I UNIVERSE WITH A UNIFORM MAGNETIC FIELD — COSMOLOGICAL IMPLICATIONS

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The process of conversion of g. waves into e.m. waves is studied in an axisymmetric Bianchi type I universe with a uniform magnetic field. A case is found in which the process is sufficiently effective to maintain a continuous interchange between photons and gravitons. This causes photons and gravitons to have the same frequency.

1. Introduction

The problem of the conversion of gravitational waves into electromagnetic waves and vice versa has been discussed from various points of view, in its own right and in view of detecting gravitational waves or in cosmological and astrophysical implications [1, 2, 3]. It has been shown that in a flat Minkowski background containing a uniform magnetic field gravitational waves are transformed into e.m. waves and v. v. at a rate growing quadratically with distance [2]. In spite of this, one must consider very high strength magnetic fields and paths of cosmological magnitude to get appreciable effects [2].

The case has also been studied in which the magnetic field is embedded in a conducting plasma with anisotropic conductivity; the results depend on the frequency of incoming gravitational waves but are not very different from the case of the empty space [4]. In this paper we study the conversion of g. waves into e.m. waves no longer in a flat background but in an anisotropic cosmological model endowed with a uniform magnetic field.

Our aim is to estimate if this process is working sufficiently to maintain photons and gravitons at the same frequency, taking into account that the inverse process must have the same rate. We proceed in this way: we consider the g. waves as first-order perturbation of the anisotropic metric with magnetic field and try to solve the Maxwell equations in this perturbed metric. As we are interested in the cosmological era preceding the recombination, we insert in the Maxwell equations (in the three cases we consider) asymptotic expressions for the metric coefficients valid near the initial singularity.

This enables us to solve analytically the Maxwell equations which in all the three cases lead to the solution of equations of the Bessel type. If we call

η=(produced flux of e.m. waves)/(incident flux of g. waves), then we give an approximate evaluation of η as a function of time.

As we know, only if the primordial magnetic field is created with a strength not exceeding the so-called $B_{\text{crit}} = 4.4 \times 10^{13}$ gauss, can it be treated on a classical ground; furthermore it seems likely that, for fields of infinite strength or of strength $B >> B_{\text{crit}}$, a field organized on a large scale would not emerge as the universe expanded out of the singularity [5].

Therefore we take as fairly unrealistic the second and the third case we consider which, also at times over the Planck time $t_g \sim 10^{43}$s, involve field strengths largely exceeding B_{crit}. In the first case we study, on the contrary, we assume an initial field of the order of B_{crit}.

2. Axisymmetric cosmologies of Bianchi type I with a uniform magnetic field—asymptotic solutions

In the second half of the sixties, after the discovery of the cosmic microwave radiation background, spatially homogeneous anisotropic cosmological models endowed with a uniform magnetic field were extensively studied [6]. The hypothesis of the existence of primordial magnetic fields was largerly favoured [5, 6] because of the necessity of understanding how galactic magnetic fields could have arisen since the big-bang creation of the universe. Also if today we seem to have a satisfactory theory of galactic magnetic fields which do not require magnetic fields frozen into the matter since the origin of the universe, the hypothesis of primordial magnetic fields is not ruled out [7]. We assume the existence of such a primordial field.

We shall consider universe models with metric

$$ds^2 = c^2dt^2 - A^2(t)(dx^2 + dy^2) - W^2(t)dz^2 \tag{2-1}$$

filled with perfect-fluid matter at rest in the coordinate system of equations (2–1) and having an equation of state

$$P = \gamma \cdot \rho_M, \tag{2-2}$$

where $0 \leq \gamma < 1$ and ρ_M is the energy density of the perfect-fluid matter. The stress-energy tensor contains both ρ_M and ρ_B, where ρ_B is the energy density of the magnetic field. Solving the Maxwell equations in the metric (2–1) in the hypothesis of a uniform magnetic field aligned along the z-axis, one obtains for the primeval field

$$\bar{B}_z = B_0(WA^2)^{-1},$$

and for the magnetic energy density

$$\rho_B = B_0^2/8\pi \cdot A^{-4} \qquad (2\text{–}3^1)$$

[1] For the definition of the 3-dimensional fields we follow L.D. Landau, E.M. Lifshitz, Classical theory of fields, Pergamon Press, Oxford — New York, 1971, p. 256, from now on referred to as L.L.

(B_0 is a constant to be fixed).

As is known, the Einstein field equations in the case of (2–1), (2–2), (2–3) have been solved analytically in four cases [6]; moreover, there are solutions valid in asymptotic conditions, near the singularity. If we define an adimensional normalized time $\tau =$ at ($a \sim 10^{-17}\ \mathrm{s}^{-1}$), the asymptotic conditions, near the initial singularity, are characterized by $\tau << 1$ (being $\tau \sim 1$ to day). We shall study three cases:

1) Axisymmetric pancake singularity;
2) Isotropic point singularity;
3) Axisymmetric hard magnetic solution.

The case 1) is defined by

$$A = (1 + \alpha \cdot \tau^{(1-\gamma)}), \quad W = \tau, \quad \rho_B = B_0^2/8\pi \left[1 - 4\alpha \cdot \tau^{(1-\gamma)}\right], \tag{2–4}$$

(α non-negative constant) $0 \leq \gamma < 1$, and is an asymptotic solution. Of the three cases it is the only one which allows for a finite magnetic energy density for $\tau \to 0$; furthermore the solution becomes isotropic for large τ. We fix the constant $B_0 \sim B_{\mathrm{crit}} = 4.4 \times 10^{13}$ gauss. The case 2), also an asymptotic solution, is given by

$$\begin{aligned} A = W = \tau^{\Gamma}(\Gamma = \frac{2}{3}(1+\gamma)), \qquad 1/3 < \gamma < 1, \\ \rho_B = B_0^2/8\pi \cdot \tau^{-4\Gamma}. \end{aligned} \tag{2–5}$$

Obviously $\rho_B \to \infty$ when $\tau \to 0$; we fix the constant $B_0 \sim 10^{-8}$ gauss (a possible value for the intergalactic field to day, but leading to a physically inconceivable magnetic field at the early times). The case 3) is at the same time an approximate solution ($\tau << 1$) and an exact solution (but not the general solution) and is given by

$$A = \tau^{1/2}, \qquad W = \tau^{\Lambda}(\Lambda = 1 - \gamma/1 + \gamma), \qquad 1/3 < \gamma < 1, \tag{2–6}$$

$\rho_B = B_0^2/8\pi \cdot \tau^{-2}$, where the constant B_0 is not arbitrary but has the value $B_0 =$ ca $/(G)^{1/2}\left[(1-\gamma)(3\gamma-1)\right]^{1/2}/2(1+\gamma)$. Also this case has $\rho_B \to \infty$ when $\tau \to 0$.

3. The Maxwell equations in the perturbed metric

We have to solve the Maxwell equations:[2]

$$\begin{aligned} \partial F_{ik}/\partial x^l + \partial F_{ei}/\partial x^k + \partial F_{kl}/\partial x^i = 0, \\ (-g)^{-1/2}\partial/\partial x^k \left[(-g)^{1/2}F^{ik}\right] = -4\pi/cj^i, \end{aligned} \tag{3–1}$$

[2]For definitions, notations (except the gravitational constant we denote by G) and all that, see L. L.

in the metric

$$g_{ik} = g^0_{ik} + h_{ik},$$
$$g^{ik} = g^{0ik} - h^{ik},$$

where g^0_{ik} are the coefficients of the metric (2–1) and h_{ik} are first-order perturbations; the solutions of (3–1) are to be considered small of the first order like the h_{ik}: squares are negligible. We choose gravitational waves travelling in the positive x-direction and two independent polarizations denoted as

$$h^3_2(x,t) \text{ and } -h^2_2(x,t) = h^3_3(x,t).$$

Being perturbations, they will be linear and we consider plane waves:

$$\begin{aligned} h^3_2(x,t) &= \varepsilon_a \exp ik(x-ct), \\ -h^2_2(x,t) = h^3_3(x,t) &= \varepsilon_b \exp ik(x-ct), \end{aligned} \tag{3–2}$$

any g. wave being a sum of terms like (3–2), we solve (3–1), separately for each of the two polarizations.

Making use of the variable τ=at, we rewrite (3–2) as

$$\begin{aligned} h^3_2(x,\tau) &= \varepsilon_a \exp\,[ik(x-c/a\cdot\tau)], \\ -h^2_2(x,\tau) = h^3_3(x,\tau) &= \varepsilon_b \exp\,[ik(x-c/a\cdot\tau)]. \end{aligned} \tag{3–2'}$$

3a. Polarization h^3_2

In virtue of the assumed plane symmetry (3–1) yield in this case

$$\begin{aligned} &\partial/\partial x(WA^2B_x) = 0; \quad \partial/\partial\tau(WA^2B_x) = 0, \\ &a/c\partial/\partial\tau(WA^2B_y) - WA^{-2}\partial/\partial x(W^{-1}A^2E_z) = 0, \\ &a/c\partial/\partial\tau(WA^2B_z) + W^{-1}\partial/\partial x(WE_y) = 0, \\ &W^{-1}A^{-2}\partial/\partial x(WE_x) = 4\pi/cj^0; \quad \partial/\partial\tau(WE_x) = 0, \\ &a/c\partial/\partial\tau(WE_y) + WA^{-2}\partial/\partial x(WA^2B_z) = 0, \\ &a/c\partial/\partial\tau(W^{-1}A^2E_z) - W^{-1}\partial/\partial x(WA^2B_y) = WA^{-2}B_0\partial h^3_2/\partial x, \end{aligned} \tag{3–1a}$$

having taken into account that $F_{12} = WA^2B_z + B_0$. Rewriting and rearranging the relevant equations, we have:

$$\begin{cases} a/c\partial/\partial\tau(WA^2B_z) + W^{-1}\partial/\partial x(WE_y) = 0, \\ a/c\partial/\partial\tau(WE_y) + WA^{-2}\partial/\partial x(WA^2B_z) = 0, \end{cases}$$

$$\tag{3–1a'}$$

$$\begin{cases} a/c\partial/\partial\tau(WA^2B_y) - WA^{-2}\partial/\partial x(W^{-1}A^2E_z) = 0, \\ a/c\partial/\partial\tau(W^{-1}A^2E_z) - W^{-1}\partial/\partial x(WA^2B_y) = WA^{-2}B_0\partial h^3_2/\partial x. \end{cases}$$

3b. Polarization $-h_2^2 = h_3^3$

In this case (3–1) yield:

$$
\begin{aligned}
&\partial/\partial x(WA^2B_x) = 0; \qquad \partial/\partial\tau(WA^2B_x) = 0,\\
&a/c\partial/\partial\tau(WA^2B_y) - WA^{-2}\partial/\partial x(W^{-1}A^2E_z) = 0,\\
&a/c\partial/\partial\tau(WA^2B_z) + W^{-1}\partial/\partial x(WE_y) = 0,\\
&W^{-1}A^{-2}\partial/\partial x(WE_x) = 4\pi/cj^0; \quad \partial/\partial\tau(WE_x) = 0,\\
&a/c\partial/\partial\tau(WE_y) + WA^{-2}\partial/\partial x(WA^2B_z) = WA^{-2}B_0\partial h_2^2/\partial x,\\
&a/c\partial/\partial\tau(W^{-1}A^2E_z) - W^{-1}\partial/\partial x(WA^2B_y) = 0.
\end{aligned}
\tag{3–1b}
$$

And, rewriting and rearranging the relevant equations:

$$
\begin{cases}
a/c\partial/\partial\tau(WA^2B_y) - WA^{-2}\partial/\partial x(W^{-1}A^2E_z) = 0,\\
a/c\partial/\partial\tau(W^{-1}A^2E_z) - W^{-1}\partial/\partial x(WA^2B_y) = 0,
\end{cases}
$$

$$
\begin{cases}
a/c\partial/\partial\tau(WA^2B_z) + W^{-1}\partial/\partial x(WE_y) = 0,\\
a/c\partial/\partial\tau(WE_y) + WA^{-2}\partial/\partial x(WA^2B_z) = WA^{-2}B_0\partial h_2^2/\partial x.
\end{cases}
\tag{3–1b'}
$$

In the subsequent sections we shall solve (3–1a') and (3–1b') in the three cases of asymptotic solution for the metric (2–1) we have discussed in Section 2, substituting the coefficients of (3–1a') and (3–1b') by their asymptotic expressions.

4. Axisymmetric pancake singularity

As we have seen in Section 2, this case is characterized by $A(\tau) = 1+\alpha\cdot\tau^{(1-\gamma)}$, $W(\tau) = \tau$ with $0 \le \gamma < 1$ and α, a non-negative constant. For small τ, (3–1a') asymptotically become:

$$
\begin{cases}
a/c\partial/\partial\tau(WA^2B_z) + \tau^{-1}\partial/\partial x(WE_y) = 0,\\
a/c\partial/\partial\tau(WE_y) + \tau\partial/\partial x(WA^2B_z) = 0,
\end{cases}
$$

$$
\begin{cases}
a/c\partial/\partial\tau(WA^2B_y) - \tau\partial/\partial x(W^{-1}A^2E_z) = 0,\\
a/c\partial/\partial\tau(W^{-1}A^2E_z) - \tau^{-1}\partial/\partial x(WA^2B_y) = ik\varepsilon_a B_0\cdot\tau\cdot\ \exp\,[ik(x - c/a\cdot\tau)].
\end{cases}
\tag{4–1a'}
$$

The systems (4–1a') have the solutions

$$
\begin{aligned}
WA^2B_y =&(A_1e^{-ikx} + B_1e^{ikx})\cdot kc/a\cdot\tau J_1(kc/a\cdot\tau)+\\
&+(-i/5\cdot kc/a\cdot\tau^3 - 1/5\cdot\tau^2)\varepsilon_a B_0\exp\,[ik(x - c/a\cdot\tau)],\\
W^{-1}A^2E_z =&-i(A_1e^{-ikx} + B_1e^{ikx})\cdot J_0(kc/a\cdot\tau)+\\
&+(i/5\,kc/a\cdot\tau^2 - 2/5\cdot\tau + 2/5\,ia/kc)\varepsilon_a B_0\exp\,[ik(x - c/a\cdot\tau)],\\
WA^2B_z =&(C_1e^{-ikx} + D_1e^{ikx})J_0(kc/a\cdot\tau),\\
WE_y =&-i(C_1e^{-ikx} + D_1e^{ikx})kc/a\cdot\tau\cdot J_1(kc/a\cdot\tau),
\end{aligned}
\tag{4–2a'}
$$

where J_0 and J_1 are Bessel functions[3]. Imposing to (4–2a') the obvious initial condition that the left members vanish at $\tau = 0$, the final result is:

$$\begin{aligned} WA^2B_y =&\varepsilon_a \cdot B_0 \left\{2/5 J_1(kc/a \cdot \tau) \cdot \tau \cdot e^{ikx} + (-i/5\, kc/a \cdot \tau^3 - 1/5 \cdot \tau^2) \cdot \right. \\ &\left. \cdot \exp\left[ik(x - c/a \cdot \tau)\right]\right\}, \\ W^{-1}A^2E_z =&\varepsilon_a B_0 \left\{-2/5 i\, a/kc\, J_0(kc/a \cdot \tau) e^{ikx} + (i/5\, kc/a \cdot \tau^2 - \right. \\ &\left. - 2/5 \cdot \tau + 2/5\, ia/kc) \exp\left[ik(x - c/a \cdot \tau)\right]\right\}, \\ WA^2B_z =&WE_y = 0 \end{aligned}$$

and then

$$\left\{ \begin{aligned} F^1_3 =&\varepsilon_a B_0 \left\{-2/5 J_1(kc/a \cdot \tau) \cdot \tau \cdot e^{ikx} + (i/5\, kc/a \cdot \tau^3 + \right. \\ &\left. +1/5\tau^2) \exp\left[ik(x - c/a \cdot \tau)\right]\right\}, \\ F^{03} =&\varepsilon_a B_0 \left\{2/5\, i\, a/kc\, J_0(kc/a \cdot \tau) \cdot \tau^{-1} \cdot e^{ikx} + (2/5 - i/5\, kc/a \cdot \tau - \right. \\ &\left. - 2/5\, i\, a/kc\, \tau^{-1}) \exp\left[ik(x - c/a \cdot \tau)\right]\right\}, \\ F^1_2 =&F^{02} = 0. \end{aligned} \right. \tag{4–3a'}$$

In the same way (3–1b') asymptotically become:

$$\left\{ \begin{aligned} a/c\partial/\partial\tau(WA^2B_y) - \tau\partial/\partial x(W^{-1}A^2E_z) = 0, \\ a/c\partial/\partial\tau(W^{-1}A^2E_z) - \tau^{-1}\partial/\partial x(WA^2B_y) = 0, \end{aligned} \right.$$

$$\left\{ \begin{aligned} &a/c\partial/\partial\tau(WA^2B_z) + \tau^{-1}\partial/\partial x(WE_y) = 0, \\ &a/c\partial/\partial\tau(WE_y) + \tau\partial/\partial x(WA^2B_z) = -ik\varepsilon_b \cdot B_0\tau \exp\left[ik(x - c/a \cdot \tau)\right], \end{aligned} \right. \tag{4–1b'}$$

with the solutions:

$$\begin{aligned} WA^2B_y =&(A_2e^{-ikx} + B_2e^{ikx})kc/a \cdot \tau \cdot J_1(kc/a \cdot \tau), \\ W^{-1}A^2E_z =& - i(A_2e^{-ikx} + B_2e^{ikx}) \cdot J_0(kc/a \cdot \tau), \\ WA^2B_z =&(C_2e^{-ikx} + D_2e^{ikx})J_0(kc/a \cdot \tau) + (-1/3 + ikc/a \cdot \tau)\varepsilon_b B_0 \cdot \\ &\cdot \exp\left[ik(x - c/a \cdot \tau)\right], \\ WE_y =& - i(C_2e^{-ikx} + D_2e^{ikx})kc/a \cdot \tau \cdot J_1(kc/a \cdot \tau) + \\ &+ (-i/3kc/a \cdot \tau^2) \exp\left[ik(x - c/a \cdot \tau)\right] \end{aligned} \tag{4–2b'}$$

and, imposing the initial conditions, we get:

$$\left\{ \begin{aligned} F^1_3 =& F^{03} = 0, \\ F^1_2 =& \varepsilon_b B_0 \left\{1/3 J_0(kc/a \cdot \tau)e^{ikx} - 1/3(ikc/a \cdot \tau + 1) \exp\left[ik(x - c/a \cdot \tau)\right]\right\}, \\ F^{02} =& \varepsilon_b B_0 \left\{i/3kc/a J_1(kc/a \cdot \tau)e^{ikx} + i/3kc/a \cdot \tau \exp\left[ik(x - c/a \cdot \tau)\right]\right\}. \end{aligned} \right. \tag{4–3b'}$$

[3] We assume the notations and definitions of V. I. Smirnov: A course of higher mathematics, Vol. III, part 2, Pergamon Press, Oxford — New York.

5. Isotropic point singularity

This case is characterized by

$$A(\tau) = W(\tau) = \tau^{\Gamma}, \text{ where } \Gamma = 2/3(1+\gamma) \text{ and } 1/3 < \gamma < 1.$$

(3–1a') become:

$$\begin{cases} a/c\partial/\partial\tau(WA^2B_z) + \tau^{-\Gamma}\partial/\partial x(WE_y) = 0, \\ a/c\partial/\partial\tau(WE_y) + \tau^{-\Gamma}\partial/\partial x(WA^2B_z) = 0, \end{cases}$$

$$\begin{aligned} &a/c\partial/\partial\tau(WA^2B_y) - \tau^{-\Gamma}\partial/\partial x(W^{-1}A^2E_z) = 0, \\ &a/c\partial/\partial\tau(W^{-1}A^2E_z) - \tau^{-\Gamma}\partial/\partial x(WA^2B_y) = ik\varepsilon_a B_0\tau^{-\Gamma} \exp\,[ik(x - c/a\cdot\tau)], \end{aligned} \tag{5–1a'}$$

and have the solutions

$$\begin{aligned} WA^2B_z =&(H_1e^{-ikx} + L_1e^{ikx})\cdot \\ &\cdot\left\{M_1 \exp\,[-ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}] + N_1 \exp\,[ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\right\}, \\ WE_y =&(H_1e^{-ikx} - L_1e^{ikx})\cdot \\ &\cdot\left\{-M_1 \exp\,[-ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}] + N_1 \exp\,[ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\right\}, \\ WA^2B_y =&(P_1e^{-ikx} + Q_1e^{ikx})\cdot \\ &\cdot\left\{R_1\cdot \exp\,[-ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}] + S_1 \exp\,[ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\right\} + \\ &+\varepsilon_a B_0 e^{ikx}\cdot\Bigg\{\bigg\{-2e^{ikc/a\cdot\tau} + \exp\,[-ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}] + \\ &+ \exp\,[ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}] - ikc/a\cdot \\ &\cdot\bigg\{\exp\,[-ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\cdot\int_0^\tau \exp\,[ikc/a(\xi^{1-\Gamma}/(1-\Gamma) - \xi)]d\xi - \\ &- \exp\,[ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\cdot\int_0^\tau \exp\,[-ikc/a(\xi^{1-\Gamma}/(1-\Gamma) + \xi)]d\xi\bigg\}\bigg\}\Bigg\}, \\ W^{-1}A^2E_z =&(-P_1e^{-ikx} + Q_1e^{ikx})\cdot \\ &\cdot\left\{-R_1 \exp\,[-ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}] + S_1 \exp\,[ikc/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\right\} + \\ &+ i\varepsilon_a B_0 kc/a\bigg\{\exp\,[ik(x - c/a(1-\Gamma)\cdot\tau^{1-\Gamma})]\cdot \\ &\cdot\int_0^\tau \exp\,[ikc/a(\xi^{1-\Gamma}/(1-\Gamma) - \xi)]d\xi - \exp\,[ik(x + c/a(1-\Gamma)\cdot\tau^{1-\Gamma})]\cdot \\ &\cdot\int_0^\tau \exp\,[-ikc/a(\xi^{1-\Gamma}/(1-\Gamma) + \xi)]d\xi\bigg\}. \end{aligned} \tag{5–2a'}$$

And, imposing the initial conditions:

$$
\begin{aligned}
WA^2B_z =& WE_y = 0,\\
WA^2B_y =& \varepsilon_a B_0 \{-2 \exp [ik(x - c/a\cdot\tau)] + \exp [ik(x - c/a(1-\Gamma)\cdot\tau^{1-\Gamma})] +\\
& + \exp [ik(x + c/a(1-\Gamma)\cdot\tau^{1-\Gamma})]\} - i\varepsilon_a B_0 kc/a\cdot\\
& \cdot\left\{ \exp [ik(x - c/a(1-\Gamma)\cdot\tau^{1-\Gamma})]\cdot\int_0^\tau \exp [ikc/a(\xi^{1-\Gamma}/(1-\Gamma) - \xi)]d\xi -\right.\\
& \left. - \exp [ik(x + c/a(1-\Gamma)\cdot\tau^{1-\Gamma})]\cdot\int_0^\tau \exp [ikc/a(\xi^{1-\Gamma}/(1-\Gamma) + \xi)]d\xi\right\}\\
W^{-1}A^2E_z =& i\varepsilon_a B_0 kc/a\cdot\\
& \cdot\left\{ \exp [ik(x - c/a(1-\Gamma)\cdot\tau^{1-\Gamma})]\cdot\int_0^\tau \exp [ikc/a(\xi^{1-\Gamma}/(1-\Gamma) - \xi)]d\xi -\right.\\
& \left. - \exp [ik(x + c/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\cdot\int_0^\tau \exp [-ikc/a(\xi^{1-\Gamma}/(1-\Gamma) + \xi)]d\xi\right\}
\end{aligned}
$$

and then:

$$
\begin{aligned}
F_2^1 =& F^{02} = 0,\\
F_3^1 =& \varepsilon_a B_0\cdot\tau^{-2\Gamma}\{2 \exp [ik(x - c/a\cdot\tau)] - \exp [ik(x + c/a(1-\Gamma)\cdot\tau^{1-\Gamma}] -\\
& - \exp [ik(x - c/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\} + i\varepsilon_a B_0 kc/a\tau^{-2\Gamma}\cdot\\
& \cdot\left\{ \exp [ik(x - c/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\cdot\int_0^\tau \exp [ikc/a(\xi^{1-\Gamma}/(1-\Gamma) - \xi)]d\xi -\right.\\
& \left. - \exp [ik(x + c/a(1-\Gamma)\cdot\tau^{1-\Gamma})]\cdot\int_0^\tau \exp [-ikc/a(\xi^{1-\Gamma}/(1-\Gamma) + \xi)]d\xi\right\},\\
F^{03} =& - i\varepsilon_a B_0 kc/a\tau^{-3\Gamma}\cdot \qquad\qquad (5\text{–}3a')\\
& \cdot\left\{ \exp [ik(x - c/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\cdot\int_0^\tau \exp [ikc/a(\xi^{1-\Gamma}/(1-\Gamma) - \xi)]...\xi -\right.\\
& \left. - \exp [ik(x + c/a(1-\Gamma)\cdot\tau^{1-\Gamma})]\cdot\int_0^\tau \exp [-ikc/a(\xi^{1-\Gamma}/(1-\Gamma) + \xi)]d\xi\right\}.
\end{aligned}
$$

In the same way, from (3–1b') we get:

$$
\begin{aligned}
F_3^1 =& F^{03} = 0,\\
F_2^1 =& - \varepsilon_b B_0\tau^{-2\Gamma}\{2 \exp [ik(x - c/a\cdot\tau)] -\\
& - \exp [ik(x + c/a(1-\Gamma)\cdot\tau^{1-\Gamma}] - \exp [ik(x - c/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\} -\\
& - i\varepsilon_b B_0 kc/a\cdot\tau^{-2\Gamma}\cdot\\
& \cdot\left\{ \exp [ik(x - c/a(1-\Gamma)\cdot\tau^{1-\Gamma}]\cdot\int_0^\tau \exp [ikc/a(\xi^{1-\Gamma}/(1-\Gamma) - \xi)]d\xi -\right.\\
& \left. - \exp [ik(x + c/a(1-\Gamma)\cdot\tau^{1-\Gamma})]\cdot\int_0^\tau \exp [-ikc/a(\xi^{1-\Gamma}/(1-\Gamma) + \xi)]d\xi\right\},
\end{aligned}
$$

$$
\begin{aligned}
F^{02} =& i\varepsilon_b B_0 kc/a \cdot \tau^{-3\Gamma} \cdot \\
& \cdot \left\{ \exp\left[-ik(x - c/a(1-\Gamma)\tau^{1-\Gamma})\right] \cdot \int_0^\tau \exp\left[ikc/a(\xi^{1-\Gamma}/(1-\Gamma) - \xi)\right] d\xi - \right. \\
& \left. - \exp\left[ik(x + c/a(1-\Gamma)\tau^{1-\Gamma})\right] \cdot \int_0^\tau \exp\left[-ikc/a(\xi^{1-\Gamma}/(1-\Gamma) + \xi)\right] d\xi \right\} .
\end{aligned}
\tag{5-3b'}
$$

6. Axisymmetric hard-magnetic solution

$A(\tau) = \tau^{1/2}$; $W(\tau) = \tau^{\Lambda}$ where $\Lambda = 1 - \gamma/1 + \gamma$ and $1/3 < \gamma < 1$. For this case, (3–1a') become

$$
\left\{
\begin{array}{l}
a/c\partial/\partial\tau(WA^2B_z) + \tau^{-\Lambda}\partial/\partial x(WE_y) = 0, \\
a/c\partial/\partial\tau(WE_y) + \tau^{\Lambda-1}\partial/\partial x(WA^2B_z) = 0,
\end{array}
\right.
$$

(6–1a'

$$
\left\{
\begin{array}{l}
a/c\partial/\partial\tau(WA^2B_y) - \tau^{\Lambda-1}\partial/\partial x(W^{-1}A^2E_z) = 0 \\
a/c0/\partial\tau(W^{-1}A^2E_z) - \tau^{-\Lambda}\partial/\partial x(WA^2B_y) = ik\varepsilon_a B_0 \cdot \tau^{\Lambda-1} \cdot \exp\left[ik(x - c/a \cdot \tau)\right]
\end{array}
\right.
$$

and the solutions are

$$
\begin{aligned}
WA^2B_z =& (\bar{E}e^{-ikx} + \bar{F}e^{ikx})(2kc/a \cdot \tau^{1/2})^{1-\Lambda} \cdot \\
& \cdot \left[\bar{G}_1 \cdot J_{1-\Lambda}(2kc/a \cdot \tau^{1/2}) + \bar{H} \cdot J_{\Lambda-1}(2kc/a \cdot \tau^{1/2})\right], \\
WE_y =& -i(-\bar{E}e^{-ikx} + \bar{F}e^{ikx})(2kc/a)^{1-\Lambda} \cdot \tau^{1/2} \cdot \\
& \cdot \left[\bar{H} \cdot J_{\Lambda}(2kc/a \cdot \tau^{1/2}) - \bar{G} \cdot J_{-\Lambda}(2kc/a \cdot \tau^{1/2})\right], \\
W^{-1}A^2E_z =& (\bar{M}e^{-ikx} + \bar{N}e^{ikx})(2kc/a\tau^{1/2})^{1-\Lambda} \cdot \\
& \cdot \left[\bar{P} \cdot J_{1-\Lambda}(2kc/a \cdot \tau^{1/2}) + \bar{Q} \cdot J_{\Lambda-1}(2kc/a \cdot \tau^{1/2})\right] + \\
& + e^{ikx} \cdot \left\{ \sum_{n=0}^{\infty} I_{0_n} b_n \cdot \tau^{n+\Lambda} + \sum_{n=0}^{\infty} c_n \cdot \tau^{n+\Lambda+1} \right\}, \\
b_n =& \frac{i(2\Lambda - 1)(-1)^n (kc/a)^{2n+1} \cdot \varepsilon_a B_0}{(\Lambda+1)_n \cdot (2\Lambda)_n} \cdot \\
& \cdot \left[1/\Lambda(2\Lambda - 1) - \sum_{m=1}^{n} (ia/kc)^m 1/m!(\Lambda)_m \cdot (2\Lambda - 1)_m \right] \\
c_n =& \frac{(-1)^n (kc/a)^{2n+2} \cdot \varepsilon_a B_0}{[(\Lambda+2)_n]^2} \left\{ 1/(\Lambda+1)^2 - \sum_{m=1}^{n} (ia/kc)^m 1/m![(\Lambda+1)_m]^2 \right\}, \\
WA^2B_y =& i(-\bar{M}e^{-ikx} + \bar{N}e^{ikx})(2kc/a)^{1-\Lambda} \cdot \tau^{\Lambda/2} \cdot \\
& \cdot \left[\bar{Q} \cdot J_{\Lambda}(2kc/a \cdot \tau^{1/2}) - \bar{P} \cdot J_{-\Lambda}(2kc/a \cdot \tau^{1/2})\right] + e^{ikx} \cdot \sum_{n=0}^{\infty} a_n \tau^{n+2\Lambda},
\end{aligned}
$$

$$a_n = \frac{(-1)^{n+1} \cdot (kc/a)^{2n+2} \cdot \varepsilon_a \cdot B_0}{(\Lambda+1)_n \cdot (2\Lambda+1)_n} \cdot \left[1/2\Lambda^2 + \sum_{m=1}^{n} (ia/kc)^m \cdot 1/m! \cdot (\Lambda)_m \cdot (2\Lambda)_m \right], \tag{6-2a'}$$

where J_p is the Bessel function of not integer subscript (see footnote 3) and, as usual $(\Lambda)_n = \Lambda \cdot (\Lambda+1) \ldots (\Lambda+n-1)$. Imposing the initial conditions, it remains:

$$\begin{aligned} WE_y &= WA^2B_z = 0, \\ W^{-1}A^2E_z &= e^{ikx} \cdot \left\{ \sum_{n=0}^{\infty} b_n \cdot \tau^{n+\Lambda} + \sum_{n=0}^{\infty} c_n \cdot \tau^{n+\Lambda+1} \right\}, \\ WA^2B_y &= e^{ikx} \cdot \left\{ \sum_{n=0}^{\infty} a_n \cdot \tau^{n+2\Lambda} \right\}, \end{aligned}$$

and then

$$\begin{aligned} F^1_2 &= F^{02} = 0, \\ F^1_3 &= -e^{ikx} \cdot \tau^{2\Lambda-1} \cdot \sum_{n=0}^{\infty} a_n \cdot \tau^n, \\ F^{03} &= e^{ikx} \left[\tau^{2\Lambda-1} \cdot \sum_{n=0}^{\infty} b_n \cdot \tau^n + \tau^{2\Lambda} \sum_{n=0}^{\infty} c_n \cdot \tau^n \right]. \end{aligned} \tag{6-3a'}$$

In the same way, we get from (3-1b'):

$$\begin{aligned} F^1_3 &= F^{03} = 0, \\ F^1_2 &= e^{ikx} \cdot \sum_{n=0}^{\infty} d_n \cdot \tau^n, \\ F^{02} &= -e^{ikx} \cdot \sum_{n=0}^{\infty} e_n \cdot \tau^n, \end{aligned} \tag{6-3b'}$$

where

$$\begin{aligned} d_n &= \frac{(-1)^{n+1} \cdot (kc/a)^{2n+2} \cdot \varepsilon_b \cdot B_0}{(\Lambda+1)_n \cdot (n+1)!} \left[1/\Lambda + \sum_{m=0}^{n} (ia/kc)^m \cdot (\Lambda)_m \right], \\ e_n &= \frac{(-1)^{n+1} \cdot (kc/a)^{2n+2} \cdot \varepsilon_b \cdot B_0}{(\Lambda+2)_n \cdot (n+1)!} \left[1/(\Lambda+1) + \sum_{m=0}^{n} (ia/kc)^m \cdot (\Lambda+1)_m \right]. \end{aligned}$$

7. Conclusions

As already mentioned in the first Section, the process we have studied and the inverse process, on the ground of general principles, must have the same rate.

Therefore our process, if working effectively, does not help to annihilate gravitons but, on the contrary, it supplies them with the same frequency of the photons, because in the mutual conversion the frequency is not altered. To estimate the effectiveness of the process we shall calculate for each case the coefficient η =(produced flux of e.m. waves)/(incident flux of g. wave). For the produced e.m. waves travelling in the positive x-direction we must calculate $c \cdot \overline{T^{01}_{em}}$ (-means average). For the first g. wave polarization (ε_a) we have:

$$c \cdot \overline{T^{01}_{em}} = -c/4\pi \overline{(ReF^1_3 \cdot ReF^{03})}$$

and for the polarization ε_b:

$$c \cdot \overline{T^{01}_{em}} = -c/4\pi \overline{(ReF^{12} \cdot ReF^{02})}.$$

Moreover, we remember that $c \cdot \overline{T^{01}_{\text{grav}}} = c^5/32\pi G \cdot K^2 \cdot \varepsilon^2_{a,b}$. The coefficient η must be evaluated each time, taking into account, when depending on the frequency of the incident g. wave of the value of the frequency at that time (considering the redshift of the particular model). Obviously, as we have inserted in the Maxwell equations asymptotic expressions of $A(\tau)$ and $W(\tau)$, the expressions we get for η are valid near the singularity. In the first case (axisymmetric pancake singularity), we have for the polarization ε_a

$$\eta \sim 4/25 \left(\frac{GB_o^2}{c^4}\right) \frac{c^2}{a^2} \tau^4.$$

η is growing with time and, for $B_0 \sim B_{\text{crit}} : \eta \to 1$ for $\tau \to 10^{-8}$ $(t = 10^9 s)$. For the polarization ε_b

$$\eta \sim 4/9 (\frac{GB_0^2}{c^4}) \cdot \frac{c^2}{a^2} \tau^2;$$

taking again $B_0 \sim B_{\text{crit}}$, $\eta \to 1$ for $\tau \to 10^{-16}$ $(t \sim 10\text{s})$. For both polarizations η does not depend on the frequency of the incident gravitational wave.

In the second case (isotropic point singularity) there is no difference between the two polarizations and we get

$$\eta \sim 16 (\frac{GB_0^2}{c^4}) \cdot \frac{1}{k^2} \tau^{-3\Gamma}.$$

Here we have a dependence on the frequency and the process is appreciable only at times inconceivably smaller than the Planck time. As an example, for $B_0 \sim 10^{-8}$ gauss and a frequency $\sim t_g^{-1}$ at $t = t_g$, one has $\eta \sim 1$ for $\tau^{-\Gamma} \sim 10^{70}$ $(1/3 < \Gamma < 1/2)$. The axisymmetric hard-magnetic solution gives η not depending on the frequency and for the polarization ε_a.

$$\eta \sim 4 (\frac{GB_0^2}{c^4}) \cdot \frac{c^2}{a^2} \cdot \tau^{4\Lambda - 3} = \frac{(1-\gamma)(3\gamma - 1)}{(1+\gamma)^2} \tau^{4\Lambda - 3}.$$

For the polarization ε_b, we have

$$\eta \sim 4(\frac{GB_0^2}{c^4})\frac{c^2}{a^2}\tau^{-2} = \frac{(1-\gamma)93\gamma - 1)}{(1+\gamma)^2}\tau^{-2}$$

$(1/3 < \gamma < 1)$, $\Lambda = (1-\gamma)/(1+\gamma)$. Through decreasing with time η, in this case, has unphysical values also at large τ.

If we consider the first case as the more realistic one (it has finite magnetic energy density at the early times and becomes isotropic at large τ), we are led to think that with sufficiently large primordial magnetic fields a continuous interchange between photons and gravitons can take place. In this way, the e.m. waves and the g. waves retain the same frequency.

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