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REPARAMETRIZATION OF SUPERGROUP: SUPERSPACE AS A VECTORSPACE

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In this paper we show that, with reparametrization of supergroup, superspace as the homogeneous space of it will be linearized, superfield will be defined uniquely, vector coordinates and spinor coordinates will play the equal rôle... We can list all possible subgroups of supergroup easily in this way of parametrization. The representation of this algebra will be given. The linearization of superspace would lead to new approaches to construct geometrical structures on it. The Abelian translation group would make easier the construction of the harmonic analysis on it. Last of all, the $SU(N)$ internal symmetry of extended superunified theories would be manifest in these models.

1. Introduction

In original works [1, 2], the supertranslation group has been introduced with parametrization by the generators Q, Q, P of the pseudo Lie algebra

$$\{Q, \bar{Q}\} = 2\sigma_\mu P^\mu ,$$
$$\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = [P_\mu, Q] = [P_\mu, \bar{Q}] = [P_\mu, P_\nu] = 0 . \tag{1.1}$$

Elements of the supertranslation group can be parametrized as:

$$G(c_\mu, \zeta, \bar{\zeta}) = \exp i(c_\mu P^\mu + \zeta Q + \bar{Q}\bar{\zeta}) . \tag{1.2}$$

The product defined on the group is not linear, not global, even not commutative [3, 4]:

$$\begin{aligned} &G(c_{1\mu}, \zeta_1, \bar{\zeta}_1) \cdot G(c_{2\mu}, \zeta_2, \bar{\zeta}_2) = \\ &= G(c_{1\mu} + c_{2\mu} + i(\zeta_2 \sigma_\mu \zeta_1 - \zeta_1 \sigma_\mu \bar{\zeta}_2), \quad \zeta_1 + \zeta_2, \bar{\zeta}_1 + \bar{\zeta}_2) . \end{aligned} \tag{1.3}$$

Then the superspace and the superfields defined on it can be introduced as:

$$\Phi(x, \vartheta, \bar{\vartheta}) = \exp [i(x_\mu P^\mu + \vartheta Q + \bar{Q}\bar{\vartheta})] \Phi(0, 0) . \tag{1.4}$$

Using the Cambell—Haussdorf identity, by such way of parametrization with $Q, \bar{Q}$ and P^μ we have three different definitions of superfield corresponding to the following three choices (see [5]):

$$\exp[i(x_\mu P^\mu + \vartheta Q + \bar{Q}\bar{\vartheta})]\,\Phi(0,0) \equiv \Phi(x, \vartheta, \bar{\vartheta}),$$
$$\exp[i(x_\mu P^\mu + \vartheta Q)] \cdot \exp(i\bar{Q}\bar{\vartheta})\Phi(0,0) \equiv \Phi_1(x, \vartheta, \bar{\vartheta}), \tag{1.5}$$
$$\exp[i(x_\mu P^\mu + \bar{Q}\bar{\vartheta})] \cdot \exp(i\vartheta Q)\Phi(0,0) \equiv \Phi_2(x, \vartheta, \bar{\vartheta}).$$

This way of parametrization has some "esthetical" shortcomings:

a) The spinortranslations do not form one-parameter group. This kind of parametrization is not as canonical as the usual techniques treated with Lie groups.

b) The supertranslation group is not commutative. The harmonic analysis and the generalized Fourier transformations have not been discussed yet on the superspace. Because the supertranslation group is not compact, but locally compact only, the commutativity would make this construction easier by the recipe given in [6].

c) Superfields are not determined uniquely.

d) When the geometrical structures on a manifold are constructed, an algebraic structure used to be given on it, so that the manifold will become a vectorspace. Concretely, one always makes an additive group isomorph with the translation group. With the previous parametrization both cannot be realized at the same time, because the supertranslation group is not Abelian. We cannot use the standard techniques to construct a linear geometry on it.

e) Geometrically speaking, up to now the superspace is considered as a fibre bundle [7, 8], not base manifold. The geometry on it is a non-linear one. So vector coordinates and spinor coordinates do not play the same rôle in our formalism.

In this study on the reparametrization of supergroup, we try to overcome all these minor esthetical shortcomings.

2. Reparametrization of supergroup and superspace

In [9] the authors have affirmed that the most general form of the commutator of the Fermi generators of degree 1 is:

$$\{Q^L, \bar{Q}^M\} = c^{LM}\sigma_\mu P^\mu. \tag{2.1}$$

According to them, if $c^{LM}=0$ then $Q=\bar{Q}=0$. So in the case of non-vanishing Fermi generators c^{LM} can always be normalized to δ^{LM}. However, it is not true, if we reexamine the analysis in which $\{Q^L, \bar{Q}^M\}$ belongs to the $(1/2, 0)(0, 1/2) = (1/2, 1/2)$ representation of Lorentz group. Because zero belongs to any representation, so in (2. 1) c^{LM} can vanish quite right. For instance, let us take the following non-vanishing Fermi generators:

$$Tsp^L = Q^L + i\bar{\vartheta}^L\sigma_\mu P^\mu,$$
$$\overline{Tsp}^L = \bar{Q}^L + i\vartheta^L\sigma_\mu P^\mu, \tag{2.2}$$

with Q and $\bar{Q}$ given in (1.1). ϑ and $\bar{\vartheta}$ are spinorcoordinate operators of the superspace (1.3). As we know from [3]

$$\{Q, \vartheta\} = \{\bar{Q}, \bar{\vartheta}\} = i\,. \tag{2.3}$$

We have the following minimal extension of Poincaré algebra

$$\begin{aligned} &[P_\mu, Tsp] = [P_\mu, \overline{Tsp}] = \{Tsp, \overline{Tsp}\}\,,\\ &\{Tsp, Tsp\} = \{\overline{Tsp}, \overline{Tsp}\} = 0\,,\\ &[M_{\mu\nu}, Tsp] = 1/2\cdot\sigma_{\mu\nu}\overline{Tsp}\,,\\ &[M_{\mu\nu}, Tsp] = 1/2\cdot\tilde{\sigma}_{\mu\nu}\overline{Tsp}\,. \end{aligned} \tag{2.4}$$

We can see that in this case Fermi generators form a Grassman algebra (while the generators given in (1.1) form a Clifford algebra). Supertranslation group will be parametrized as:

$$G(a_\mu, \zeta, \bar{\zeta}) = \exp\left[i(a_\mu P^\mu + \zeta Tsp + \overline{Tsp}\bar{\zeta})\right]. \tag{2.5}$$

The product defined on this group turns to be commutative, linear and global:

$$G(a_{1\mu}, \zeta_1, \bar{\zeta}_1)\cdot G(a_{2\mu}, \zeta_2, \bar{\zeta}_2) = G(a_{1\mu} + a_{2\mu}, \zeta_1 + \zeta_2, \bar{\zeta}_1 + \bar{\zeta}_2)\,. \tag{2.6}$$

Superfield will be determined uniquely if the superspace is parametrized with $Tsp, \overline{Tsp}$ and P^μ:

$$\Phi(x, \vartheta, \bar{\vartheta}) = \exp\left[i(x_\mu P^\mu + \vartheta Tsp + \overline{Tsp}\bar{\vartheta})\right]\Phi(0, 0)\,. \tag{2.7}$$

Let us parametrize the elements of the minimal spinor extension of Poincaré group with generators $Tsp, \overline{Tsp}, P^\mu$ and $M_{\mu\nu}$ by $\{\Lambda, a_\mu, \zeta\}$. We can get easily the set of all possible subgroups of it, when we fix each parameter:

1. $\{1, 0, 0\} = 1$: it is the trivial group, the unit of the supergroup.
2. $\{\Lambda, 0, 0\} \in \mathfrak{L}$: it is the usual Lorentz group.
3. $\{\Lambda, a_\mu, 0\} \in \mathfrak{P}$: it is the usual Poincaré group.
4. $\{1, a_\mu, \zeta\} \in \mathfrak{S}T$: it is the commutative supertranslation group.
5. $\{1, a_\mu, 0\} \in T$: it is the usual space time translation group.
6. $\{1, 0, \zeta\} \in Tsp$: it is the new spinor translation group.
7. $\{\Lambda, 0, \zeta\} \in \mathfrak{S}\mathfrak{L}$: it is the new super Lorentz group.

It is worth noting that in this way of parametrization we get two new groups Tsp and $\mathfrak{S}\mathfrak{L}$. The consideration of these is an interesting work, and it will be discussed elsewhere.

The superspace now is the set of numbers $(x_\mu, \vartheta, \bar{\vartheta})$ transforming under the action of the element $\{\Lambda, a_\mu, \zeta\}$ of supergroup as follows:

$$\begin{aligned} &x_\mu \to \Lambda^\nu_\mu x_\nu + a_\mu\,,\\ &\vartheta \to A(\Lambda)\vartheta + \zeta\,,\\ &\bar{\vartheta} \to \bar{A}(\Lambda)\bar{\vartheta} + \bar{\zeta}\,. \end{aligned} \tag{2.8}$$

So we come to a natural and general definition of supergroup: *Supergroup is the group of inhomogeneous linear transformations acting on superspace and leaving Minkowski space and spinorspace invariant.*

3. Representation of Tsp algebra

Let us consider the massive case: Take $\vec{P}=0$ and $P_0=m$ $Tsp \cdot \overline{Tsp}$ is Casimir operator then, because it commutes with all other generators. So we have:

i) *Irreducible multiplets with* $\overline{Tsp} \cdot Tsp\,|\varphi\rangle=0$

There are two possibilities:

a) $Tsp\,|m, J, J_3\rangle_0^l=0$: $|m, J, J_3\rangle_0^l$ is the Grassman vacuum with left-handed chirality. This vacuum degenerates with four states forming a complete basis, which spans a 4-dimensional representation space:

$$|m, J, J_3\rangle_0^l; \quad Tsp_\alpha|m, J, J_3\rangle_0^l; \quad Tsp_\alpha Tsp_\beta|m, J, J_3\rangle_0^l . \tag{3.1}$$

b) $Tsp|m, J, J_3\rangle_0^r=0$: $|m, J, J_3\rangle_0^r$ is the Grassman vacuum with right-handed chirality. This vacuum degenerates with four states forming a complete basis, which spans a 4-dimensional representation space:

$$|m, J, J_3\rangle_0^r; \quad \overline{Tsp}_\alpha|m, J, J_3\rangle_0^r; \quad \overline{Tsp}_\alpha \overline{Tsp}_\beta|m, J, J_3\rangle_0^r . \tag{3.2}$$

ii) *Irreducible multiplets with* $\overline{Tsp} \cdot Tsp|\varphi\rangle=0$

It occupies 8 states forming a complete basis. It is an octet with neutral chirality:

$$\begin{gathered}
Tsp|m, J, J_3\rangle_0^n \neq 0; \quad \overline{Tsp}|m, J, J_3\rangle_0^n \neq 0: \\
|m, J, J_3\rangle_0^n; \quad Tsp_\alpha|m, J, J_3\rangle_0^n; \quad \overline{Tsp}_{\dot\alpha}|m, J, J_3\rangle_0^n; \\
\overline{Tsp}_{\dot\alpha} Tsp_\beta|m, J, J_3\rangle_0^n; \quad \overline{Tsp}_{\dot\alpha} \overline{Tsp}_{\dot\beta}|m, J, J_3\rangle_0^n; \\
Tsp_\alpha Tsp_\beta|m, J, J_3\rangle_0^n .
\end{gathered} \tag{3.3}$$

In the space of functions on the supergroup, we represent the generators of the minimal spinor extension of Poincaré group as:

$$\begin{aligned}
&P_\mu=i\partial_\mu , \\
&M_{\mu\nu}=i(x_\mu\partial_\nu-x_\nu\partial_\mu)+1/2 \cdot \left(\sigma_{\mu\nu}\bar\vartheta \frac{\partial}{\partial\vartheta}-\tilde\sigma_{\mu\nu}\vartheta \frac{\partial}{\partial\bar\vartheta}\right), \\
&Tsp_\alpha=i\partial/\partial\vartheta_\alpha; \quad \overline{Tsp}_{\dot\alpha}=i\partial/\partial\bar\vartheta_{\dot\alpha} .
\end{aligned} \tag{3.4}$$

Generally speaking, the 16-component superfield is reducible and it can be reduced into the sum of an irreducible quartet with left-handed chirality (independent of $\bar{\vartheta}$), and an irreducible quartet with right-handed chirality and an octet with neutral chirality. At the same time, it is the product of a superfield with right-handed chirality and a superfield with left-handed chirality:

$$\begin{aligned}\Phi(x_\mu, \vartheta, \bar{\vartheta}) &= \Phi^r(x, \vartheta) + \Phi^l(x, \bar{\vartheta}) + \Phi^n(x, \vartheta, \bar{\vartheta}) = \\ &= \Phi'^r(x, \vartheta) \cdot \Phi'^l(x, \bar{\vartheta}) .\end{aligned} \tag{3.5}$$

Let us note that the product of superfields of the same chirality is a superfield of that chirality.

4. Discussion

The construction of Lagrange field theory for *Tsp* algebra is straightforward by the standard method proposed by Salam and Stradee [2]. Here we do not discuss it in detail.

From a geometrical viewpoint, the constructon of geometrical structures on the superspace as a vectorspace is a very interesting work. For this purpose we would define a certain scalar product on superspace as:

$$\langle Z, Z' \rangle = g_{mn} Z^m \cdot Z^n \qquad \text{where} \qquad Z^m = (x_\mu, \vartheta, \bar{\vartheta}) .$$

First we consider the flat superspace, in which g_{mn} is a global supermetric tensor. Only from the scalar nature and from the symmetric property of this product with ϑ and $\bar{\vartheta}$ we come to

$$\langle Z, Z \rangle = f(x_\mu x^\mu, \vartheta, \bar{\vartheta}) = x_\mu x^\mu . \tag{4.1}$$

Proof: From the scalar nature of this product, we must pair ϑ with $\bar{\vartheta}$ in any terms of this product. So:

$$\langle Z, Z \rangle = f(x_\mu x^\mu, \vartheta\bar{\vartheta}) .$$

But because the product is symmetric, if in a certain term of the product there is a $\vartheta\bar{\vartheta} \cdot C(x, \vartheta, \bar{\vartheta})$, there must be $\bar{\vartheta}\vartheta \cdot C(x, \vartheta, \bar{\vartheta})$ terms in the product. However, the two terms destroy each another.

So: *Distance in superspace is the distance is Minkowski space, but the angle in superspace is not that one in Minkowski space.*

So we can see that the spinor coordinates would give contributions to the curvature of manifolds in the superspace. Indeed, if we give a hypersurface by the following equation:

$$\vartheta = \vartheta(x); \quad \bar{\vartheta}(x) = \bar{\vartheta} . \tag{4.2}$$

We can come to a curved Minkowski space as a physical manifold in the superspace with the following metric tensor:

$$
\begin{aligned}
g_{\mu\nu}(x) &= g_{\mu\nu} + g_{\mu\alpha}\cdot\partial\vartheta^{\alpha}(x)/\partial x^{\nu} + g_{\mu\dot\alpha}\partial\bar\vartheta^{\dot\alpha}(x)/\partial x^{\nu} + \\
&+ g_{\alpha\nu}\partial\vartheta^{\alpha}(x)/\partial x^{\mu} + g_{\dot\alpha\nu}\partial\bar\vartheta^{\dot\alpha}(x)/\partial x^{\mu} + \\
&+ g_{\alpha\beta}\partial\vartheta^{\alpha}(x)/\partial x^{\mu}\cdot\partial\vartheta^{\beta}(x)/\partial x^{\nu} + g_{\dot\alpha\dot\beta}\partial\bar\vartheta^{\dot\alpha}(x)/\partial x^{\mu}\cdot\partial\bar\vartheta^{\dot\beta}(x)/\partial x^{\nu} + \\
&+ g_{\dot\beta\alpha}\partial\bar\vartheta^{\dot\beta}(x)/\partial x^{\mu}\cdot\partial\bar\vartheta^{\alpha}(x)/\partial x^{\nu} + g_{\alpha\dot\beta}\partial\vartheta^{\alpha}(x)/\partial x^{\mu}\cdot\partial\bar\vartheta^{\dot\beta}(x)/\partial x^{\nu}.
\end{aligned}
\tag{4.3}
$$

Specially, if we choose:

$$
g_{mn} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i & 0 & 0
\end{pmatrix}
\tag{4.4}
$$

With the assumption that the scalar product is bilinear and symmetric with ϑ and $\bar\vartheta$. Then (4. 3) gets the following form:

$$
\begin{aligned}
g_{\mu\nu}(x) &= g_{\mu\nu} + i(\partial\vartheta(x)/\partial x^{\mu}\cdot\partial\bar\vartheta(x)/\partial x^{\nu} + \\
&+ \partial\bar\vartheta(x)/\partial x^{\mu}\cdot\partial\vartheta(x)/\partial x^{\nu}) \equiv g_{\mu\nu} + S\,.
\end{aligned}
\tag{4.5}
$$

The S-term will cause the curvature of the space. We stop the discussion with the remark that: *In the superspace we can get all configurations of gravity corresponding to the set of the possible* $\vartheta = \vartheta(x)$; $\bar\vartheta = \bar\vartheta(x)$ *in it.*

In conclusion, we note that with *Tsp* algebra, the $SU(N)$ internal symmetry will be manifest.

Consider the extended superunified algebra in the general form:

$$
\begin{aligned}
&[P_\mu, P_\nu] = [P_\mu, B_l] = [P_\mu, Q^L] = [M_{\mu\nu}, B_l] = 0\,, \\
&[M_{\mu\nu}, P_\rho] = i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu)\,, \\
&[M_{\mu\nu}, Q^L] = 1/2\cdot\sigma_{\mu\nu}Q^L\,, \\
&\{Q^L_\alpha, Q^M_\beta\} = \varepsilon_{\alpha\beta}\Sigma(a_l)^{LM}\cdot B_l\,, \\
&\{Q^L_\alpha, \bar Q^M_{\dot\alpha}\} = c^{LM}(\sigma_\mu)_{\alpha\dot\alpha}P^\mu\,, \\
&[B_l, B_m] = i\Sigma c^k_{lm}B_k\,, \\
&[B_l, Q^L] = \Sigma s_l^{LM}\cdot Q^M\,.
\end{aligned}
\tag{4.6}
$$

When $c^{LM}=0$, we get the *Tsp* algebra.

Using Jacobi identity (B_l, Q^L, Q^M) we come to:

$$\Sigma c^{LM}\cdot \bar{s}_l^{MN}+\Sigma s_l^{LM}\cdot c^{MN}=0 \tag{4.7}$$

or in the matrix form: $c\cdot \bar{s}_l=-\bar{s}\cdot c$.

In usual models, with $c^{LM}=\delta^{LM}$, we have $\bar{s}_l=-s_l$. As we now, s_l matrices are the representations of the internal symmetry group. So the internal symmetry group must be orthogonal. We could get the $SU(N)$ symmetry only after a lengthy manipulation with so called self-duality (see [10]).

In our *Tsp* algebra $c^{LM}=0$, (4.7) satisfies automatically. So we can get a manifest $SU(N)$ superunified theory. This kind of symmetry fits better with reality.

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