ACHROMATIC COMBINATIONS OF BIREFRINGENT PLATES

Part II. An Achromatic Quarter-Wave Plate

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1. Introduction

A combination of birefringent plates which can be used for producing circularly polarised light over a fairly large range of wave-lengths has been described in Part I. And it has there been pointed out that the arrangement cannot be properly described as an achromatic quarter-wave plate. A quarter-wave plate when rotated between crossed nicols will show two extinction positions; the achromatic combination described will show none. And to understand under what conditions a combination of plates will have the properties of a single birefringent plate—let alone a plate which is in addition achromatic—it is desirable to have a general method for combining the action of a pile of birefringent plates when their principal planes are not parallel. We shall follow a method differing slightly from that which may be found described by Pockels.[1]

2. General Considerations

The problem of designing an achromatic elliptic polariser is the problem of transforming a particular state of polarization represented by P_1 (on the Poincaré sphere) to another particular state P_1'. But, of an achromatic retardation plate we require that it transforms *every* P to a corresponding P′ obtained by rotating the sphere by an angle δ about an equatorial diameter.

Passage of monochromatic polarised light through a succession of birefringent plates corresponds to a succession of rotations of the Poincaré sphere about a series of equatorial diameters. Any succession of rotations of the sphere can, however, be compounded into one single resultant rotation. Hence the action of a combination of birefringent plates corresponds to a rotation ϕ of the Poincaré sphere about some diameter EE′—which will in general be inclined to the equatorial plane; this means, of course, that the combination is equivalent to a single plate having elliptic birefringence—

E and E′ being the orthogonal elliptic vibrations propagated without change of form, and between which a phase retardation ϕ is incurred.

The condition that the combination should act as a purely birefringent plate (for a particular wave-length λ) is that the axis EE′ of the resultant rotation should be on the equator; so that, if 2ω denote the lattitude of E, the faster ellipse, $2\omega = 0$. And if the retardation of the plate is to have a required value δ, the magnitude of the resultant rotation ϕ should be equated to this value.

Both the magnitude ϕ as well as the axis EE′ of the resultant rotation will in general alter with the wave-length, since the magnitudes (though not the axes) of the component rotations are a function of wave-length. Hence, for the combination to be considered achromatic in the immediate neighbourhood of the wave-length λ, some more conditions have to be imposed; when the wave-length is increased by $\Delta\lambda$ the values of the composite phase retardation ϕ, as well as the latitude 2ω and longitude $2l$ of E, should remain unaltered at least to the first order of approximation.

Instead of following matrix methods, we shall use a well-known geometrical construction (illustrated in Fig. 1) for finding the resultant of any

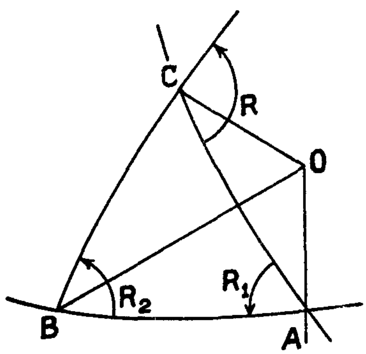

FIG. 1. Composition of Rotations: $2R_1 + 2R_2 = 2R$.

two rotations. If ABC be a spherical triangle described on a sphere whose centre is O then: *a rotation about AO through twice the internal angle at A, followed by a rotation about BO through twice the internal angle at B is equivalent to a rotation about CO through twice the external angle at C.*

3. A Birefringent Combination with Variable Retardation

We shall now show that a combination of three birefringent plates, the first and last of which have their corresponding principal planes parallel and their retardations identical, will behave as a single purely birefringent plate of retardation 2δ, whose fast vibration direction is inclined at an angle c_1 to that of the first plate, where 2δ and c_1 are given by:

$$\cos \delta = \cos 2\delta_1 \cos \delta_2 - \sin 2\delta_1 \sin \delta_2 \cos 2c \qquad (a)$$

and

$$\cot 2c_1 = \operatorname{cosec} 2c\,(\sin 2\delta_1 \cot \delta_2 + \cos 2\delta_1 \cos 2c) \qquad (b)$$

In the above relations $2\delta_1$ is the common retardation of the first and last plates, $2\delta_2$ that of the central plate, and c the angle between the fast vibration directions of the central plate and the other two.

Before proceeding to prove these relations, we may mention that by choosing 2 quarter-wave plates as the first and last plates, and a half-wave plate as the central plate, we can, by rotating the central plate, vary the retardation of the combination continuously from 0 to 2π; the principal planes of the combination will always be inclined at 45° to those of the first plate.

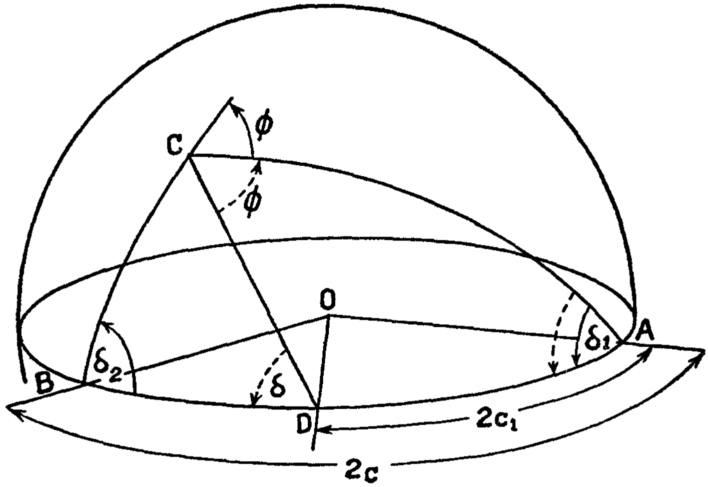

FIG 2.

Referring to Fig. 2, a rotation $2\delta_1$ about AO followed by a rotation $2\delta_2$ about BO, where A and B give the orientation of the fast axes of the first and middle plates respectively, is equivalent to a rotation 2ϕ about CO—where CAB $= \delta_1$, CBA $= \delta_2$ and $\pi -$ ACB $= \phi$. Let us draw an arc CD such that ACD $= \phi$, D being the point of intersection of this arc with the equator. Considering now the triangle ACD, the rotation 2ϕ about C (representing the combined action of the first 2 plates) followed by a rotation $2\delta_1$ about A (which now represents the action of the third plate) is equivalent to a rotation about D through twice the angle external to ADC. The combination is therefore equivalent to a birefringent plate of retardation 2δ whose fast axis is inclined at c_1 to that of the first plate—where $\delta = \pi - \text{ADC}$, and $2c_1 = \text{arc AD}$.

The expressions which we shall derive for δ and c_1 do not depend on assuming that D lies within AB.

From the triangle ADC, denoting arc CD by d

$$\cos \delta_1 = \cos \phi \cos \delta + \sin \phi \sin \delta \cos d \tag{1}$$

From the triangle DBC

$$\cos \delta_2 = \cos 2\phi \cos \delta + \sin 2\phi \sin \delta \cos d \tag{2}$$

Eliminating d by multiplying (1) by $2 \cos \phi$ and subtracting from (2), we obtain:

$$\cos \delta = 2 \cos \phi \cos \delta_1 - \cos \delta_2$$

The value of $\cos\phi$ to be substituted is obtained from the triangle ABC.

$$\cos \phi = \cos \delta_1 \cos \delta_2 - \sin \delta_1 \sin \delta_2 \cos 2c$$

Hence

$$\cos \delta = \cos 2\delta_1 \cos \delta_2 - \sin 2\delta_1 \sin \delta_2 \cos 2c$$

which is the required expression (a).

To get an expression for c_1, we use a well-known trigonometrical theorem, which gives from triangle ABC, on denoting are AC by b

$$\cot 2c \sin b = \cot \text{ACB} \sin \delta_1 + \cos b \cos \delta_1$$

and from triangle ACD

$$\cot 2c_1 \sin b = \cot \text{ACD} \sin \delta_1 + \cos b \cos \delta_1$$

or

$$\sin b (\cot 2c_1 - \cot 2c) = 2 \sin \delta_1 \cot \phi$$

Substituting

$$\sin b = \sin \delta_2 \cdot \frac{\sin 2c}{\sin \phi}$$

we obtain,

$$\sin 2c (\cot 2c_1 - \cot 2c) = \frac{2 \sin \delta_1}{\sin \delta_2} \cos \phi$$

i.e.,

$$\sin 2c \cot 2c_1 - \cos 2c = \frac{2 \sin \delta_1}{\sin \delta_2} (\cos \delta_1 \cos \delta_2 - \sin \delta_1 \sin \delta_2 \cos 2c)$$

or

$$\sin 2c \cot 2c_1 = \sin 2\delta_1 \cot \delta_2 + \cos 2\delta_1 \cos 2c$$

which is the required expression (*b*).

4. Conditions for Achromatism

To make a combination of the type described above achromatic in the immediate neighbourhood of the wave-length λ we may differentiate the expressions on the right-hand side of the equations (*a*) and (*b*), and equate to zero. Since however our object is to attain achromatism over a finite range of wave-lengths we proceed in a different manner.

Let λ' and λ'' be two wave-lengths on either side of λ for which the retardations of each individual plate are multiplied by a factor $(1 - f)$ and $(1 + f)$ respectively. Values corresponding to these two wave-lengths will be denoted by corresponding single- and double-primed symbols.

We impose the conditions that the retardation $2\delta'$ and $2\delta''$ of the combination for these two wave-lengths be equal to one another and to 2Δ, the required retardation of the achromatic combination; also that the orientations of the fast axes of the combination for these two wave-lengths should be the same, *i.e.*, $c_1' = c_1''$. We then get three equations involving the three unknowns δ_1, δ_2 and c:

$$\cos \Delta = \cos 2\delta_1' \cos \delta_2' - \sin 2\delta'_1 \sin \delta_2' \cos 2c \tag{1}$$

$$\cos \Delta = \cos 2\delta_1'' \cos \delta_2'' - \sin 2\delta_1'' \sin \delta_2'' \cos 2c \tag{2}$$

$$\sin 2\delta_1' \cot \delta_2' + \cos 2\delta_1' \cos 2c = \sin 2\delta_1'' \cot \delta_2'' + \cos 2\delta_1'' \cos 2c \tag{3}$$

From these three relations we may get two relations involving only the unknowns δ_1 and δ_2.

Thus eliminating $\cos 2c$ from (1) and (2)

$$\frac{\cos \delta_2' \cos 2\delta_1' - \cos \Delta}{\sin 2\delta_1' \sin \delta_2'} = \frac{\cos \delta_2'' \cos 2\delta_1'' - \cos \Delta}{\sin 2\delta_1'' \sin \delta_2''} \tag{4}$$

For the second equation, we eliminate $\cos 2c$ as well as $\cos \Delta$ between the three equations. Thus equating the right-hand expression of (1) and (2), and then substituting the value of $\cos 2c$ from (3) we get:

$$\frac{\cos 2\delta_1' \cos \delta_2' - \cos 2\delta_1'' \cos \delta_2''}{\sin 2\delta_1' \sin \delta_2' - \sin 2\delta_1'' \sin \delta_2''} = \frac{\sin 2\delta_1' \cot \delta_2' - \sin 2\delta_1'' \cot \delta_2''}{\cos 2\delta_1'' - \cos 2\delta_1'}$$

which on simplification yields,

$$(1 - \cos 2\delta_1' \cos 2\delta_1'')(\cos \delta_2' + \cos \delta_2'')$$

$$= \sin 2\delta_1' \sin 2\delta_1'' \frac{\sin \delta_2' \cos \delta_2'' + \cos \delta_2' \sin \delta_2''}{\sin \delta_2' \sin \delta_2''}$$

Substituting

$$\delta_2' = (1 - f)\,\delta_2 \text{ and } \delta_2'' = (1 + f)\,\delta_2,$$

we get

$$(1 - \cos 2\delta_1' \cos 2\delta_1'')\, 2 \cos \delta_2 \cos f \delta_2 = \frac{\sin 2\delta_1' \sin \delta_1'' \cdot \sin 2\delta_2}{\frac{1}{2}(\cos f \delta_2 - \cos \delta_2)} \quad (5)$$

We have to find values of δ_1 and δ_2 which simultaneously satisfy (4) and (5). The task is considerably simplified when we note that equation (5) is identically satisfied for $\delta_2 = \pi/2$. The corresponding value of δ_1 may be found from (4), which now becomes:

$$\sin f \frac{\pi}{2} \cdot \sin 2(\delta_1' + \delta_1'') = \cos \Delta (\sin 2\delta_1'' - \sin 2\delta_1')$$

Substituting

$$\delta_1' = (1 - f)\,\delta_1 \text{ and } \delta_1'' = (1 + f)\,\delta_1,$$

this gives

$$\sin f \,.\, 2\delta_1 = \frac{\sin f \cdot \frac{\pi}{2}}{\cos \Delta} \cdot \sin 2\delta_1 \quad \text{(I)}$$

An expression for $\cos 2c$ is got by eliminating Δ from (1) and (2).

$$\cos 2c = \frac{\sin f \frac{\pi}{2}}{\cos f \frac{\pi}{2}} \cdot \frac{\cos 2\delta_1' + \cos 2\delta_1''}{\sin 2\delta_1' - \sin 2\delta_1''}$$

or

$$\cos 2c = -\frac{\tan f \frac{\pi}{2}}{\tan f \cdot 2\delta_1} \quad \text{(II)}$$

For any required retardation 2Δ and for any choice of the arbitrary parameter f which determines the range of achromatism, the retardation $2\delta_1$ of the first and last plates of the achromatic combination may be found from I; since this is a transcendental equation it has to be solved by an iteration procedure. The orientation c of the central plate, which should be a half-wave plate, may then be obtained from II. The orientation c_1' of the principal planes of the combination may, if required, be calculated from

$$\sin 2c \cot 2c_1' = \sin (1 - f)\, 2\delta_1 \tan f \frac{\pi}{2} + \cos (1 - f)\, 2\delta_1 \cos 2c \quad \text{(III)}$$

The retardation and the orientation of the principal planes of the achromatic retardation plate, taken as a whole, will disperse slightly with wave-length. The combination will possess exactly the same retardation 2Δ and the same orientation c_1' of its fast vibration direction, for two wave-lengths; these wave-lengths, if we neglect the dispersion of birefringence, will be $(1+f)\lambda$ and $(1-f)\lambda$; and we may, somewhat arbitrarily, consider the range of achromatism as extending from $(1+\sqrt{2}f)\lambda$ to $(1-\sqrt{2}f)\lambda$.

We shall present the numerical solutions only for the case of an achromatic quarter-wave plate—for which we must substitute $2\Delta = 90°$. If we choose $f = \cdot 18$ the solution of I can be shown to be $2\delta_1 = 115° \, 42'$; so that from II, $\cos 2c = -\cdot 7639$ or $2c = 139° \, 48'$. The corresponding value of c_1' is got from III, which yields $\cot 2c_1' = \cdot 5489$ or $2c_1' = 61° \, 14'$.

On the other hand if $f = \cdot 1414$ we get

$$2\delta_1 = 115° \, 30', \; 2c = 140° \, 26', \; 2c_1' = 61° \, 30'$$

Thus an achromatic quarter-wave plate is obtained by superposing three plates of the same birefringent material (the dispersion of the birefringence of which need not be negligible). The central plate should be a half-wave plate for the wave-length λ in the middle of the spectral range to be covered. The first and last plates should have their principal planes in parallel orientation and their retardations of the same magnitude—equal to, say, $2\delta_1$ for the wave-length λ. With respect to the common fast vibration direction of the first and last plates, let the inclinations of the fast vibration direction of the central plate, and of the combination taken as a whole, be c and c_1' respectively. Then for covering the range of wave-lengths from about $1\cdot 25\,\lambda$ to $\cdot 75\,\lambda$ the set of values, $2\delta_1 = 115° \, 42'$, $c = 69° \, 54'$ may be used—for which $c_1' = 30° \, 37'$. The achromatism is not high but can be increased by restricting the range to be covered. Thus for covering the region from $1\cdot 2\,\lambda$ to $\cdot 8\,\lambda$ the values suitable are:

$$2\delta_1 = 115° \, 30', \; c = 70° \, 13',$$

for which

$$c_1' = 30° \, 45'$$

We may mention that the range over which these combinations may be considered achromatic would have to be altered if the dispersion of the birefringence is appreciable, but their essentially achromatic nature would not be changed. For retardation plates of mica, the dispersion of birefringence is negligible.[2]

Experimental Verification

Two plates of mica were cleaved, whose thicknesses were in the ratio of 180 : 115·75, the former being nearly a half-wave plate for the D lines of sodium. The second plate was cut into two portions which were then superposed and cemented together at a corner—the principal planes of the two portions being made exactly coincident while the cement was still wet. After the cement was dry, the half-wave plate—with a drop of copal varnish on either side—was inserted *between* the other two plates at roughly the calculated angle. The combination was placed on a mirror and, with the aid of another mirror, white light from a point source was made to pass normally through a polaroid and the combination of plates and then back again—after which it reached the eye. The orientation of the central plate was then gradually altered while the varnish was still wet—and a position was found where the image of the point source could be almost completely extinguished by rotating the polaroid. It was also verified that the combination thus prepared showed two practically black extinction positions when rotated between crossed polaroids.

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Summary

An achromatic quarter-wave plate is obtained by superposing three birefringent plates of the same material; the first and last should have the same retardation $2\delta_1$, their fast vibration directions being parallel to one another but inclined at a specific angle c to that of the central plate—of retardation π. The desired range of achromatism determines the optimum values of $2\delta_1$ and c (which, in turn, will determine the orientation of the effective principal planes of the combination). As an example, using mica retardation plates prepared for Hg 5461, the range from 4100 Å to 6800 Å is covered with $2\delta_1 = 115° 42'$ and $c = 69° 54'$.

Further, for a particular wave-length, a birefringent compensator of variable retardation (0 to 2π) is obtained by interposing a half-wave plate that can be rotated in its own plane, between two quarter-wave plates that have their fast vibration directions parallel.

The results follow from the Poincaré sphere by geometrically compounding successive rotations.

References

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2. Mathieu .. *Bull. Soc. Franc. Min.*, 1934, **57**, 233.