CURVATURE TENSORS IN RIEMANNIAN MANIFOLD—II

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ABSTRACT

In a recent paper author and Mishra [1] have studied the recurrent properties of conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor and projective curvature tensor in a Riemannian manifold. In another paper author and Mishra [2] have defined a new curvature tensor and obtained its relativistic significance. In this paper the geometrical properties of the new tensor in a Riemannian manifold have been discussed.

INTRODUCTION

LET us consider n-dimensional Riemannian manifold V_n with Riemannian metric tensor g. Let R (X, Y, Z) be the curvature tersor of V_n

$$R(X, Y, Z) \text{ def } D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z. \tag{1.1}$$

Let us put

$$'R(X, Y, Z, W) = g\big(R(X, Y, Z), W\big). \tag{1.2}$$

Then for Riemannian manifold, we have

$$'R(X, Y, Z, W) = -\,'R(Y, X, Z, W) \tag{1.3}$$

$$'R(X, Y, Z, W) = -\,'R(X, Y, W, Z) \tag{1.4}$$

$$'R(X, Y, Z, W) = \,'R(Z, W, X, Y) \tag{1.5}$$

$$'R(X, Y, Z, W) + \,'R(Y, Z, X, W) + \,'R(Z, X, Y, W) = 0. \tag{1.6}$$

In our previous paper [2] a new curvature tensor has been defined to obtain its relativistic significance

$$W_2(X, Y, Z) \text{ def } R(X, Y, Z) + \frac{1}{n-1}\,[g(X, Z)\, r(Y) - g(Y, Z)\, r(X)] \tag{1.7}$$

where Ric (X, Y) def $g(r(X), Y)$, Ric denoting the Ricci tensor. Let us put

$$'W_2(X, Y, Z, W) = g(W_2(X, Y, Z), W) \tag{1.8}$$

then for a Riemannian manifold, we have

$$'W_2(X, Y, Z, W) = -\,'W_2(Y, X, Z, W) \tag{1.9}$$

$$'W_2(X, Y, Z, W) + 'W_2(Y, Z, X, W) + 'W_2(Z, X, Y, W) = 0. \tag{1.10}$$

We know [1] that manifold V_n is called a recurrent manifold if

$$(D_T R)(X, Y, Z) = B(T)\, R(X, Y, Z) \tag{1.11}$$

where B (T) is the recurrence parameter.

It is alled a Ricci recurrent if

$$(D_T \mathrm{Ric})(X, Y) = B(T)\, \mathrm{Ric}(X, Y). \tag{1.12}$$

The projective curvature tensor is given by

$$W(X, Y, Z) = R(X, Y, Z) + \frac{1}{n-1}[\mathrm{Ric}(X, Z)\, Y - \mathrm{Ric}(Y, Z)\, X]. \tag{1.13}$$

The tensors

$$\begin{aligned} V(X, Y, Z) &= R(X, Y, Z) - \frac{1}{n-2}[\mathrm{Ric}(Y, Z)\, X - \mathrm{Ric}(X, Z)\, Y \\ &\quad - g(X, Z)\, r(Y) + g(Y, Z)\, r(X)] + \frac{R}{(n-1)(n-2)} \\ &\quad \times [g(Y, Z)\, X - g(X, Z)\, Y] \end{aligned} \tag{1.14}$$

$$\begin{aligned} L(X, Y, Z) &= R(X, Y, Z) - \frac{1}{n-2}[\mathrm{Ric}(Y, Z)\, X - \mathrm{Ric}(X, Z)\, Y \\ &\quad - g(X, Z)\, r(Y) + g(Y, Z)\, r(X)] \end{aligned} \tag{1.15}$$

and

$$C(X, Y, Z) = R(X, Y, Z) - \frac{R}{n(n-1)} [g(Y, Z) X - g(X, Z) Y] \quad (1.16)$$

are called conformal, conharmonic and concircular curvature tensor respectively [1].

2. *Definition* (2.1).—We shall call the manifold V_n to be W_2-recurrent if

$$(D_T W_2)(X, Y, Z) = B(T) W_2(X, Y, Z). \quad (2.1)$$

THEOREM (2.1): If a Riemannian manifold is W_2-recurrent and Ricci-recurrent, then for the same recurrence parameter, it is recurrent.

Proof.—From (1.7), we have

$$(D_T W_2)(X, Y, Z) = (D_T R)(X, Y, Z) + \frac{1}{n-1} [g(X, Z)(D_T r)(Y) - g(Y, Z)(D_T r)(X)]. \quad (2.2)$$

Let the manifold be W_2-recurrent and Ricci-recurrent, then (2.2) becomes

$$B(T) W_2(X, Y, Z) = (D_T R)(X, Y, Z) + \frac{B(T)}{n-1} [g(X, Z) r(Y) - g(Y, Z) r(X)] \quad (2.3)$$

which after rearrangement gives

$$B(T) \left[W_2(X, Y, Z) - \frac{1}{n-1} \{g(X, Z) r(Y) - g(Y, Z) r(X)\} \right] = (D_T R)(X, Y, Z). \quad (2.4)$$

If we substitute (1.7) in (2.4), we get

$$B(T) R(X, Y, Z) = (D_T R)(X, Y, Z) \quad (2.5)$$

which proves that the manifold is recurrent.

Theorem (2.2) : For a Riemannian manifold, we have

$$W_2(X, Y, Z) = V(X, Y, Z) + \frac{1}{n-2}[\mathrm{Ric}(Y, Z) X - \mathrm{Ric}(X, Z) Y] - \frac{1}{(n-1)(n-2)}[g(X, Z) r(Y) - g(Y, Z) r(X)] - \frac{R}{(n-1)(n-2)}[g(Y, Z) X - g(X, Z) Y] \tag{2.6}$$

$$W_2(X, Y, Z) = L(X, Y, Z) + \frac{1}{n-2}[\mathrm{Ric}(Y, Z) X - \mathrm{Ric}(X, Z) Y] - \frac{1}{(n-1)(n-2)}[g(X, Z) r(Y) - g(Y, Z) r(X)]. \tag{2.7}$$

$$W_2(X, Y, Z) = C(X, Y, Z) + \frac{1}{n-2}[g(X, Z) r(Y) - g(Y, Z) r(X)] + \frac{R}{n(n-1)}[g(Y, Z) X - g(X, Z) Y] \tag{2.8}$$

and

$$W_2(X, Y, Z) = W(X, Y, Z) + \frac{1}{n-1}[g(X, Z) r(Y) - g(Y, Z) r(X) + \mathrm{Ric}(Y, Z) X - \mathrm{Ric}(X, Z) Y] \tag{2.9}$$

Hence, from (2.6), if the manifold is conformal recurrent and Ricci-recurrent, then for the same recurrence parameter, it is W_2-recurrent.

Proof.—The proof follows the pattern of Theorem (2.1).

Note (2.2).—If we replace the conformal recurrent condition by conharmonic recurrent, concircular recurrent and projective recurrent conditions and use equations (2.7), (2.8) and (2.9) respectively, then we get three more theorems like Theorem (2.2). Similarly, if we take that the manifold is W_2-recurrent and Ricci-recurrent then for the same recurrence parameter, with the help of equations (2.6), (2.7), (2.8) and (2.9) it is conformal recurrent, conharmonic recurrent, concircular recurrent and projective recurrent respectively, to give four more theorems like Theorem (2.2).

THEOREM (2.3) : For a Riemannian manifold, we have

$$W_2(X, Y, Z) = \frac{1}{n-1}[nR(X, Y, Z) + (n-2)L(X, Y, Z)] - W(X,Y,Z) \tag{2.10}$$

$$W_2(X, Y, Z) = \frac{1}{n-1}[(n-2)V(X, Y, Z) + nC(X, Y, Z)] - W(X, Y, Z) \tag{2.11}$$

Proof.— Using equation (1.7), (1.13), (1.14), (1.15) and (1.16) we get the result.

3. In this section we shall study the n-ply recurrent properties of the curvature tensors in the Riemannian manifold.

Definition (3.1).—The Riemannian manifold will be called n-ply recurrent if

$$(D_{X_1}D_{X_2} \ldots D_{X_n} R)(X, Y, Z) = a(X_1, X_2 \ldots X_n) R(X, Y, Z) \tag{3.1}$$

and n-ply Ricci-recurrent if

$$(D_{X_1}D_{X_2} \ldots D_{X_n} \mathrm{Ric})(Y, Z) = a(X_1, X_2 \ldots X_n) \mathrm{Ric}(Y, Z) \tag{3.2}$$

where $a(X_1, X_2 \ldots X_n)$ is called a recurrence tensor of order $(0, n)$.

Definition (3.2).— We shall call a manifold W_2-n-ply recurrent if

$$(D_{X_1}D_{X_2} \ldots D_{X_n} W_2)(X, Y, Z) = a(X_1 \ldots X_n) W_2(X, Y, Z). \tag{3.3}$$

From (1.7), we have

$$(D_{X_1}D_{X_2} \ldots D_{X_n} W_2)(X, Y, Z)$$
$$= (D_{X_1}D_{X_2} \ldots D_{X_n} R)(X, Y, Z)$$

$$+ \frac{1}{n-1} [g(X, Z)(D_{X_1} D_{X_2} \ldots D_{X_n} r)(Y) - g(Y, Z)(D_{X_1} D_{X_2} \ldots D_{X_n} r)(X)]. \tag{3.4}$$

Let the manifold be W_2-n-ply recurrent and n-ply Ricci-recurrent then after rearranging (3.4) becomes

$$a(X_1, X_2 \ldots X_n)\Big[W_2(X, Y, Z) - \frac{1}{n-1}\{g(X, Z) r(Y) - g(Y, Z) r(X)\}\Big] = (D_{X_1} D_{X_2} \ldots D_{X_n} R)(X, Y, Z). \tag{3.5}$$

Substituting from (1.7) in (3.5), we get

$$a(X_1, X_2, \ldots, X_n) R(X, Y, Z) = (D_{X_1} D_{X_2} \ldots D_{X_n} R)(X, Y, Z) \tag{3.6}$$

which proves that the manifold is n-ply recurrent. Hence we have the theorem.

THEOREM (3·1) : If a Riemannian manifold is W_2-n-ply recurrent and Ricci-n-ply recurrent, then for the same recurrence tensor it is n-ply recurrent.

We can similarly have a set of theorems like theorem (2.2) in this case also.

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