QUASICONFORMAL MAPPINGS AND GLOBAL INTEGRABILITY OF THE DERIVATIVE

By

KARI ASTALA AND PEKKA KOSKELA

To Professor F. W. Gehring on his 65th birthday

1. Introduction

Quasiconformal mappings are homeomorphisms for which, by definition, the infinitesimal distortion of conformality is uniformly bounded. More precisely, a homeomorphism $f: D \to D'$ with $D, D' \subset \mathbf{R}^n$ is K-quasiconformal if $f \in W^1_{n,\mathrm{loc}}(D)$ and

$$(1) \qquad |f'(x)|^n \leq K J_f(x)$$

holds for a.e. $x \in D$. By results of Mori [M], Bojarski [B] and Gehring [G2],[G3] the bound (1) on the infinitesimal level restricts the behaviour of f under larger scales and, in particular, leads to bounds on local distortion of metric quantities like distance, length and area. For instance, if $E \subset D$ is a compact subset and $\alpha = K^{1/(1-n)}$, we have the estimate

$$(2) \qquad |f(x) - f(y)| \leq M_1 |x - y|^\alpha,$$

whenever $x, y \in E$. Moreover, if $A \subset E$ is measurable, the inequality for the Lebesgue n-measures

$$(3) \qquad |fA| \leq M_2 |A|^\beta$$

holds for some exponent $\beta > 0$ depending only on n and K. Equivalently

$$(4) \qquad \int_E (J_f(x))^p \, dm \leq M_3 < \infty, \qquad p = p(n,K) > 1.$$

Here in (2)–(4) the constants M_i depend on f and E. Finally, even though the length or rectifiability of a curve $\gamma \subset D$ need not be preserved under f, one obtains from (3), (4) bounds for the Hausdorff dimension $\dim_H(f(\gamma)) < n$ for a rectifiable curve γ, see [G3].

In general, the estimates (2), (3) or (4) do not hold in the whole domain D and hence it is natural to ask when do the local distortion bounds lead, in turn, to global ones. In the case of the Hölder continuity the following important notion was introduced by Gehring and Martio [GM1], see also [BP].

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1.1. Definition. Let $D \subset \mathbf{R}^n$ be a proper subdomain. We say that D satisfies a quasihyperbolic boundary condition, (qhbc) for short, if there are constants c_1, c_2 such that

$$k_D(x, x_0) \leq c_1 \log \frac{d(x_0, \partial D)}{d(x, \partial D)} + c_2 \tag{5}$$

for some (fixed) point $x_0 \in D$ and for all $x \in D$.

In (5) $d(x, \partial D)$ denotes the Euclidean distance to the boundary ∂D and k_D the quasihyperbolic metric

$$k_D(x_1, x_2) = \inf\left\{ \int_\gamma \frac{ds}{d(x, \partial D)} : x_1, x_2 \in \gamma \right\}.$$

Then, as shown by Gehring and Martio, if D' satisfies the condition (qhbc) and $f: D \to D'$ is quasiconformal, the estimate (2) holds in all balls $B \subset D$ with α, M_1 independent of B. Under additional conditions on D this yields then the global Hölder continuity of f, see [GM1].

Recently, the global area distortion was studied by Martio and Väisälä [MV] who proved that $J_f \in L^p(D)$ for some $p > 1$ or, in view of (1), that $|f'(x)| \in L^p(D)$ for some $p > n$, if $f: D \to D'$ is quasiconformal with D bounded, uniform and D' a John domain. (Note that trivially $J_f \in L^1(D)$ when D' is bounded or has finite volume.) The purpose of this paper is to look for the geometric properties of D or D' that characterize the higher integrability of the derivative in the whole domain D. Unexpectedly, one arrives at the same answers as in the case of the Hölder continuity.

1.2. Theorem. *Let $D' \subset \mathbf{R}^n$ be a domain satisfying a quasihyperbolic boundary condition. If $D \subset \mathbf{R}^n$ and $f: D \to D'$ is quasiconformal, then*

$$\int_D |f'(x)|^p \, dm < \infty$$

for some $p > n$. Here the exponent p depends only on n, D' and the dilatation $K(f)$.

In the converse direction we have

1.3. Theorem. *Let $D \subset \mathbf{R}^n$ be a bounded uniform domain and $f: D \to D'$ quasiconformal. Then $|f'(x)| \in L^p(D)$ for some $p > n$ if and only if D' satisfies a quasihyperbolic boundary condition.*

As an interesting special case we conclude that for univalent functions in the unit disk the global area distortion (3) is equivalent to Hölder continuity.

1.4. Corollary. *Let f be conformal in the disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ with $f\Delta \subset \mathbf{C}$. Then the following conditions are equivalent:*

(a) $\int_\Delta |f'(z)|^p \, dm < \infty$ *for some* $p > 2$.

(b) $|f(z) - f(w)| \leq M|z - w|^\alpha$ *for some constants* $M < \infty$, $0 < \alpha \leq 1$, *and* $\forall z, w \in \Delta$.

After presenting the necessary preliminaries in sections 2 and 3 we give the proofs of the above results, with some generalizations, in section 4. We shall also briefly consider there the lower global integrability of $|f'(x)|$, i.e. the case $p < n$, and study how far the property "for all domains D and all quasiconformal $f: D \to D'$, $|f'| \in L^p(D)$ for some $p > n$" characterizes the domains with quasihyperbolic boundary condition. In the last section we consider the connections between the Hölder continuity and the higher integrability for the more general class of quasiregular mappings.

Notation. Throughout this paper D denotes a domain in $\mathbf{R}^n, n \geq 2$, and $L^p(D)$, $1 < p < \infty$, the Banach space of measurable functions $u : D \to \mathbf{R} \cup \{-\infty, \infty\}$ for which the norm $\|u\|_{L^p(D)} = (\int_D |u|^p dm)^{1/p}$ is finite.

For any pair $E, F \subset \bar{D}$ of disjoint sets we define the modulus of E and F relative to D by

$$\operatorname{mod}(E,F;D) = \operatorname{mod}(\Delta(E,F;D)),$$

where $\Delta(E,F;D)$ is the family Γ of curves joining E and F in D, and $\operatorname{mod}(\Gamma)$ is the modulus of Γ; see [V, 6.1]. Furthermore, we abbreviate $\operatorname{mod}(E,F;\mathbf{R}^n)$ to $\operatorname{mod}(E,F)$.

We say that D is a John domain, cf. [MS], if there is a constant $b > 0$ and a point $x_0 \in D$ (called the John center of D) such that each $x \in D$ can be joined to x_0 by a curve $\gamma : [0,a] \to D$, parametrized by arc length, such that $\gamma(0) = x$, $\gamma(a) = x_0$ and $B(\gamma(t), bt) \subset D$ for all $t \in [0,a]$. We call D uniform [MS] if there is a constant $b \geq 1$ such that for all $x, y \in D$ we find a curve γ joining x to y in D with

$$l(\gamma) \leq b|x - y| \quad \text{and} \quad B\Big(\gamma(t), \frac{1}{b}\min\{t, l(\gamma) - t\}\Big) \subset D;$$

here $t \in [0, l(\gamma)]$ is the arc length parameter of γ and $l(\gamma)$ its length. Recall that by [GM1, 2.18, 3.9 and 3.11] a bounded uniform domain is a John domain, a John domain satisfies the condition (qhbc), and, respectively, a domain satisfying the condition (qhbc) is bounded.

Finally, the operator norm of a linear mapping A is denoted by $|A|$, and we also use $|x|$ for the length of a vector x. The constant ω_{n-1} is the $(n-1)$-measure of S^{n-1}.

2. Hölder continuity and Minkowski dimension

We begin by reviewing the basic notions and ingredients needed for the proofs of Theorems 1.2–1.4 and recall first the definition of local Lip_α-functions, due to Gehring and Martio [GM1]. If $D \subset \mathbf{R}^n$ is a domain and f a continuous function on D, we say that f belongs to the class $\mathrm{loc\,Lip}_\alpha(D)$, $0 < \alpha \leq 1$, if there is a constant $M < \infty$ such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

whenever x, y lie in a ball B contained in D. Note that, in general, this need not imply that the Hölder estimate holds for all pairs of points $x, y \in D$. For a quasiconformal f the $\mathrm{loc\,Lip}_\alpha$-property controls the boundary distortion of the mapping.

2.1. Lemma ([GM1, 3.4]). *Let $f: D \to D'$ be a K-quasiconformal mapping and $0 < \alpha \leq K^{1/(1-n)}$. Then $f \in \mathrm{loc\,Lip}_\alpha(D)$ if and only if there is a constant $M < \infty$ with*

$$d(f(x), \partial D') \leq M d(x, \partial D)^\alpha, \qquad x \in D.$$

In addition, with $\mathrm{loc\,Lip}_\alpha$ classes one can nearly characterize the domains that satisfy the quasihyperbolic boundary condition.

2.2. Theorem ([GM1, 3.17, 3.20]).

(a) *If $D' \subset \mathbf{R}^n$ is a domain with a quasihyperbolic boundary condition then for any domain D and for any quasiconformal $f: D \to D'$, $f \in \mathrm{loc\,Lip}_\alpha(D)$ for some $\alpha > 0$; here $\alpha = \alpha(n, K, D')$.*

(b) *Conversely, if $f: D \to D'$ is quasiconformal, if $f \in \mathrm{loc\,Lip}_\alpha(D)$ and if D satisfies a quasihyperbolic boundary condition then so does D'.*

However, the assumption $f \in \mathrm{loc\,Lip}_\alpha(D)$ alone in the converse part of 2.2 does not imply that the image domain D' satisfies a quasihyperbolic boundary condition and hence both assumptions in (b) are really needed. Indeed, it turns out that $f \in \mathrm{loc\,Lip}_\alpha(D)$ for each quasiconformal $f: D \to D'$, $\alpha = \alpha(f) > 0$, also if D' is a bounded *M-QED* domain; by Example 2.5 below, these two classes of domains do not coincide. Here, following the terminology of [GM2], we call D' an *(M-)quasiextremal distance domain* or an *M-QED domain* if for any pair $K_0, K_1 \subset D$ of disjoint continua

$$\mathrm{mod}(K_0, K_1) \leq M \,\mathrm{mod}(K_0, K_1; D). \tag{6}$$

2.3. Theorem. *Let f be a K-quasiconformal mapping of a domain D onto a bounded M-QED domain D'. Then $f \in \mathrm{loc\,Lip}_\alpha(D)$ for $\alpha = (KM)^{1/(1-n)}$.*

Proof. We shall show, in fact, that it suffices to assume the inequality (6) for a fixed, non-degenerate continuum $K_1' \subset D'$.

Suppose that the inequality (6) holds for some K_1' and for all continua $K_0' \subset D'$ disjoint from K_1'. Write $K_1 = f^{-1}(K_1')$. By Lemma 2.1 it suffices to show that there is a constant c with $d(f(x), \partial D') \le c\, d(x, \partial D)^\alpha$ for all $x \in D$. Since D' is bounded we may assume that $d(x, \partial D) < b_0 = d(K_1, \partial D)$.

Let $x \in D$ with $d(x, \partial D) < b_0$ and write $x' = f(x)$. Pick a point $y \in \partial D$ with $|x - y| = d(x, \partial D)$ and let F be the half open line segment joining x and y in D. Clearly $d(x', \partial D') \le \operatorname{dia}(F')$, where $F' = f(F)$. Hence it suffices to show that $\operatorname{dia}(F') \le c|x-y|^\alpha$. By [V, 7.5]

$$\operatorname{mod}(F, K_1; D) \le \omega_{n-1}\left(\log \frac{b_0}{|x-y|}\right)^{1-n},$$

and thus the quasiconformality of f implies

$$\operatorname{mod}(F', K_1'; D') \le K\omega_{n-1}\left(\log \frac{b_0}{|x-y|}\right)^{1-n}.$$

Fix $z_1 \in K_1'$, and pick a Möbius transformation ψ mapping z_1 to ∞. Moreover, select a point $z_2 \in K_1'$ with $|z_2 - z_1| = \operatorname{dia}(K_1')/2$. Then for any $x' \neq y' \in F'$

$$\frac{|\psi(z_2) - \psi(x')|}{|\psi(x') - \psi(y')|} = \frac{|z_2 - x'||z_1 - y'|}{|x' - y'||z_2 - z_1|} \le 2\, \frac{\operatorname{dia}(D')^2}{\operatorname{dia}(K_1')|x' - y'|}.$$

It follows that

$$\frac{d(\psi K_1', \psi F')}{\operatorname{dia}(\psi F')} \le 4\, \frac{\operatorname{dia}(D')^2}{\operatorname{dia}(K_1')\operatorname{dia}(F')}.$$

This estimate together with [G1, Theorem 4] and the conformal invariance of the modulus yields

$$\begin{aligned} \operatorname{mod}(F', K_1') &= \operatorname{mod}(\psi F', \psi K_1') \\ &\ge \omega_{n-1}\left(\log \lambda_n\left(1 + 4\, \frac{\operatorname{dia}(D')^2}{\operatorname{dia}(K_1')\operatorname{dia}(F')}\right)\right)^{1-n} \\ &\ge \omega_{n-1}\left(\log(c/\operatorname{dia}(F'))\right)^{1-n}, \end{aligned}$$

where

$$c = 5\lambda_n\, \frac{\operatorname{dia}(D')^2}{\operatorname{dia}(K_1')}$$

and λ_n depends only on n.

Approximating F' from inside by continua $K_0' \subset D'$ the condition (6) and the above inequality imply

$$\operatorname{mod}(F', K_1'; D') \ge \omega_{n-1}/M\left(\log(c/\operatorname{dia}(F'))\right)^{n-1}.$$

We conclude that

$$\log \frac{b_0}{|x-y|} \leq (KM)^{1/(n-1)} \log \frac{c}{\operatorname{dia}(F')},$$

and the claim follows.

Remark. It is possible to modify the above argument to prove the following extension of [GM1, 3.30] concerning uniform domains:

Let D and D' be bounded *M-QED* domains. If f is a K-quasiconformal mapping of D onto D', then both f and f^{-1} are uniformly Hölder continuous with exponent $\alpha = (MK)^{1/(1-n)}$.

We turn then to the other basic ingredient in the study of global integrability, the boundary dimension. While in most contexts the Hausdorff dimension is the correct tool to study the size of a set, in our case we must use the analogous but more geometric Minkowski dimension: If E is a compact set in $\mathbf{R}^n$, $0 < \delta \leq n$ and $r > 0$, set

$$M^\delta(E;r) = \inf\left\{kr^\delta : E \subset \bigcup_1^k B(x_i, r)\right\}.$$

The *Minkowski content* of E is now

$$M^\delta(E) = \limsup_{r\to 0} M^\delta(E;r).$$

Thus in the definition of the Minkowski content the set E is covered by balls all of equal radii r whereas, in comparison, the standard δ-dimensional Hausdorff measure H^δ allows coverings with arbitrary radii $\leq r$; hence $H^\delta(E) \leq M^\delta(E)$ for any set E. Moreover, as is easily seen, M^δ is not a measure, but still in many cases the Minkowski dimension

$$\dim_M(E) = \inf\{\delta : M^\delta(E) < \infty\}$$

describes the geometry of E better than the Hausdorff dimension

$$\dim_H(E) = \inf\{\delta : H^\delta(E) < \infty\}.$$

A remarkable connection between the Minkowski dimension and the Hölder continuity was found by Smith and Stegenga [SS], who proved the following using an argument due to Jones and Makarov:

2.4. Theorem. *If D is a domain in $\mathbf{R}^n$ which satisfies the quasihyperbolic boundary condition, then*

(7) $$\dim_M(\partial D) < n.$$

For us this result will be useful in finding the precise relation between global Hölder continuity and integrability of the derivative.

On the other hand, the class of bounded *M-QED* domains does not satisfy (7) as the next example shows. Combining with 2.4 (or by elementary calculation) we have a bounded *M-QED* domain which does not satisfy the quasihyperbolic boundary condition. And conversely, Becker and Pommerenke constructed a simply connected plane domain D with the quasihyperbolic boundary condition which is not a quasidisk and hence not a *M-QED* domain, cf. [BP],[GM2].

2.5. Example. Let $F \subset [0,1]$ be a compact subset with Hausdorff dimension equal to 1 but length $H^1(F) = 0$; such examples are easily constructed, for instance, by taking the union of Cantor sets $F_k \subset [1/k, 1/(k-1)]$ with $\dim_H(F_k) = 1 - 1/k$. Let $F^n = F \times \cdots \times F$ (n times) and set $D = B^n(2\sqrt{n}) \setminus F^n$. Since the projections of F^n along the coordinate axis have vanishing $(n-1)$-measures, D is *QED*: This follows from the ACL-characterization of Sobolev functions, see e.g. [Z, 2.1.4], and the equality of the modulus and the variational capacity [H, 5.5]. However, clearly $\dim_M(\partial D) = n$.

Another example can be obtained with the choice

$$F = \{1/\log k : k \in \mathbf{N}, k \geq 2\}.$$

By a simple estimate one can show that $\dim_M(F) = 1$ and $\dim_M(F^n) = n$. Since F^n is countable, $D = B^n(2\sqrt{n}) \setminus F^n$ is *QED*.

3. Average derivative

The global integrability of the derivative of a quasiconformal mapping f depends on the distortion properties of the Jacobian J_f. For conformal mappings these can be obtained from the well-known Koebe distortion theorem. However, since a quasiconformal mapping has derivatives only in the generalized or Sobolev sense, pointwise estimates cannot hold in the quasiconformal case. Therefore we need a substitute, the *average derivative* a_f introduced in [AG].

3.1. Definition. Let f be a quasiconformal mapping in a proper subdomain $D \subset \mathbf{R}^n$ and set $B(x) = B(x, d(x, \partial D)/2)$. Then the average derivative is defined by

$$a_f(x) = \exp\left(\frac{1}{n|B(x)|} \int_{B(x)} \log J_f(y)\, dm(y)\right), \qquad x \in D.$$

Note that if $n = 2$ and f is conformal, then $a_f(x) = |f'(x)|$ by the mean value theorem, and that for any quasiconformal mapping the integrals of $|f'(x)|$ and $a_f(x)$ are comparable, see 3.4 below. Moreover, the counterpart of the Koebe distortion theorem holds for a_f.

3.2. Theorem ([AG]). *If f is K-quasiconformal in a domain $D \subset \mathbf{R}^n$, then*

$$c_1 \frac{d(f(x),\partial fD)}{d(x,\partial D)} \le a_f(x) \le c_2 \frac{d(f(x),\partial fD)}{d(x,\partial D)},$$

where c_1, c_2 depend only on n and K.

To compare more closely the derivatives $|f'|$ and a_f we need to recall the local integrability properties of $|f'|$. This is most conveniently done in terms of the A_p-weights of Muckenhoupt; a similar (but considerably more technical) argument could, of course, be presented with the help of the known distortion theorems of quasiconformal mappings.

Let $Q \subset \mathbf{R}^n$ be a cube and $w \ge 0$ integrable on Q. We say that w belongs to the class $A_p = A_p(Q)$, $1 < p < \infty$, if the inequality

$$\frac{1}{|Q'|} \int_{Q'} w \, dm \le C \left(\frac{1}{|Q'|} \int_{Q'} w^{-1/(p-1)} \, dm \right)^{1-p}$$

holds for each parallel cube $Q' \subset Q$ and for a fixed constant $C < \infty$. Then the following conditions are known to be equivalent, cf. [GCRF]:

(8.a) $$w \in A_p(Q) \qquad \text{for some } 1 < p < \infty.$$

(8.b) There exist constants $c_1 < \infty$ and $\epsilon > 0$, such that for each parallel n-cube $Q' \subset Q$,

$$\frac{1}{|Q'|} \int_{Q'} w^{1+\epsilon} \, dm \le c_1 \left(\frac{1}{|Q'|} \int_{Q'} w \, dm \right)^{1+\epsilon}.$$

In addition, the constants in (8.a) and (8.b) depend only on each other and not on the particular cube Q or function w.

According to Gehring's L^p-integrability result the derivative of a quasiconformal mapping belongs locally to A_p. More precisely,

3.3. Theorem (Gehring, [G3]). *Let $D \subset \mathbf{R}^n$ be a domain, f be K-quasiconformal on D and $Q \subset D$ a cube with* $\operatorname{dia}(fQ) \le d(fQ, \partial fD)$. *Then there is an exponent $p_0 > n$ and a constant $b < \infty$, both depending only on n and K, such that*

$$\left(\frac{1}{|Q'|} \int_{Q'} |f'(x)|^{p_0} \, dm \right)^{1/p_0} \le b \left(\frac{1}{|Q'|} \int_{|Q'|} |f'(x)| \, dm \right)$$

for each parallel n-cube $Q' \subset Q$.

Combining these estimates we obtain

3.4. Theorem. *There exists an $\epsilon = \epsilon(n,K) > 0$ such that whenever f is K-quasiconformal in a domain $D \subset \mathbf{R}^n$ then*

$$c_1 \int_D a_f(x)^p \, dm \le \int_D |f'(x)|^p \, dm \le c_2 \int_D a_f(x)^p \, dm$$

holds for all $-\epsilon < p < n + \epsilon$. *Here the constants* c_1, c_2 *depend only on* n, K, *and* p.

Proof. We start with a distortion inequality due to Gehring, see [V, 18.1]. There exists an increasing function $\theta : (0,1) \to \mathbf{R}_+$, depending only on the dimension n and the dilatation K, such that $\lim_{t \to 0} \theta(t) = 0$ and

$$\frac{|f(x) - f(y)|}{d(f(x), \partial f D)} \le \theta\left(\frac{|x-y|}{d(x, \partial D)}\right), \qquad x, y \in D, \tag{9}$$

is satisfied by all K-quasiconformal mappings f of the domain D. Consequently, if we choose the constant $\lambda = \lambda(n, K)$ small enough, the assumptions of 3.3 are satisfied for a cube $Q \subset D$ whenever $\operatorname{dia}(Q) \le \lambda d(Q, \partial D)$. Then

$$\left(\frac{1}{|Q'|} \int_{Q'} |f'|^{p_0} \, dm\right)^{1/p_o} \le b\left(\frac{1}{|Q'|} \int_{Q'} |f'| \, dm\right)$$

for all parallel cubes $Q' \subset Q$. Moreover, in view of (8), $|f'| \in A_{p_1}(Q)$ for some $p_1 < \infty$.

Next, let $Q \subset D$ be a cube with

$$\frac{1}{4} \lambda \operatorname{dist}(Q, \partial D) \le \operatorname{dia}(Q) \le \lambda \operatorname{dist}(Q, \partial D) \tag{10}$$

and let x_0 be the center of Q. From (9) and the Koebe distortion theorem 3.2 we see that

$$a_f(x) \le c_1 a_f(y), \qquad x, y \in Q$$

and that

$$\frac{1}{|Q|} \int_Q |f'|^n dm \le \frac{K|f(Q)|}{|Q|} \le c_2 \left(\frac{d(f(x_0), \partial f D)}{d(x_0, \partial D)}\right)^n \le c_3 a_f(x_0)^n.$$

On the other hand, as in [AG, 2.8] we get $(d(f(x_0), \partial f D)/d(x_0, \partial D))^n \le c_4 |Q|^{-1} \int_Q |f'|^n \, dm$ and hence

$$a_f(x_0)^n \le c_5 \frac{1}{|Q|} \int_Q |f'|^n \, dm.$$

Finally, applying these estimates, Hölder's inequality, and Gehring's theorem 3.3 we obtain

$$\begin{aligned} a_f(x_0) &\le c_5 \left(\frac{1}{|Q|} \int_Q |f'|^{p_0} \, dm\right)^{1/p_0} \le b c_5 \frac{1}{|Q|} \int_Q |f'| \, dm \\ &\le c_6 \left(\frac{1}{|Q|} \int_Q |f'|^{1/(1-p_1)} \, dm\right)^{1-p_1}. \end{aligned}$$

However, Hölder's inequality shows also that for any integrable function g and any $\epsilon > 0$, $(|Q|^{-1}\int_Q |g(x)|^{-\epsilon}\, dm(x))^{-1/\epsilon} \le |Q|^{-1}\int_Q |g(x)|\, dm(x)$. Therefore

$$\left(\frac{1}{|Q|}\int_Q |f'|^{1/(1-p_1)}\, dm\right)^{1-p_1} \le \left(\frac{1}{|Q|}\int_Q |f'|^n\, dm\right)^{1/n} \le c_3 a_f(x_0).$$

Here and above all the constants depend only on n and K. In conclusion, these inequalities imply that if $\epsilon = \epsilon(n,K) = \min\{p_0 - n, 1/(p_1 - 1)\}$ and $-\epsilon < p < n + \epsilon$, then

$$\frac{1}{C}\int_Q a_f(x)^p\, dm \le \int_Q |f'(x)|^p\, dm \le C\int_Q a_f(x)^p\, dm(x). \tag{11}$$

By the well-known Whitney decomposition argument, see [S, p. 16], we can express D as a union of cubes Q_i, such that (10) holds for each Q_i. Hence the theorem follows from (11).

3.5. Corollary. *Let $\epsilon = \epsilon(n,K) > 0$ be as in Theorem* 3.4, *let $-\epsilon < p < n + \epsilon$ and suppose $f: D \to D'$ is K-quasiconformal. Then*

$$\frac{1}{C}\int_{D'} (a_{f^{-1}})^{n-p}\, dm \le \int_D (a_f)^p\, dm \le C\int_{D'} (a_{f^{-1}})^{n-p}\, dm,$$

where $C = C(n,K)$.

Proof. Since $J_f(x) \le |f'(x)|^n \le KJ_f(x)$ a.e. $x \in D$ and

$$\int_D J_f^q\, dm = \int_D J_{f^{-1}}(f)^{1-q} J_f\, dm = \int_{D'} (J_{f^{-1}})^{1-q}\, dm,$$

the claim follows from 3.4.

4. Integrability of $|f'|$

We can now study the global distortion properties of a quasiconformal mapping $f: D \to D'$. Recall that $|f'(x)| \in L^n(D)$ if and only if D' has finite volume and that $|f'(x)| \in L^p(D)$ for a $p > n$ whenever f satisfies globally the area distortion estimate (3). Analogously, one can ask what are the properties of D or D' that imply $|f'(x)| \in L^p(D)$ for some $p < n$. Indeed, since $L^{p_1}(D) \cap L^{p_2}(D) \subset L^r(D)$ when $p_1 < r < p_2$, we see that if $|D'| < \infty$ and $f: D \to D'$ is quasiconformal, there is an interval $I \subset \mathbf{R}$ containing n such that $|f'| \in L^p(D)$ for all $p \in I$. In general, however, I may reduce to the single point $I = \{n\}$.

4.1. Example. Let $D = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0, |z - 1| > 1/2\}$ and $f(z) = 1/\log z$. Then f is conformal in D, fD is bounded (and hence $|f'| \in L^2(D)$) but $|f'| \notin L^p(D)$ if $p \ne 2$.

Thus it is natural to look under what geometric conditions we can find numbers $p_1 = p_1(n,K) < n$ and $p_2 = p_2(n,K) > n$ such that $|f'| \in L^p(D)$ whenever

$f: D \to D'$ is K-quasiconformal and $p_1 < p < p_2$. The case $K = 1$ and $D' = \Delta$, the unit disk of the complex plane, is well known: If $f: D \to \Delta$ is conformal, it follows from elementary distortion theorems that $|f'(z)| \in L^p(D)$ for $4/3 < p < 3$. Here the lower bound 4/3 is known to be sharp and Brennan [Br] conjectured that $p < 4$ is the best upper bound; so far the best known estimate is $p < 3.399$ due to Pommerenke [P].

In this section we first study the higher integrability of $|f'|$ where more detailed information can be obtained and then turn to the case $p < n$. As a corollary of 4.5 and 4.10 we then have the following far-reaching extension of the case of the unit disk Δ.

4.2. Theorem. *Let $\Omega \subset \mathbf{R}^n$ be a John domain and let $K \geq 1$. Then there are exponents $p_1 = p_1(n,K,\Omega) < n$ and $p_2 = p_2(n,K,\Omega) > n$ such that for any domain $D \subset \mathbf{R}^n$ and any K-quasiconformal $f: D \to \Omega$ we have*

$$\int_D |f'(x)|^p \, dm < \infty, \qquad p_1 < p < p_2.$$

4.A. Higher integrability. We begin with a simple consequence of the integrability of $|f'|^p$, $p > n$, which surprisingly turns out to characterize the quasiconformal mappings with this property when $\dim_M(\partial D') < n$.

4.3. Lemma. *Let f be quasiconformal on a domain $D \subset \mathbf{R}^n$. If $|f'| \in L^p(D)$ for some $p > n$, then $f \in \operatorname{loc}\operatorname{Lip}_\alpha(D)$ with $\alpha = 1 - n/p$.*

Proof. If f_i, $i = 1, \ldots, n$, denote the coordinate mappings of f, then by the Sobolev's imbedding theorem, see e.g. [BI, 1.7],

$$|f_i(x) - f_i(y)| \leq C(p,n)|x-y|^{1-n/p}\left(\int_B |\nabla f_i|^p \, dm\right)^{1/p}$$

whenever $x, y \in B$, a ball contained in D. Since $|\nabla f_i(x)| \leq |f'(x)|$, $1 \leq i \leq n$, the claim follows. ■

We can now prove the main result of this section.

4.4. Theorem. *Let D' be a bounded domain in $\mathbf{R}^n$ with $\dim_M(\partial D') < n$. If $f: D \to D'$ is K-quasiconformal, then the following conditions are equivalent:*

(a) *$f \in \operatorname{loc}\operatorname{Lip}_\alpha(D)$ for some $0 < \alpha \leq 1$.*

(b) *$|f'| \in L^p(D)$ for some $p > n$.*

Here α and p depend only on each other, n, K and $\dim_M(\partial D')$.

Proof. By Lemma 4.3 we only need to show that (a) implies (b).

If $g = f^{-1}$ is the inverse mapping, then according to Theorem 3.4 and Corollary 3.5

$$\int_D |f'(x)|^p \, dm \leq C(n,K) \int_{D'} a_g(x)^{n-p} \, dm$$

for $n \le p < n + \epsilon(n,K)$. Moreover, $a_g(x) \ge c_1 d(g(x),\partial D)/d(x,\partial D')$ by 3.2 and from 2.1 with $f = g^{-1}$ we obtain $d(g(x),\partial D)^\alpha \ge c_2 d(x,\partial D')$ because $f \in \operatorname{loc}\operatorname{Lip}_\alpha(D)$. Thus

$$\int_D |f'(x)|^p\, dm \le c_3 \int_{D'} d(x,\partial D')^{-\delta}\, dm$$

where $\delta = (n-p)(1-1/\alpha) \ge 0$.

Finally, we use the condition $\dim_M(\partial D') = \beta < n$. As shown in [MVu, 3.1], this is equivalent to requiring that for each $\delta < n - \beta$ we have

$$|\{x \in D' : d(x,\partial D') < r\}| \le |\partial D' + B^n(r)| \le c_4(\delta) r^\delta,$$

for all $r \le r_0 = \operatorname{dia}(\partial D')$. Hence, by change of variables,

$$\begin{aligned}\int_{D'} d(x,\partial D')^{-\delta}\, dm &= \int_0^\infty |\{x \in D' : d(x,\partial D')^{-\delta} > t\}|\, dt \\ &= r_0^{-\delta}|D'| + \delta \int_0^{r_0} |\{x \in D' : d(x,\partial D') < t\}|\, t^{-\delta-1}\, dt \\ &< \infty\end{aligned}$$

when $0 < \delta < n - \beta$. Therefore we conclude that $|f'(x)| \in L^p(D)$ if $n < p < n + \min\{\epsilon, (n-\beta)/(1/\alpha - 1)\}$.

Proof of Theorem 1.2. If D' is a domain in $\mathbf{R}^n$ with a quasihyperbolic boundary condition, then $\dim_M(\partial D') < n$ according to Theorem 2.4 of Smith and Stegenga and, moreover, each quasiconformal $f : D \to D'$ belongs to some class $\operatorname{loc}\operatorname{Lip}_\alpha(D)$, $0 < \alpha \le 1$, by Theorem 2.2. As D' is bounded [GM1, 3.11], the claim follows from Theorem 4.4.

Since John domains satisfy a quasihyperbolic boundary condition, Theorem 4.4 also extends the theorem of Martio and Väisälä [MV]:

4.5. Corollary. *Let $D' \subset \mathbf{R}^n$ be a John domain and $f : D \to D'$ K-quasiconformal. Then*

$$\int_D |f'(x)|^p\, dm < \infty$$

for an exponent $p = p(n,K,D') > n$.

Yet another consequence, useful for domains of infinite connectivity, can be obtained from 4.4.

4.6. Corollary. *Let D' be a bounded M-QED domain in $\mathbf{R}^n$ and suppose that $\dim_M(\partial D') < n$. Then for each K-quasiconformal $f : D \to D'$, $|f'(x)| \in L^p(D)$ with $p = p(n,K,D') > n$.*

We do not know whether the assumption $\dim_M(\partial D') < n$ in Theorem 4.4 is really necessary. In many situations it can be dropped and we give a general formulation of this phenomenon; Theorems 1.3 and 1.4 will then be special cases of this result.

4.7. Theorem. *Let $D \subset \mathbf{R}^n$ be a domain with a quasihyperbolic boundary condition and let $f: D \to D'$ be quasiconformal. Then the following are equivalent:*
(a) $|f'(x)| \in L^p(D)$ *for some* $p > n$.
(b) $f \in \operatorname{loc\,Lip}_\alpha(D)$ *for some* $0 < \alpha \leq 1$.
(c) *D' satisfies a quasihyperbolic boundary condition.*

Proof. If $|f'| \in L^p(D)$ for some $p > n$, then by Sobolev's imbedding, Lemma 4.3, $f \in \operatorname{loc\,Lip}_\alpha(D)$. Thus (a) implies (b).

Since D satisfies a quasihyperbolic boundary condition, the implication (b) $\Rightarrow$ (c) follows from Theorem 2.2(b). Finally, if (c) holds, $f \in \operatorname{loc\,Lip}_\alpha(D)$ for some $0 < \alpha < 1$ by Theorem 2.2(a). Since also the boundary $\partial D'$ has Minkowski dimension strictly less than n, by Theorem 2.4, we see from Theorem 4.4 that (c) implies (a).

To summarize the above results, the higher integrability of the derivative of a quasiconformal mapping $f: D \to D'$ essentially depends only on the geometry of the domain D'. One is therefore led to axiomatize this property.

4.8. Definition. We say that a domain $\Omega \subset \mathbf{R}^n$ is a *Gehring-domain*, if for all $K \geq 1$ there is a number $p = p(K) > n$ such that

$$\int_D |f'(x)|^p \, dm < \infty$$

for each domain D and each K-quasiconformal mapping $f: D \to \Omega$.

It is clear that Gehring domains all have finite volume. Moreover, in the case of finitely connected plane domains this class admits a geometric characterization.

4.9. Proposition. *Let D be a finitely connected domain in the plane* $\mathbf{R}^2$. *Then D is a Gehring domain if and only if D satisfies a quasihyperbolic boundary condition.*

Proof. Since a finitely connected plane domain is conformally equivalent to a bounded uniform domain, the claim follows from Theorems 1.2 and 1.3.

There are, however, Gehring domains even in the plane that do not satisfy the quasihyperbolic boundary condition. In fact, the latter *M-QED* domain $D \subset \mathbf{R}^n$ introduced in 2.5 gives a particular example.

We conclude this section with two open problems.

(A) Is Theorem 4.4 true without the assumption $\dim_M(\partial D') < n$?
(B) Does there exist a geometric characterization for Gehring domains?

4.B. Lower integrability. If the domain $D' \subset \mathbf{R}^n$ has finite volume, $\int_D |f'(x)|^n \, dm \leq K\int_D J_f(x)\, dm = K|D'|$ whenever $f: D \to D'$ is K-quasiconformal. If also $|D| < \infty$, $|f'| \in L^p(D)$ for all $p < n$ by Hölder's inequality. However, for a general domain D we get lower integrability only under special assumptions on the image D'.

4.10. Theorem. *Let D' be a John domain. Then for each $K \geq 1$ there is an exponent $p < n$ such that for any domain D and any K-quasiconformal $f: D \to D'$,*

$$\int_D |f'(x)|^p \, dm < \infty.$$

Proof. Let x_0 be the John center of D'. We prove first that if g is K-quasiconformal on D' with $gD' \subset \mathbf{R}^n$, then

$$d(g(x), \partial gD') \leq M d(x, \partial D')^{-\zeta}, \qquad x \in D', \tag{12}$$

where $M = c_1 d(g(x_0), \partial gD')$ and $c_1 < \infty$, $\zeta > 0$ depend only on n, K and the constants of D'.

Indeed, if $x \in D'$, let $\gamma: [0, a] \to D'$ be a path parametrized by arclength with $\gamma(0) = x$, $\gamma(a) = x_0$ and

$$B(\gamma(t), bt) \subset D', \qquad 0 \leq t \leq a.$$

Moreover, set $t_0 = a$ and define $t_k = (1 - b/2)t_{k-1}$, $k \in \mathbf{N}$. As $|\gamma(t_k) - \gamma(t_{k-1})| \leq (b/2)t_{k-1} \leq \frac{1}{2}d(\gamma(t_{k-1}), \partial D')$, by the inequality (9) we have $|g(\gamma(t_k)) - g(\gamma(t_{k-1}))| \leq c_0 d(g(\gamma(t_{k-1})), \partial gD')$ where $c_0 = c_0(n, K)$. Consequently,

$$\begin{aligned} d(g(\gamma(t_k)), \partial gD') &\leq (1 + c_0) d(g(\gamma(t_{k-1})), \partial gD') \\ &\leq (1 + c_0)^k d(g(x_0), \partial gD'). \end{aligned}$$

If now $d(x, \partial D') < 3a$, choose the smallest positive integer k such that $3t_k \leq d(x, \partial D')$. Then

$$d(x, \partial D') < 3t_{k-1} = 3a\lambda^{k-1}, \qquad \lambda = (1 - b/2) < 1,$$

and as $|x - \gamma(t_k)| \leq t_k \leq \frac{1}{3}d(x, \partial D')$, $|x - \gamma(t_k)| < \frac{1}{2}d(\gamma(t_k), \partial D')$. Thus

$$\begin{aligned} d(g(x), \partial gD') &\leq (1 + c_0) d(g(\gamma(t_k)), \partial gD') \\ &\leq (1 + c_0)^{k+1} d(g(x_0), \partial gD') \\ &\leq M_1 d(x, \partial D')^{-\zeta} \end{aligned}$$

where $M_1 = (1 + c_0)^2(3a)^\zeta d(g(x_0), \partial gD')$ and $\log(1 + c_0) = \zeta\log(1/\lambda)$. If $d(x, \partial D') \geq 3a$, $|x - x_0| \leq a \leq \frac{1}{3}d(x, \partial D')$ and we have $|x - x_0| \leq \frac{1}{2}d(x_0, \partial D')$. Hence inequality (9) yields

$$d(g(x), \partial gD') \leq (1 + c_0)d(g(x_0), \partial gD') \leq M_2 d(x, \partial D')^{-\zeta}$$

where

$$M_2 = (1 + c_0)(3/2)^\zeta d(x_0, \partial D')^\zeta d(g(x_0), \partial gD')$$

in this case. In brief, these estimates give (12) with $M = \max\{M_1, M_2\}$.

To complete the proof we apply Theorem 3.2 and formula (12) with $g = f^{-1}$ to get $a_{f^{-1}}(x) \leq c_1 d(x, \partial D')^{-1-\zeta}$ and then Theorems 3.4, 3.5 to estimate

$$\int_D |f'|^p \, dm \leq c_2 \int_{D'} a_{f^{-1}}(x)^{n-p} \, dm \leq c_3 \int_{D'} d(x, \partial D)^{-\epsilon} \, dm \tag{13}$$

where $\epsilon = (n - p)(1 + \zeta) > 0$. By [MVu, 6.4] the Minkowski dimension of the boundary of a John domain is always strictly less than n and so the last integral in (13) converges when $0 < \epsilon < n - \dim_M(\partial D')$, i.e. when $n - (1 + \zeta)^{-1}(n - \dim_M(\partial D')) < p < n$.

5. Higher integrability for quasiregular mappings

Suppose that D is a bounded domain and let $f: D \to D'$ be a quasiregular mapping. In the previous section we saw that if f lies in $\operatorname{loc}\operatorname{Lip}_\alpha(D)$ for some $0 < \alpha \leq 1$, if it is one-to-one and if, in addition, D is sufficiently smooth then $|f'| \in L^p(D)$ for some $p > n$. Here we establish the following analogue of this result for general quasiregular mappings.

5.1. Theorem. *Let $f: D \to D'$ be a quasiregular mapping that lies in $\operatorname{loc}\operatorname{Lip}_\alpha(D)$ for an $\alpha \in (0,1]$. If D is bounded and if $\dim_M(\partial D) < n\alpha$, then $|f'| \in L^p(D)$ for some $p > n$.*

Before turning into the proof of Theorem 5.1 we produce an example which shows that in the class of all quasiregular mappings to have $|f'| \in L^p(D), p > n$, it is not sufficient only to assume $f \in \operatorname{loc}\operatorname{Lip}_\alpha(D)$ for some $0 < \alpha \leq 1$. Indeed, we shall show that in the plane the condition $\alpha > \dim_M(\partial D) - 1$ is necessary for higher integrability; note that $\lambda - 1 \to \lambda/2$ as $\lambda \to 2$.

5.2. Example. Let $F \subset [0,1]$ be a self-similar fractal set satisfying the open set condition with $\dim_H(F) = \lambda/2$, $1 < \lambda < 2$, see [Hu]. Then also the Minkowski dimension $\dim_M(f) = \lambda/2$; in [MVu, 4.19] this was shown in the case where F is isotropic, but this extra assumption is unnecessary: Indeed, by [Hu] F supports a measure μ such that $c_1 R^{\lambda/2} \leq \mu(B(x_0, R)) \leq c_2 R^{\lambda/2}$ for all $x_0 \in F$ and $R \leq \operatorname{dia}(F)$. Moreover, if t is given we cover the set $\{x \in \mathbf{R} : d(x, F) < t\}$ by $k(t)$

balls $B(x_i, t)$, $x_i \in F$, such that at each point no more than two of them intersect and hence

$$\begin{aligned} |\{x \in \mathbf{R} : d(x,F) < t\}| &\leq 2k(t)t \leq c_1 t^{1-\lambda/2} \sum_{i=1}^{k(t)} \mu(B(x_i,t)) \\ &\leq 2c_1 t^{1-\lambda/2}\mu(F) \\ &= c_3 t^{1-\lambda/2}. \end{aligned}$$

Therefore, by [MVu, 3.1], $\lambda/2 = \dim_H(F) \leq \dim_M(F) \leq \lambda/2$.

Set $D = B^2(2)\setminus(F \times F) \subset \mathbf{C}$. Now $\dim_M(\partial D) = \lambda$ and we shall show that if $|f'| \in L^2(D)$ for all (non-constant) analytic f in $\operatorname{loc}\operatorname{Lip}_\alpha(D)$ then $\alpha > \lambda - 1$. Namely, as $H^1(F) = 0$, it follows that any function f analytic in D and satisfying $|f'| \in L^2(D)$ is absolutely continuous on almost every line parallel to a coordinate axis. Hence such an f extends to an analytic function of $B^2(2)$. Therefore, if $|f'| \in L^2(D)$ for each analytic $f \in \operatorname{loc}\operatorname{Lip}_\alpha(D)$, $F \times F$ is removable for all α-Hölder continuous analytic functions on $\mathbf{C}\setminus(F \times F)$ and hence, by [Ga, 4.5], $\lambda = \dim_M(F \times F) < 1 + \alpha$.

The proof of Theorem 5.1 is based on the following lemmas. We consider continuous real valued functions u in $W^1_{n,\mathrm{loc}}(D)$ that satisfy the Caccioppoli type inequality

$$\int_Q |\nabla u|^n \, dm \leq \frac{c}{|Q|} \int_{2Q} |u - u_{2Q}|^n \, dm \tag{14}$$

whenever $2Q \subset D$; here the constant c is independent of Q and u_{2Q} denotes the mean value of u over the cube $2Q$.

5.3. Lemma. *Suppose that u satisfies* (14). *Then there is a constant c and an exponent $p > n$ independent of u such that*

$$\left(\frac{1}{|Q|}\int_Q |\nabla u|^p \, dm\right)^{1/p} \leq c\left(\frac{1}{|Q|}\int_{2Q} |\nabla u|^n \, dm\right)^{1/n} \tag{15}$$

for any cube Q with $2Q \subset D$.

Proof. The Sobolev–Poincaré inequality and (14) yield

$$\begin{aligned} \left(\frac{1}{|Q|}\int_Q |\nabla u|^n \, dm\right)^{1/n} &\leq c_1\left(\frac{1}{|Q|^2}\int_{2Q} |u(x) - u_{2Q}|^n \, dm\right)^{1/n} \\ &\leq c_2\left(\frac{1}{|Q|}\int_{2Q} |\nabla u|^{n/2} \, dm\right)^{2/n} \end{aligned}$$

whenever $2Q \subset D$. The inverse Hölder inequality due to F. W. Gehring, as stated in [BI, 4.2], implies then (15).

5.4. Lemma. *Suppose that D is a bounded domain with* $\dim_M(\partial D) = \lambda < n$. *If* $u \in \operatorname{loc}\operatorname{Lip}_\alpha(D)$ *satisfies* (14) *and if* $\alpha > \lambda/n$, *then* $\nabla u \in L^p(D)$ *for some* $p > n$.

Proof. Let Q be a cube with $4Q \subset D$. Then (14) and (15) imply

$$\int_Q |\nabla u|^p \, dm \le c_1 |Q| \left(\frac{1}{|Q|} \int_{2Q} |\nabla u|^n \, dm \right)^{p/n}$$
$$\le c_2 |Q|^{1-2p/n} \left(\int_{4Q} |u(x) - u_{4Q}|^n \, dm \right)^{p/n}.$$

Since $4Q \subset D$ is uniform and $u \in \operatorname{loc}\operatorname{Lip}_\alpha(D)$, this yields by [GM1, 2.24]

$$\int_Q |\nabla u|^p \, dm \le c_3 \operatorname{dia}(Q)^{n-p(1-\alpha)}. \tag{16}$$

Now let W be the Whitney decomposition of D as in [S, p. 16] and denote by N_i the number of cubes Q in W of sidelength 2^{-i}, $i \in \mathbf{N}$. Since D is bounded and $\dim_M(\partial D) = \lambda < n$, [MVu, 3.9] implies that for any $\lambda' > \lambda$, $N_i \le c 2^{\lambda' i}$, $i = 1, 2, \ldots$, where c is independent of i. Hence (16) gives for each $\lambda' > \lambda$

$$\int_D |\nabla u|^p \, dm \le c_4 \sum_{i=1}^{\infty} 2^{i(\lambda' - n + p(1-\alpha))}.$$

The sum converges whenever $p < n + (n\alpha - \lambda)/(1 - \alpha)$ and thus the claim follows.

Proof of Theorem 5.1. Each coordinate mapping u_i, $i = 1, \ldots, n$, of the quasiregular mapping f satisfies (14) by [GLM, p. 54]. Hence the claim follows from Lemma 5.4.

Remark. As is well-known, weak solutions of quasilinear elliptic equations of the type

$$\nabla \cdot A(x, \nabla u) = 0$$

with appropriate conditions on A (see [GLM]) satisfy (14), hence the conclusion of Lemma 5.4 holds for these weak solutions as well.

Note added in proof. B. Hanson and P. Koskela have recently answered the problem (A) of section 4 in the negative.

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Kari Astala
Institut Mittag-Leffler
Auravägen 17
S-182 62 Djursholm, Sweden
and
University of Helsinki
Department of Mathematics
Hallituskatu 15
SF-00100 Helsinki, Finland

Pekka Koskela
Institut Mittag-Leffler
Auravägen 17
S-182 62 Djursholm, Sweden
and
University of Jyväskylä
Department of Mathematics
Seminaarinkatu 15
SF-40100 Jyväskylä, Finland

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