Characterization of compactly supported refinable splines

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We prove that a compactly supported spline function ϕ of degree k satisfies the scaling equation $\phi(x) = \sum_{n=0}^{N} c(n)\phi(mx - n)$ for some integer $m \geq 2$, if and only if $\phi(x) = \sum_n p(n)B_k(x - n)$ where $p(n)$ are the coefficients of a polynomial $P(z)$ such that the roots of $P(z)(z-1)^{k+1}$ are mapped into themselves by the mapping $z \to z^m$, and B_k is the uniform B-spline of degree k. Furthermore, the shifts of ϕ form a Riesz basis if and only if P is a monomial.

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1. Introduction

Splines, as well as refinable functions, have been widely used in the areas of numerical solution of differential equations, computer aided geometric design and wavelet theory. In this note we give a complete characterization of compactly supported refinable splines.

Definition 1.1
A non-zero compactly supported function $\phi : \mathbb{R} \to \mathbb{C}$ is a *spline* if there exists an integer $L \geq 2$ and points $x_0 < x_1 < \cdots < x_L$ such that ϕ is supported on $[x_0, x_L]$ and for each $1 \leq j \leq L$ and $x \in [x_{j-1}, x_j)$,

$$\phi(x) = P_j(x), \tag{1.1}$$

where P_j is a polynomial of degree k_j.

The points x_j, $j = 0, \ldots, L$, are called *knots* of ϕ and the *degree* of ϕ is $k = \max\{k_1, \ldots, k_L\}$. A knot x_j is called an *active knot* of ϕ if its derivative $\phi^{(l)}$ is discontinuous at x_j for some l. We shall assume that all the knots are active knots.

A special class of splines are the uniform B-splines which are defined inductively by

$$B_0(x) := \chi_{[0,1)}(x) := \begin{cases} 1 & x \in [0,1), \\ 0 & \text{otherwise}, \end{cases} \tag{1.2}$$

and for $k \geq 1$

$$B_k := B_{k-1} * B_0, \tag{1.3}$$

where $*$ denotes the operation of convolution. Then for any integers $k \geq 0$, B_k has $k-1$ continuous derivatives with active knots at $0, \ldots, k+1$ and it is a polynomial of degree k between the knots.

The Fourier transform of an integrable function ϕ will be denoted by $\hat{\phi}$ and is defined by

$$\hat{\phi}(u) := \int_{-\infty}^{\infty} \phi(x) e^{2\pi i x u} dx.$$

Definition 1.2
Let $m \geq 2$ be an integer. A function f is *m-refinable* if there exist positive integers $N_1 < N_2$ and a finite sequence of complex number $a(n)$, $n = N_1, \ldots, N_2$, called the *scaling sequence*, such $a(N_1) \neq 0$, $a(N_2) \neq 0$ and f satisfies the following *scaling (or refinement) equation*

$$f(x) = \sum_{n=N_1}^{N_2} a(n) f(mx - n). \tag{1.4}$$

Note that every m-refinable function may be reduced to *standard form* by letting $\phi(x) = f(x + (N_1/m - 1))$. Then (1.4) becomes

$$\phi(x) = \sum_{n=0}^{N} c(n) \phi(mx - n), \tag{1.5}$$

where $c(n) := a(n + N_1)$ and $N := N_2 - N_1$. We shall assume that all m-refinable functions have been reduced to standard form unless otherwise stated. We shall also call (1.5) a *scaling equation of length N*. The polynomial $C(z) := c(0) + c(1)z + \cdots + c(N)z^N$ is called the *scaling polynomial* of ϕ.

By taking Fourier transforms of the functions in (1.5), we have

$$\hat{\phi}(u) = \frac{1}{m} C(e^{2\pi i u/m}) \hat{\phi}\left(\frac{u}{m}\right). \tag{1.6}$$

The uniform B-spline B_k is an m-refinable compactly supported spline for any $m \geq 2$. Indeed, B_k satisfies the scaling equation

$$B_k(x) = \sum_{n=0}^{N} b_{k,m}(n) B_k(mx - n), \tag{1.7}$$

where $N := (k+1)(m-1)$, and the scaling coefficients $b_{k,m}(n)$ are given recursively by

$$b_{0,m}(n) := \begin{cases} 1 & n = 0, \ldots, m-1, \\ 0 & \text{otherwise}, \end{cases} \tag{1.8}$$

and for $k = 1, 2, \ldots,$

$$b_{k,m} := b_{k-1,m} * b_{0,m} := \frac{1}{m} \sum_{j=0}^{m-1} b_{k-1,m}(n-j). \tag{1.9}$$

The scaling sequences $(b_{k,m}(n))_{n \in \mathbb{Z}}$ are the discrete uniform B-splines. The scaling relation (1.7) with the scaling coefficients given in (1.8) and (1.9) is established by induction on k using (1.3). If we denote the scaling polynomial of B_k by

$$B_{k,m}(z) := \sum_{n=0}^{N} b_{k,m}(n) z^n, \quad z \in \mathbb{C}, \tag{1.10}$$

then the above recursive definition of $b_{k,m}(n)$ is equivalent to

$$B_{k,m}(z) = \frac{1}{m^k}(1 + z + \cdots + z^{m-1})^{k+1}. \tag{1.11}$$

This can also be established directly by taking the Fourier transforms of the functions in the scaling equation (1.7).

Refinable functions are studied in computer aided geometric design via subdivision methods [2] and also in wavelet theory [3]. The refinability makes it possible to iterate a fixed numerical scheme to generate a curve or surface by computers, and provides a simple numerical decomposition and reconstruction algorithm for image compression. Refinable functions with compact supports give efficient numerical schemes and better time localization in image compression.

The interpolatory subdivision scheme, an iterative interpolation process, is a special subdivision scheme. The corresponding refinable function obtained from such a scheme is a fundamental function. A function ϕ is called a fundamental function if there is an integer i, so that

$$\phi(j) = \delta_{i,j}, \quad \text{for all } j \in \mathbb{Z}.$$

Deslauriers and Dubuc [4] considered such a scheme which led them to the construction of 2-refinable compactly supported fundamental functions. These functions are usually not splines. It is well known that refinable fundamental functions can be constructed from the uniform B-splines (see [6]), but they usually have infinite supports.

In [2], Cavaretta et al. made an extensive study of the theory and the applications of stationary subdivision. Among other things, they gave a characterization of compactly supported refinable cardinal splines (a cardinal spline is a linear combinations of shifts, i.e. integer translates, of a fixed uniform B-spline). It would be of interest to give a complete characterization of all compactly supported refinable splines. The main result of this paper gives such a characterization.

It is of particular interest in wavelet theory to cónstruct compactly supported refinable functions with orthonormal shifts. In [3], Daubechies gave a general construction of a class of 2-refinable compactly supported functions with orthonormal shifts by constructing scaling sequences with certain properties. The corresponding refinable functions, which are defined via their Fourier transforms, usually do not have analytic forms. On the other hand, refinable functions with orthonormal shifts constructed by Battle and Lemarié [1,5] from uniform B-splines are piecewise polynomials. However, they usually have infinite supports.

Splines are widely used in computer aided geometric design and in the numerical solution of differential and integral equations. Recently more general ideas on refinability, nested spaces and decomposition of spaces have also been used in these areas. Therefore it would be useful to have compactly supported refinable splines which have orthonormal shifts, or which are fundamental functions. This partially motivates our study here and an immediate corollary of the main results of this paper confirms that, except the trivial cases, there are no such splines.

In section 2 we introduce some preliminaries and state the main theorem whose proof is given in section 3.

2. Preliminaries and statement of results

We introduce necessary concepts, derive preliminary results and state the characterization theorem for m-refinable splines.

Lemma 2.1
If ϕ is an m-refinable compactly supported function satisfying a scaling equation of length N, then the smallest interval containing its support is $[0, N/(m-1)]$.

Proof
If ϕ has compact support let $[a, b]$ be the smallest interval containing the support of ϕ. The scaling equation (1.5) implies

$$[a, b] \subset \bigcup_{n=0}^{N} \left[\frac{n+a}{m}, \frac{n+b}{m}\right].$$

Therefore $a \geq 0$ and $b \leq N/(m-1)$. □

Lemma 2.2
If ϕ is an m-refinable compactly supported spline function for any $m \geq 2$, then the knots of ϕ are integers.

Proof

Lemma 2.1 implies that $x_i \geq 0$ for each i. Suppose that there exists a non-integer knot and let $0 < x_i$ be the smallest non-integer knot. Then (1.5) implies

$$\phi\left(\frac{x_i}{m}\right) = c(0)\phi(x_i) + \sum_{n=1}^{N} c(n)\phi(x_i - n).$$

The points $x_i - n$ are not knots since they are non-integers and they are smaller than x_i and x_i was assumed to be the smallest non-integer knot. Since x_i is a knot and $x_i - n$ are not knots x_i/m is a knot and is smaller than x_i. Furthermore, x_i/m is a non-integer knot, since x_i is a non-integer, and is smaller than x_i. This contradicts the assumption that x_i is the smallest non-integer knot and completes the proof. □

Definition 2.1

Let $m \geq 2$ be an integer. A polynomial $P(z)$ is *m-closed* if its roots are mapped into themselves (counting multiplicity) by the mapping $z \to z^m$.

Lemma 2.3

A polynomial $P(z)$ is m-closed if and only if $P(z)$ divides $P(z^m)$.

Proof

Suppose that $P(z)$ divides $P(z^m)$. Then $P(z^m) = Q(z)P(z)$ for some polynomial $Q(z)$. Therefore, if λ is a root of $P(z)$ of multiplicity μ then λ^m is also a root of $P(z)$ of multiplicity at least μ. Hence $P(z)$ is m-closed.

Conversely, suppose that $P(z)$ is m-closed. Now the rational function $P(z^m)/P(z)$ has at most isolated poles at the roots of $P(z)$. If λ is a root of $P(z)$ of multiplicity μ, then λ^m is a root of $P(z)$ of multiplicity at least μ. Hence $(z^m - \lambda^m)^\mu$ is a factor of $P(z^m)$. Since $(z-\lambda)^\mu$ divides $(z^m - \lambda^m)^\mu$, $P(z^m)$ has a factor $(z-\lambda)^\mu$. Hence $P(z^m)/P(z)$ is a polynomial. □

Definition 2.2

The *linear combination* of shifts of a function f by a polynomial $P(z) = p(0) + p(1)z + \cdots + p(d)z^d$ is the function

$$g(x) = \sum_{n=0}^{d} p(n)f(x-n). \tag{2.1}$$

Clearly g is a linear combination of shifts of ϕ by $P(z)$ if and only if

$$\hat{g}(u) = P(e^{2\pi i u})\hat{\phi}(u).$$

We now state the main theorem.

Theorem 2.1
Suppose that ϕ is a compactly supported spline function of degree k and $m \geq 2$ is an integer. Then ϕ is an m-refinable function satisfying the scaling equation (1.5) if and only if there exists a polynomial $P(z)$ such that $P(z)(z-1)^{k+1}$ is m-closed and ϕ is the linear combination of shifts of B_k by $P(z)$. Further, the shifts of ϕ form a Riesz basis if and only if $P(z)$ is a monomial.

If f is an m-refinable compactly supported spline function satisfying the scaling equation (1.4), then its shift $f(\cdot + N_1/(m-1))$ satisfies a scaling equation of the form (1.5). Therefore the following are direct consequences of theorem 2.1.

Corollary 2.1
A compactly supported spline function ϕ of degree k satisfies the scaling equation (1.4) if and only if there exists a polynomial $P(z)$ such that $P(z)(z-1)^{k+1}$ is m-closed and ϕ is the linear combination of shifts of $B_k(\cdot - N_1/(m-1))$ by $P(z)$.

Corollary 2.2
The shifts of a compactly supported m-refinable spline function of degree k form a Riesz basis if and only if it is of the form $B_k(\cdot - j/(m-1))$ for $j \in \mathbb{Z}$.

Corollary 2.3
Let ϕ be a compactly supported m-refinable spline function. Then ϕ and its shifts form an orthonormal set if and only if $\phi(x) = B_0(x - j/(m-1))$ for some $j \in \mathbb{Z}$; and ϕ is a fundamental function if and only if $\phi(x) = B_0(x - j/(m-1))$ for some $j \in \mathbb{Z}$ or $\phi(x) = B_1(x - j/(m-1))$ for some $j \in \mathbb{Z}$.

The proof of theorem 2.1 based on properties of the Fourier transform of ϕ and its relationship to the scaling polynomial $C(z)$ of ϕ is given in the next section.

3. Proof of the main theorem

We first establish some auxiliary results.

Lemma 3.1
For a compactly supported spline function ϕ of degree k, $\hat{\phi}(u) \neq 0$ for almost all u, and

$$\hat{\phi}(u) = \sum_{j=0}^{k} u^{j-k-1} T_j(u), \tag{3.1}$$

where each $T_j(y), j = 0, \ldots, k$, is a polynomial of the form

$$T_j(u) = \sum_{l=0}^{L} t_{l,j} e^{2\pi i x_l u}, \tag{3.2}$$

and x_l, $l = 0, \ldots, L$, are the active knots of ϕ.

Proof
Suppose that ϕ is given by (1.1). Then

$$\hat{\phi}(u) = \sum_{j=1}^{L} \int_{x_{j-1}}^{x_j} P_j(x) e^{2\pi i x u} dx.$$

The result then follows by integration by parts. □

Lemma 3.2
Let ϕ be an m-refinable compactly supported spline function of degree k with scaling polynomial $C(z)$ and Fourier transform given in (3.1). Then each T_j is a trigonometric polynomial satisfying the equation

$$T_j(u) = m^{k-j} C(e^{2\pi i u/m}) T_j\left(\frac{u}{m}\right). \tag{3.3}$$

Proof
By lemmas 2.1 and 2.2, the knots of ϕ, $0 = x_0, \ldots, x_L = N/(m-1)$ are integers. Hence each T_j is a trigonometric polynomial. Substituting the expression for $\hat{\phi}(u)$ in (3.1) into (1.6) we obtain

$$\sum_{j=0}^{k} u^{j-k-1} T_j(u) = \sum_{j=0}^{k} m^{k-j} C(e^{2\pi i u/m}) u^{j-k-1} T_j\left(\frac{u}{m}\right).$$

The relation (3.3) then follows by equating the coefficients of powers of u since C and T_j are periodic of period 1 and have only finitely many zeros in the interval $[0,1]$. □

We are now in the position to prove the main result.

Proof of theorem 2.1
Suppose that ϕ is an m-refinable compactly supported spline function of degree k satisfying the scaling equation (1.5). Then the Fourier transform $\hat{\phi}$ is given in (3.1). First we show that all but one of the T_j's are identically zero. For any $0 \le j \le k$, (1.6) and (3.3) show that $T_j(u)$ satisfies

$$T_j(u) = m^{(k-j+1)p} \frac{\hat{\phi}(u)}{\hat{\phi}(u/m^p)} T_j\left(\frac{u}{m^p}\right), \quad p = 0, 1, \ldots. \tag{3.4}$$

Choose l such that T_l is not identically zero. Suppose the Taylor series expansions of $\hat{\phi}$ and T_l about $u=0$ are

$$\hat{\phi}(u) = \alpha u^{\nu} + O(u^{\nu+1})$$

and

$$T_l(u) = \alpha_l u^{\nu_l} + O(u^{\nu_l+1}),$$

respectively. Then for $u \neq 0$ such that $\hat{\phi}(u) \neq 0$,

$$T_l(u) = \lim_{p\to\infty} \frac{\alpha_l}{\alpha} m^{(k-l+1+\nu-\nu_l)p} u^{\nu_l-\nu} \hat{\phi}(u). \tag{3.5}$$

It follows that

$$\nu_l = k - l + 1 + \nu$$

and

$$T_l(u) = \frac{\alpha_l}{\alpha} u^{k-l+1} \hat{\phi}(u).$$

Hence

$$\frac{\alpha}{\alpha_l} u^{l-k-1} T_l(u) = \sum_{j=0}^{k} u^{j-k-1} T_j(u).$$

Therefore, T_j is identically zero for all $j \neq l$ and it follows from (3.1) that

$$\hat{\phi}(u) = \frac{T_l(u)}{u^{k-l+1}}.$$

By defining

$$T(z) := \sum_{j=0}^{L} t_{j,l} z^{x_j}, \quad z \in \mathbb{C}, \tag{3.6}$$

we obtain

$$\hat{\phi}(u) = \frac{T(e^{2\pi i u})}{u^{k-l+1}}, \tag{3.7}$$

for some l, $0 \leq l \leq k$.

Since $\hat{\phi}(u)$ is bounded, (3.7) shows that $T(z)$ must have a root or order at least $k-l+1$ at $z=1$. Furthermore, (3.3) implies

$$T(z^m) = m^{k-l} C(z) T(z), \quad z \in \mathbb{C}, \tag{3.8}$$

where $C(z)$ is the scaling polynomial of ϕ. Hence, $T(z)$ is m-closed.

We define the polynomial $P(z)$ by

$$P(z) = \frac{T(z)}{(z-1)^{k-l+1}}. \tag{3.9}$$

Then $P(z)(z-1)^{k-l+1}$ is m-closed. Further,

$$\hat{\phi}(u) = P(e^{2\pi iu})\frac{(e^{2\pi iu}-1)^{k-l+1}}{u^{k-l+1}} = P(e^{2\pi iu})\hat{B}_{k-l}(u),$$

showing that ϕ is the linear combination of shifts of the B-spline B_{k-l} by $P(z)$. Since ϕ is of degree k, it follows that $l=0$.

Conversely, suppose that ϕ is the linear combination of shifts of B_k for some integer $k \geq 0$ by a polynomial $P(z)$ for which $P(z)(z-1)^{k+1}$ is m-closed. Then

$$\hat{\phi}(u) = P(e^{2\pi iu})\hat{B}_k(u) \tag{3.10}$$

and

$$\frac{\hat{\phi}(mu)}{\hat{\phi}(u)} = \frac{1}{m^{k+1}}\frac{P(e^{2\pi imu})(e^{2\pi imu}-1)^{k+1}}{P(e^{2\pi iu})(e^{2\pi iu}-1)^{k+1}}$$

is a polynomial. Hence ϕ is m-refinable.

Finally, if ϕ is the linear combination of shifts of B_k by $P(z)$, then (3.10) gives

$$\sum_{n\in\mathbb{Z}} |\hat{\phi}(u+n)|^2 = |P(e^{2\pi iu})|^2 \sum_{n\in\mathbb{Z}} |\hat{B}_k(u+n)|^2, \quad u \in \mathbb{R}.$$

Hence

$$A \leq \sum_{n\in\mathbb{Z}} |\hat{\phi}(u+n)|^2 \leq B, \quad u \in \mathbb{R},$$

for some positive constants A and B, if and only if $P(z)$ has no root on the unit circle. Since $P(z)(z-1)^{k+1}$ is m-closed, all the roots of $P(z)$ are either zero or lie on the unit circle. Hence $P(z)$ has no roots on the unit circle if and only if $P(z)$ is a monomial. □

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