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Drawing with Complex Numbers

It is not commonly realized that the algebra of complex numbers can be used in an elegant way to represent the images of ordinary 3-dimensional figures, orthographically projected to the plane. We describe these ideas here, both using simple geometry and setting them in a broader context.

Consider orthogonal projection in Euclidean n-space onto an m-dimensional subspace. We may as well choose coördinates so that this is the standard projection $P: \mathbb{R}^n \to \mathbb{R}^m$ onto the first m variables. Fix a nondegenerate simplex $\sum$ in $\mathbb{R}^n$. Two such simplices are said to be *similar* if one can be obtained from the other by a Euclidean motion together with an overall scaling. This article answers the following question. Given $n + 1$ points in $\mathbb{R}^m$, when can these points be obtained as the images under P of the vertices of a simplex similar to $\sum$?

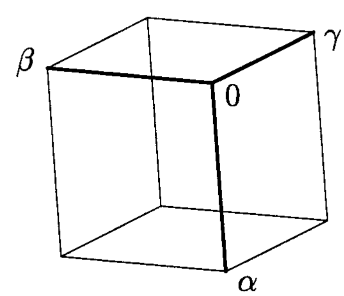
When $n = 3$ and $m = 2$, then P is the standard *orthographic* projection (as often used in engineering drawing), and we are concerned with how to draw a given tetrahedron. We shall show, for example, that four points α, β, γ, δ in the plane are the orthographic projections of the vertices of a *regular* tetrahedron if and only if

$$(\alpha + \beta + \gamma + \delta)^2 = 4(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \qquad (1)$$

where α, β, γ, δ are regarded as *complex* numbers! Similarly, suppose a cube is orthographically projected and normalised so that a particular vertex is mapped to the origin. If α, β, γ are the images of the three neighbouring vertices, then

$$\alpha^2 + \beta^2 + \gamma^2 = 0 \qquad (2)$$

again as a *complex* equation. Conversely, if this equation is satisfied, then one can find a cube whose orthographic image is given in this way. Since parallel lines are seen as parallel in the drawing, equation (2) allows one to draw the general cube:

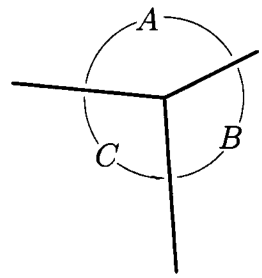


In this example, $\alpha = 2 - 26i$
$\beta = -23 + 2i$
$\gamma = 14 + 7i$

The result for a cube is known as *Gauss's fundamental theorem of axonometry*—see [3, p. 309] where it is stated without proof. In engineering drawing, one usually fixes three *principal* axes in Euclidean three-space, and then an orthographic projection onto a plane transverse to these axes is known as an *axonometric* projection (see, for example, [8, Chapter 17]). Gauss's theorem may be regarded as determining the degree of foreshortening along the principal axes for a general axonometric projection. The projection corresponding to taking α, β, γ to be the three cube roots of unity is called *isometric* projection because the foreshortening is the same for the three principal axes.

In an axonometric drawing, it is conventional to take the image axes at mutually obtuse angles:

[1]Supported by the Australian Research Council.

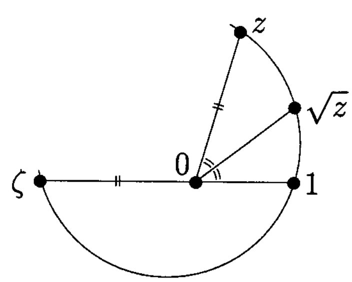


If $|\alpha| = a$, $|\beta| = b$, $|\gamma| = c$, then equation (2) is equivalent to the sine rule for the triangle with sides α^2, β^2, γ^2, namely

$$\frac{a^2}{\sin 2A} = \frac{b^2}{\sin 2B} = \frac{c^2}{\sin 2C}.$$

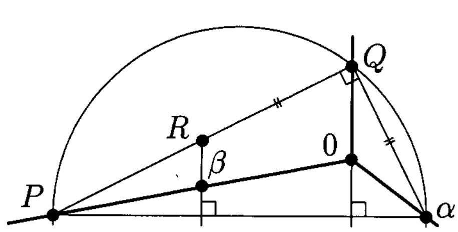
In this form, the fundamental theorem of axonometry is due to Weisbach, and was published in Tübingen in 1844 in the *Polytechnische Mitteilungen* of Volz and Karmasch. Equivalent statements can be found in modern engineering drawing texts (e.g., [7, p. 44]).

Equation (2) may be used to give a ruler-and-compass construction of the general orthographic image of a cube. If we suppose that the image of a vertex and two of its neighbours are already specified, then (2) determines (up to a two-fold ambiguity) the image of the third neighbour. The construction is straightforward, except perhaps for the construction of a complex square root, for which we advocate the following as quite efficient:



First, ζ is constructed by marking the real axis at a distance $\|z\|$ from the origin. Then, a circle is constructed passing through the three points ζ, 1, and z. Finally, the angle between 1 and z is bisected and $\sqrt{z}$ appears where this bisector meets the circle.

In engineering drawing, it is more usual that the images of the three principal axes are prescribed or chosen by the designer and one needs to determine the relative degree of foreshortening along these axes. There is a ruler-and-compass construction given by T. Schmid in 1922 (see, for example, [8, §17.17–17.19]):



In this diagram, the three principal axes and α are given. By drawing a perpendicular from α to one of the principal axes and marking its intersection with the remaining principal axis, we obtain P. The point Q is obtained by drawing a semicircle as illustrated. The point R is on the resulting line and equidistant with α from Q. Finally, β is obtained by dropping a perpendicular as shown. It is easy to see that this construction has the desired effect—in Euclidean three-space, rotate the right-angled triangle with hypotenuse $P\alpha$ about this hypotenuse until the point Q lies directly above 0, in which case R will lie directly above β and the third vertex will lie somewhere over the line through 0 and Q. One may verify the appropriate part of Weisbach's condition

$$\frac{a^2}{\sin 2A} = \frac{b^2}{\sin 2B} \tag{3}$$

by the following calculation. Without loss of generality we may represent all these points by complex numbers normalised so that $Q = 1$. Then it is straightforward to check that

$$R = 1 + i - i\alpha, \qquad P = \frac{\alpha(\alpha + \bar{\alpha}) + 2(1 - \alpha - \bar{\alpha})}{\alpha - \bar{\alpha}},$$
$$\beta = \frac{\alpha(\alpha + \bar{\alpha}) + 2(1 - \alpha - \bar{\alpha})}{2 - \alpha - \bar{\alpha}}\, i,$$

and therefore that

$$\alpha^2 + \beta^2 = 4\frac{(\alpha - 1)(\overline{\alpha - 1})(\alpha + \bar{\alpha} - 1)}{(\alpha + \bar{\alpha} - 2)^2}.$$

That $\alpha^2 + \beta^2$ is real is equivalent to (3).

To prove Gauss's theorem more directly, consider three vectors in $\mathbb{R}^3$ as the columns of a 3×3 matrix. This matrix is orthogonal if and only if the three vectors are orthonormal. It is equivalent to demand that the three rows be orthonormal. However, any two orthonormal vectors in $\mathbb{R}^3$ may be extended to an orthonormal basis. Thus, the condition that three vectors

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

in $\mathbb{R}^2$ be the images under $P\colon \mathbb{R}^3 \to \mathbb{R}^2$ of an orthonormal basis of $\mathbb{R}^3$, is that

$$(x_1 \quad x_2 \quad x_3) \qquad \text{and} \qquad (y_1 \quad y_2 \quad y_3)$$

be orthonormal in $\mathbb{R}^3$. Dropping the requirement that the common norm be 1, we obtain

$$x_1{}^2 + x_2{}^2 + x_3{}^2 = y_1{}^2 + y_2{}^2 + y_3{}^2 \qquad \text{and} \qquad x_1y_1 + x_2y_2 + x_3y_3 = 0.$$

Writing $\alpha = x_1 + iy_1$, $\beta = x_2 + y_2$, $\gamma = x_3 + y_3$, these two equations are the real and imaginary parts of (2). To deduce the case of a regular tetrahedron as described by equation (1) from the case of a cube as described by equation (2), it suffices to note that equation (1) is translation-invariant and that a regular tetrahedron may be inscribed in a cube. Thus, we may take $\delta = \alpha + \beta + \gamma$ and observe that (1) and (2) are then equivalent.

It is easy to see that the possible images of a particular tetrahedron Σ in $\mathbb{R}^3$ under an arbitrary Euclidean motion followed by the projection P form a 5-dimensional space—the group of Euclidean motions is 6-dimensional, but translation orthogonal to the plane leaves the image unaltered. It therefore has codimension 3 in the 8-dimensional space of all tetrahedral images (2 degrees of freedom for each vertex). Allowing similar tetrahedra rather than just congruent ones reduces the codimension to 2. Therefore, two real equations are to be expected. Always, these two real equations combine as a single *complex* equation such as (1) or (2). At first sight, this is perhaps surprising; and even more so when the same phenomenon occurs for $P: \mathbb{R}^n \to \mathbb{R}^2$ for arbitrary n.

For $n = 3$, there is a proof of Gauss's theorem which brings in complex numbers at the outset. Consider the space H of Hermitian 2×2 matrices with zero trace, i.e., matrices of the form

$$X = \begin{pmatrix} w & u + iv \\ u - iv & -w \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3.$$

We may identify H with $\mathbb{R}^3$, and, in so doing, $-\det X$ becomes the square of the Euclidean length. The group G of invertible 2×2 complex matrices of the form

$$\Lambda = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$$

acts linearly on H by $X \mapsto \Lambda X \bar{\Lambda}^t$. Moreover,

$$\det(\Lambda X \bar{\Lambda}^t) = (|a|^2 + |b|^2)^2 \det X,$$

so G acts by similarities. It is easy to check that all similarities may be obtained in this way. (This trick is essentially as used in Hamilton's theory of quaternions and is well known to physicists. In modern parlance it is equivalent to the isomorphism of Lie groups $\text{Spin}(3) \cong \text{SU}(2)$.) Therefore, an arbitrary orthographic image of a cube may be obtained by acting with Λ on the standard basis

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and then picking out the top right-hand entries. We obtain

$$\Lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bar{\Lambda}^t = \begin{pmatrix} * & a^2 - b^2 \\ * & * \end{pmatrix} \mapsto a^2 - b^2 = \alpha$$

$$\Lambda \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \bar{\Lambda}^t = \begin{pmatrix} * & i(a^2 + b^2) \\ * & * \end{pmatrix} \mapsto i(a^2 + b^2) = \beta$$

$$\Lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{\Lambda}^t = \begin{pmatrix} * & 2ab \\ * & * \end{pmatrix} \mapsto 2ab = \gamma$$

and therefore $\alpha^2 + \beta^2 + \gamma^2 = 0$, as required. Conversely, this is exactly the condition that α, β, γ may be written in this form. (Compare the half-angle formulae: if $s^2 + c^2 = 1$, then $s = 2t/(1 + t^2)$ and $c = (1 - t^2)/(1 + t^2)$ for some t.) That Gauss [3, p. 309] makes the same observation concerning the form of α, β, γ suggests that perhaps he also had this reasoning in mind.

In general, the following terminology concerning the standard projection $P: \mathbb{R}^n \to \mathbb{R}^m$ is useful. We shall say that $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ are *normalised eutactic* if and only if there is an orthonormal basis $u_1, u_2, \ldots, u_n$ of $\mathbb{R}^n$ with $v_j = Pu_j$ for $j = 1, 2, \ldots, n$. We shall say that $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ are *eutactic* if and only if $\mu v_1, \mu v_2, \ldots, \mu v_n$ are normalised eutactic for some $\mu \neq 0$. The proof of Gauss's theorem using orthogonal matrices clearly extends to yield the following result.

Theorem *The points $z_1, z_2, \ldots, z_n \in \mathbb{C} = \mathbb{R}^2$ are eutactic if and only if*

$$z_1^2 + z_2^2 + \cdots + z_n^2 = 0$$

and not all z_j are zero.

There is an alternative proof for $n = 4$ based on the isomorphism

$$\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2),$$

and, indeed, this is how we came across the theorem in the first place.

However, a more direct route to complex numbers, and one which applies in all dimensions, is based on the observation that $\text{Gr}_2^+(\mathbb{R}^2)$, the Grassmannian of oriented two-planes in $\mathbb{R}^n$, is naturally a *complex* manifold. When $n = 3$, this Grassmannian is just the two-sphere and has a complex structure as the Riemann sphere. In general, consider the mapping

$$\mathbb{CP}_{n-1} \setminus \mathbb{RP}_{n-1} \xrightarrow{\pi} \text{Gr}_2^+(\mathbb{R}^n)$$

induced by $\mathbb{C}^n \ni z \mapsto iz \wedge \bar{z}$. In other words, a complex vector $z = x + iy \in \mathbb{C}^n$ is mapped to the two-dimensional oriented subspace of $\mathbb{R}^n$ spanned by x and y, the real and imaginary parts of z. Let $\langle\,,\rangle$ denote the standard inner product on $\mathbb{R}^n$ extended to $\mathbb{C}^n$ as a complex bilinear form. Then, $\langle z, z\rangle = 0$ imposes two real equations

$$\|x\|^2 = \|y\|^2 \qquad \text{and} \qquad \langle x, y\rangle = 0$$

on the real and imaginary parts. In other words, x, y is proportional to an orthonormal basis for $\text{span}\{x, y\}$. Hence, if z and w satisfy $\langle z, z\rangle = 0 = \langle w, w\rangle$ and define the same oriented two-plane, then $w = \lambda z$ for some $\lambda \in \mathbb{C}\setminus\{0\}$. The nonsingular complex quadric

$$K = \{[z] \in \mathbb{CP}_{n-1} \text{ s.t. } \langle z, z\rangle = 0\}$$

avoids $\mathbb{RP}_{n-1} \subset \mathbb{CP}_{n-1}$, and we have shown that $\pi|_K$ is injective. It is clearly surjective. The isomorphism

$$\pi: K \xrightarrow{\simeq} \text{Gr}_2{}^+(\mathbb{R}^n)$$

respects the natural action of $\text{SO}(n)$ on K and $\text{Gr}_2{}^+(\mathbb{R}^n)$. The generalised Gauss theorem follows immediately, for, rather than asking about the image of a general orthonormal basis under the standard projection $P: \mathbb{R}^n \to \mathbb{R}^2$, we may, equivalently, ask about the image of the standard basis $e_1, e_2, \ldots, e_n$ under a general orthogonal projection onto an oriented two-plane $\Pi \subset \mathbb{R}^n$. Any such Π is naturally complex, the action of i being given by rotation by 90° in the positive sense. If Π is represented by $[z_1, z_2, \ldots, z_n] \in K$ as above and we use $x, y \in \Pi$ to identify Π with $\mathbb{C}$, then $e_j \mapsto z_j$ and

$$z_1^2 + z_2^2 + \cdots + z_n^2 = \langle z, z\rangle = 0,$$

as required. Conversely, a solution of this complex equation determines an appropriate plane Π.

For the case of a general tetrahedron or simplex and for general m and n, it is more convenient to start with Hadwiger's theorem [4] or [2, page 251] as follows. The proof is obtained by extending our orthogonal matrix proof of Gauss's theorem.

Theorem (Hadwiger) *Assemble $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ as the columns of an $m \times n$ matrix V. These vectors are normalised eutactic if and only if $VV^t = 1$ (the $m \times m$ identity matrix).*

Proof If $v_1, v_2, \ldots, v_n$ are normalised eutactic, then assembling a corresponding orthonormal basis of $\mathbb{R}^n$ as the columns of an $n \times n$ matrix, we have $V = PU$ and $U^tU = 1$ (the $n \times n$ identity matrix). Therefore, $UU^t = 1$ and

$$VV^t = PUU^tP^t = PP^t = 1,$$

as required. Conversely, if $VV^t = 1$, then the columns of V^t may be completed to an orthonormal basis of $\mathbb{R}^n$, i.e., $V^t = U^tP^t$ for $UU^t = 1$. Now, $U^tU = 1$ and $V = PU$, as required. □

The case of a general simplex is obtained essentially by a change of basis as follows. Suppose $a_1, a_2, \ldots, a_n, a_{n+1}$ are the vertices of a non-degenerate simplex $\sum$ in $\mathbb{R}^n$ whose centre of mass is at the origin. In other words, the $n \times (n+1)$ matrix A has rank n and $Ae = 0$ where e is the column vector all of whose $n + 1$ entries are 1. Form the $(n + 1) \times (n + 1)$ symmetric matrix

$$Q = A^t(AA^t)^{-2}A,$$

noting that rank $A = n$ implies that the *moment matrix* AA^t is invertible.

Theorem *Given $b_1, b_2, \ldots, b_n, b_{n+1} \in \mathbb{R}^m$ assembled as the columns of an $m \times (n+1)$ matrix B, these vectors are the images under orthogonal projection $P: \mathbb{R}^n \to \mathbb{R}^m$ of the vertices of a simplex congruent to $\sum$ if and only if*

$$BQB^t = 1. \tag{4}$$

Proof The vertices of a simplex congruent to $\sum$ are the columns of a matrix $UA + ae^t$ for some orthogonal matrix U and translation vector $a \in \mathbb{R}^n$. Also, note that $Qe = 0$. Thus, if $B = P(UA + ae^t)$, then

$$\begin{aligned} BQB^t &= PUAQA^tU^tP^t \\ &= PUAA^t(AA^t)^{-2}AA^tU^tP^t \\ &= PUU^tP^t = PP^t = 1, \end{aligned}$$

as required.

Conversely, $Qe = 0$ implies that (4) is translation invariant. So, without loss of generality, we may suppose that $b_1 + b_2 + \cdots + b_n + b_{n+1} = 0$, that is to say, $Be = 0$. Writing out (4) in full gives

$$BA^t(AA^t)^{-1}(BA^t(AA^t)^{-1})^t = 1$$

so, by Hadwiger's theorem, there is an orthogonal matrix U so that

$$BA^t(AA^t)^{-1} = PU.$$

Thus,

$$BA^t(AA^t)^{-1}A = PUA \quad \text{and} \quad Be = 0.$$

Certainly, $B = PUA$ is a solution of these equations; but it is the only solution, because $A^t(AA^t)^{-1}A$ has rank n and e is not in the range of this linear transformation. □

Corollary (case m = 2) *Points $z_1, z_2, \ldots, z_n, z_{n+1} \in \mathbb{C}$ are the images under orthogonal projection of the vertices of a simplex similar to $\sum$ if and only if*

$$z^tQz = 0$$

where z is the column vector with components $z_1, z_2, \ldots, z_n, z_{n+1}$.

It is, of course, possible to compute Q explicitly for any given example. If the simplex $\sum$ has some degree of symmetry, however, we can often circumvent such computation. Consider, for example, the case of a *regular* simplex. From the corollary above, we know that the image of such a simplex in the plane is characterised by a complex homogeneous quadratic polynomial. The symmetries of the regular simplex ensure that this polynomial must be invariant under $\mathcal{S}_{n+1}$, the symmetric group on $n + 1$ letters. Hence, it must be expressible in terms of the elementary symmetric polynomials. Equivalently, it must be a linear combination of

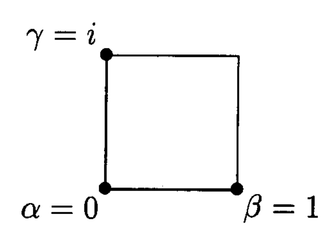
$$(z_1 + z_2 + \cdots + z_n + z_{n+1})^2 \quad \text{and} \quad z_1^2 + z_2^2 + \cdots + z_n^2 + z_{n+1}^2.$$

Up to scale, there is only one such combination that is translation-invariant, namely

$$(z_1 + z_2 + \cdots + z_n + z_{n+1})^2 - (n+1)(z_1^2 + z_2^2 + \cdots + z_n^2 + z_{n+1}^2). \tag{5}$$

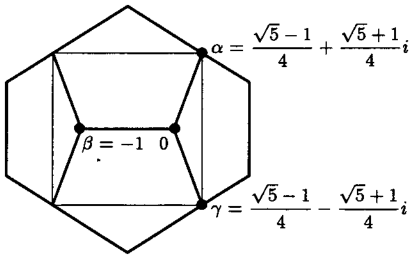
It follows that the vanishing of this polynomial is an equation that characterises the possible images of a regular simplex under orthogonal projection into the plane. The special case $n = 2$ characterises the equilateral triangles in the plane [1, Problem 15 on page 79].

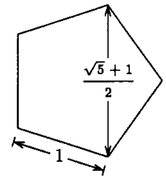
Equation (2) characterising the orthographic images of a cube, may be deduced by similar symmetry considerations. If a particular vertex is mapped to the origin and its neighbours are mapped to α, β, γ, then, since each of these neighbouring vertices is on an equal footing, the polynomial in question must be a linear combination of $(\alpha + \beta + \gamma)^2$ and $\alpha^2 + \beta^2 + \gamma^2$. To find out which linear combination, we need only consider a particular projection, for example:



In this example, $(\alpha + \beta + \gamma)^2 = 2i$ and $\alpha^2 + \beta^2 + \gamma^2 = 0$. Up to scale, therefore, (2) is the correct equation.

The case of a regular dodecahedron is similar. Using the fact that a cube may be inscribed in such a dodecahedron [5], we may deduce a particular projection:





with $(\alpha + \beta + \gamma)^2 = (7 - 3\sqrt{5})/2$ and $\alpha^2 + \beta^2 + \gamma^2 = (2 - \sqrt{5})/2$. In this particular case,

$$(\alpha + \beta + \gamma)^2 + (\sqrt{5} - 1)(\alpha^2 + \beta^2 + \gamma^2) = 0.$$

Therefore, this is the correct equation in the general case. It may be used as the basis of a ruler-and-compass construction of the general orthographic projection of a regular dodecahedron.

It is interesting to note that if *all* the vertices of a Platonic solid are orthographically projected to $z_1, z_2, \ldots, z_N \in \mathbb{C}$, then necessarily

$$(z_1 + z_2 + \cdots + z_N)^2 = N(z_1{}^2 + z_2{}^2 + \cdots + z_N{}^2). \quad (6)$$

Only for a tetrahedron, when (6) coincides with (1), is this condition also sufficient. To verify (6) for the other Platonic solids, first note that it is translation-invariant. Therefore, it suffices to impose $z_1 + z_2 + \cdots + z_N = 0$ and show that $z_1{}^2 + z_2{}^2 + \cdots + z_N{}^2 = 0$. The case of a cube now follows immediately, as its vertices may be grouped as two regular tetrahedra. The dodecahedral case may be dealt with by grouping its vertices into five regular tetrahedra. The regular octahedron is amenable to a similar trick, but not the icosahedron. Rather than resorting to direct computation, a uniform proof may be given as follows.

As before, assemble the vertices of the given solid Σ as the columns of a matrix A, now of size $3 \times N$, and consider the moment matrix $M \equiv AA^t$. Observe that

$$(1 \quad i \quad 0)\, M \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = z_1{}^2 + z_2{}^2 + \cdots + z_N{}^2.$$

The moment matrix is positive definite and symmetric. In other words, it defines a metric on $\mathbb{R}^3$, manifestly invariant under the symmetries of Σ. If Σ is regular—or, more generally, enjoys the symmetries of a regular solid (e.g., a cuboctahedron or rhombicosidodecahedron)—then its symmetry group acts irreducibly on $\mathbb{R}^3$. Thus, M must be proportional to the identity matrix and the result follows. For a general solid Σ, the two complex numbers

$$\pm\sqrt{z_1{}^2 + z_2{}^2 + \cdots + z_N{}^2}$$

are the foci of the ellipse

$$(x \quad y)\, R \begin{pmatrix} x \\ y \end{pmatrix} = 1,$$

where R is the inverse of the quadratic form obtained by restricting M to the plane of projection.

This reasoning also works in higher dimensions, where it shows (as conjectured to us by H.S.M. Coxeter) that the orthogonally projected images in the plane of the N vertices of any non-degenerate regular polytope, real or complex, will satisfy equation (6). This includes regular polygons in the plane, where the projection is vacuous. As already remarked, for polyhedra other than simplices, a quadratic equation such as (6) is no longer sufficient to characterise the orthogonal image up to scale. In general, there will also be some linear relations. For a non-degenerate N-tope in $\mathbb{R}^n$ there will be $N - n - 1$ such relations. The simplest example is a square in $\mathbb{R}^2$, which is characterised by the complex equations

$$(\alpha + \beta + \gamma + \delta)^2 = 4(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \quad \text{and} \quad \alpha + \gamma = \beta + \delta.$$

It is interesting to investigate further the relationship between a non-degenerate simplex Σ in $\mathbb{R}^n$ and its quadratic form $Q = A^t(AA^t)^{-2}A$. Recall that A is the $n \times (n + 1)$ matrix whose columns are the vertices of Σ. There are several other formulae for or characterisations of Q. Let S denote the $(n + 1) \times (n + 1)$ symmetric matrix

$$\mathbf{1} - \frac{1}{n+1}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

It is the matrix of orthogonal projection in $\mathbb{R}^{n+1}$ in the direction of the vector e. We maintain that Q is characterised by the equations

$$QG = S \quad \text{and} \quad Qe = 0,$$

where $G \equiv A^tA$. Certainly, if these equations hold, then they are enough to determine Q, because the matrix G has rank n and e is not in its range. The second equation is evident, and the first equation with Q replaced by $A^t(AA^t)^{-2}A$ and G by A^tA reads

$$A^t(AA^t)^{-1}A = S.$$

To see that this holds it suffices to observe that it is clearly true after postmultiplication by A^t or e. We may equally well characterise Q by means of the equations

$$GQ = S \quad \text{and} \quad Qe = 0$$

These equations relate G and Q geometrically: both matrices annihilate e, whilst on the hyperplane orthogonal to e they are mutually inverse. This is to say that G and Q are *generalised inverses* [6] of each other. Thus we write

$$Q = G^\dagger = (A^tA)^\dagger = A^\dagger A^{\dagger t}$$

where $A^\dagger$ is the generalised inverse of A. In this case, $A^\dagger = A^t(AA^t)^{-1}$. This also shows how to compute Q more directly in certain cases. The matrix G has direct geometric interpretation as the various inner products of the vectors $a_1, a_2, \ldots, a_n, a_{n+1}$. In the case of a regular simplex, for example, we know that $\|a_i\|^2$ is independent of i, that $\|a_i - a_j\|^2$ is independent of $i \neq j$, and that $a_1 + a_2 + \ldots + a_n + a_{n+1} = 0$. We may deduce that, with a suitable overall scale, $G = S$. Since $S^\dagger = S$, it follows that $Q = S$. This is a direct derivation of (5).

It is clear geometrically that G determines Σ up to congruency. Therefore, so does Q. In other words, the possible quadratic forms Q that can arise give a natural parametrisation of the non-degenerate simplices up to congruency. As to which Q do arise, certainly they enjoy the following properties:

- Q is a real $(n + 1) \times (n + 1)$ symmetric matrix.
- $Qe = 0$, and only multiples of e are in the kernel of Q.
- All other eigenvalues of Q are positive.

Conversely, these properties characterise the possible Q that can arise: given such a Q, we may take A^t to have as its columns a system of mutually orthogonal eigenvectors for the non-zero eigenvalues of Q, each being scaled to have length equal to the square-root of the reciprocal of the corresponding eigenvalue. It follows easily that $Q = A^t(AA^t)^{-2} A$.

It is also possible to repeat this analysis in pseudo-Euclidean spaces. The only difference is that the condition that the non-zero eigenvalues of Q be positive is replaced by a condition on sign precisely reflecting the original signature of the inner product.

Finally we should mention some possible applications. There is much current interest in *computer vision*. In particular, there is the problem of recognising a wire-frame object from its orthographic image. The results we have described can be used as test on such an image, for example to see whether a given image could be that of a cube, or to keep track of a moving shape. It is clear that such tests could be implemented quite efficiently. Another possibility is in the manipulation of CADD[2] data. Rather than storing an image as an array of vectors in $\mathbb{R}^3$, it may sometimes be more efficient to store certain tetrahedra within such an image by means of the corresponding quadratic form. For orthographic imaging this may be preferable.

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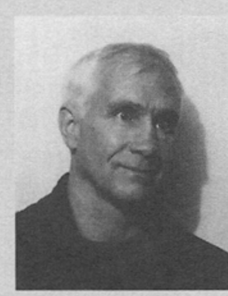

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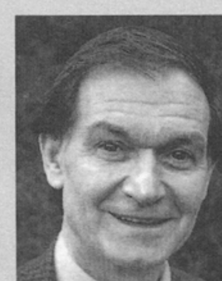

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[2]Computer Aided Drafting and Design.