The Emperor's New Clothes: Full Regalia, G string, or Nothing?

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Although my age may be an impediment, I would like to try to play the role of the little boy [1]. The occasion is the cover of the recent issue (Vol. 5, No. 4) of the Mathematical Intelligencer, or rather, more precisely, its caption: "The cover shows the seventeen doubly periodic symmetries of the plane. While these were known to ancient Egyptian craftsmen, the proof that these exhaust all possibilities was provided by George Pólya in 1924. . . . For more information see the article by Jean Pederson [sic] in this issue." Indeed, the very nice Pedersen article [11] provides not only more information, but also a *correct* quotation from [21], with full attribution and reference. The quotation is worth repeating:

> One can hardly overestimate the depth of geometric imagination and inventiveness reflected in these patterns. Their construction is far from being mathematically trivial. The art of ornament contains in implicit form the oldest piece of higher mathematics known to us. To be sure, the conceptual means for a complete abstract formulation of the underlying problem, namely the mathematical notion of a group of transformations, was not provided before the nineteenth century; and only on this basis is one able to prove that the 17 symmetries already implicitly known to the Egyptian craftsmen exhaust all possibilities. Strangely enough the proof was carried out only as late as 1924 by George Pólya, now teaching at Stanford.

Now this paragraph is the Emperor's edict ([21], pages 103–104); it had been generally acknowledged as being obviously true ever since he first issued it in 1938, in practically identical formulation [20]. One could argue about what "implicit" means in this context (and whether it should not have been kept in the caption of the Intelligencer's title page)—but there is a much more straightforward reason for pointing at what the Emperor is wearing: the available information indicates that nothing even remotely resembling "all 17 classes of symmetry groups" was known to the Egyptian craftsmen, implicitly or otherwise! (Or, if they had that knowledge, they were completely successful in not leaving any traces of it.) After surveying all the books on Egyptian ornamentation I could lay my hands on (in particular, the ones mentioned in [21], including the work of Owen Jones [7] which, on page 93 of [18], is said to "contain them all"), I have yet to find even a single example of the occurrence of one of the *five* classes of groups that include 3-fold rotations (p3, p3m1, p31m, p6, p6m; see Figure 1). Naturally, my "proof" that the correct number is 12 and not 17 could be ruined in the next issue by a reader informing us that in the umpteenth volume of "Excavations at Tel Abu-Meghukhakh", on page 971, there is a p6m pattern; seeing that we are talking about several thousands of years of activity, it would be surprising if no such pattern had been accidentally formed—but this would provide the Emperor, at best, with a G string. Cleary, in a situation like the one we are discussing here, the burden of proof should be on the side that claims the existence . . .

I have no means of *knowing* why the Emperor made his assertion. A *guess* would be that he was swayed by the Prime Minister, whose well known, influential and deserving book [18] brought the symmetry analysis of ornaments to the attention of mathematicians (Pólya's work was not noticed by most of them). That guess is motivated by the fact that the two have had very similar views on the centrality of group theory and on other topics, were well acquainted with each other, and—more tangibly—we find the Prime Minister's name mentioned in the Index of [21] more often than any other contemporary, [18] is explicitly mentioned

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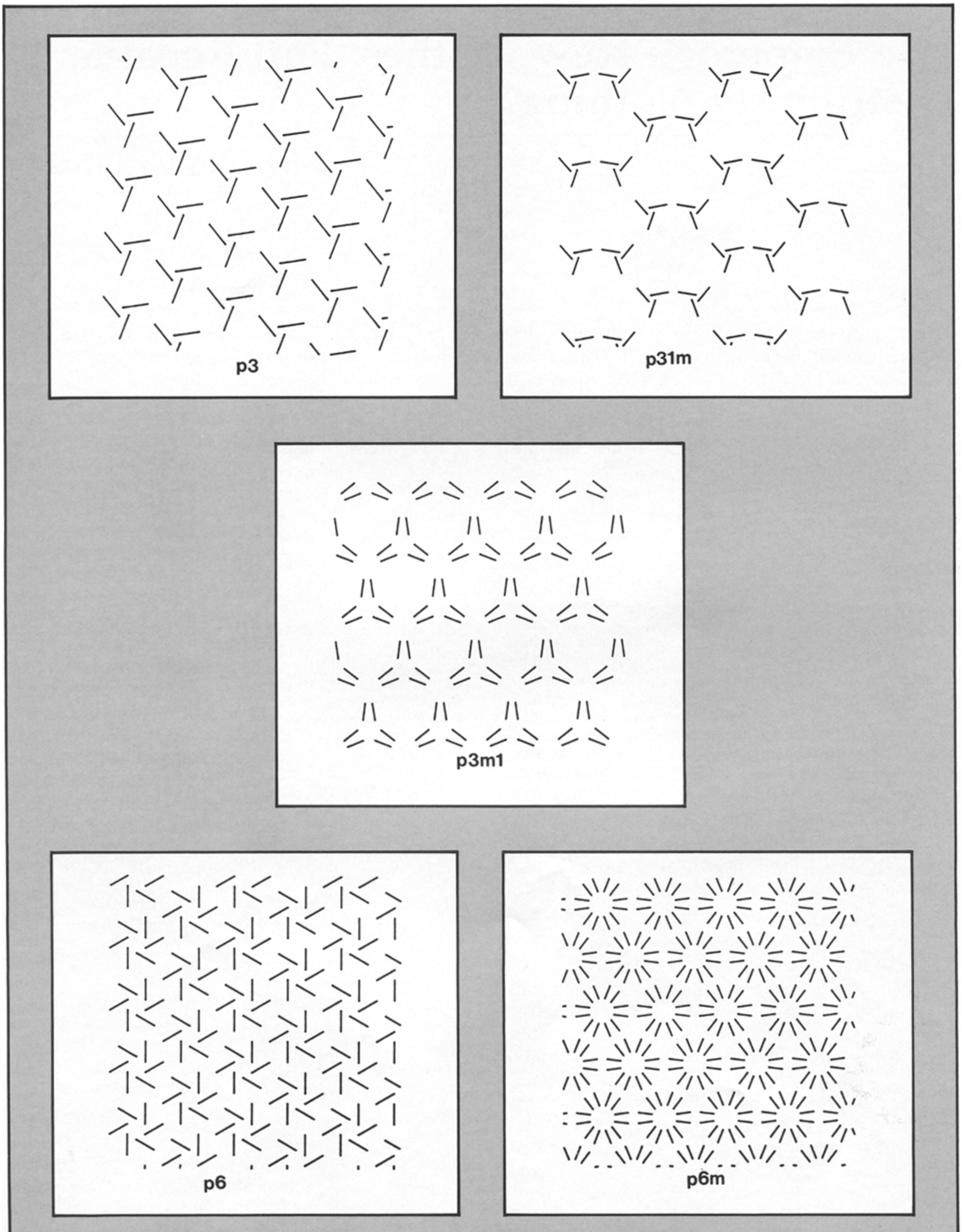

Figure 1. Ornaments formed by line segments, with the five symmetry groups that are not documented on any ornament from ancient Egypt.

in the Preface of [21], and two of the illustrations of [21] are taken from [18] (for one of these, the attribution of its symmetry group is explained in detail—with the same factual errors in [20] and [21]). But it is only the spirit of [18] that can be blamed, not any one decisive statement. The difference between the two is that [21] is rather explicit, while [18] is fuzzy, and beats around the bush. Indeed, in [18] we find, in the Introduction and in the chapter on symmetries of ornaments, such obviously suggestive but not legally binding assertions as these (freely translated from pages 1, 76):

> Long before people considered permutations, they constructed mathematical figures that have close connections to the theory of groups, and can be grasped only using group-theoretical concepts: the regular patterns, which can be brought to self-coincidence through motions and reflections. Together with music, they form a main topic of higher mathematics in antiquity.
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> . . .
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> Thus ornamentation proves itself as a geometric art. . . . Through the possibility of using effective mathematical methods, the creative power of ornamentation is very great; . . . The examples of surface decorations which I shall give stem mostly from Egypt, since there is the source of all later ornamentation. . . . [Then he goes on, approvingly quoting from [13], page 5]: . . .'Practically it is very difficult, or almost impossible, to point out decoration which is proved to have originated independently, and not to have been copied from the Egyptian stock.'

Significantly, all the examples of ornaments with various symmetry groups given on pages 91 to 95 of [18] are taken from Egyptian art—except one: that one is the only illustration which contains 3-fold rotations. In fairness to the Prime Minister it must be noted that on page 2 of [18] he states: "It is to be regretted that the Egyptian and Arabic ornaments have never been examined for their geometric content, ..." But even when the situation changed (by the finding in the Ph.D. thesis [7] of one of his students that the ornaments in the Alhambra belong to only 11 of the 17 groups), the tone and wording of the general statements was left unchanged in the fourth edition of [18].

The tailors' role is obviously played by Sir Flinders Petrie. Explicitly in [13], and in different words in many of his other publications, he indicated that ancient Egypt is the source of all worthwhile art. He clearly felt that the art of ornamentation, and the world in general, have been degenerating ever since. But as Sir Flinders was no mathematician, it is rather misleading to impute to his statement the mathematical implications that inevitably arise from the way he is quoted in [18]. Since the Prime Minister was a mathematician, was more than casually interested in the mathematical content of art, visited Egypt to study the ornaments (see [2], page 18), and was certainly able to tell the difference between groups that contain 3-fold rotations and those that do not—it seems also fair to say that even if he was not purposely creating a deception, he did get carried away by his preconceived ideas and was not very mindful of the impression his formulations were bound to create.

But even so, the Emperor himself is—after all—responsible for what he wears. . .

The moral of the story is: don't believe all you read, don't enjoy the sights you do not see, and do not buy the Brooklyn bridge—whoever be the seller.

* * *

Unfortunately, another misrepresentation from [18] and [21] is widespread and keeps being repeated—hampering the understanding of ancient and other ornaments, as well as impeding contemporary mathematics. It is the idea that the motivation of ancient artists and craftsmen (Egyptians, Chinese, Moors, . . .) was the same concern for symmetry that we express through symmetry groups. Therefore, according to this view, the only proper way to analyze these or any other repeating ornaments is by the use of symmetry groups; the very essence of orderliness is in the underlying groups. On page 79 of [18], in the discussion of the group p6m, there is the following statement quite clearly revealing the attitude [my translation].

> . . .Figure 4 is an excellent illustration of the prevailing symmetry relations. It shows a mosaic from the temple of Isis in Pompeii. A clumsy artist (since it is unlikely that he was a "new-wave" musician who loves dissonances) applied mathematical decorations which were taken from other ornaments and which show false symmetries: a circle with a five-rayed figure in a hexagon, a pair of interlaced ovals which admit 90° rotation at a center of 2-fold rotation. . . .

The scorn here seems to me to be as inappropriate as it was an effect of selective vision: since Egyptians could do no wrong, it was all right for them to position five-pointed stars in a square lattice arrangement, to reduce 90° rotational symmetries by placing various stripes or decorations with different symmetries, to impose color schemes incompatible with the underlying symmetries, etc.—all very clearly visible in [7] or [14], the sources quoted in [18] (see Figure 2).

But the main problem is with the very idea of symmetry. There is no basis whatsoever to assume that symmetry—as an isometric mapping of the ornament onto itself—was anywhere or at any time motivating artists or craftsmen. Even if we were to believe (as the authors of [18] and [21] and many others do) that symmetries can be used to *explain* the ornaments, that has absolutely no implication on what the creators of these ornaments had in mind. Any of the periodic symmetry groups have as a prerequisite the infinite extent of the ornament; surely no Islamic artist would have dared even to think in such a sacrilegious way about ornaments he can create. Or can anyone imagine the Pharaonic architect explaining his wishes for decoration to

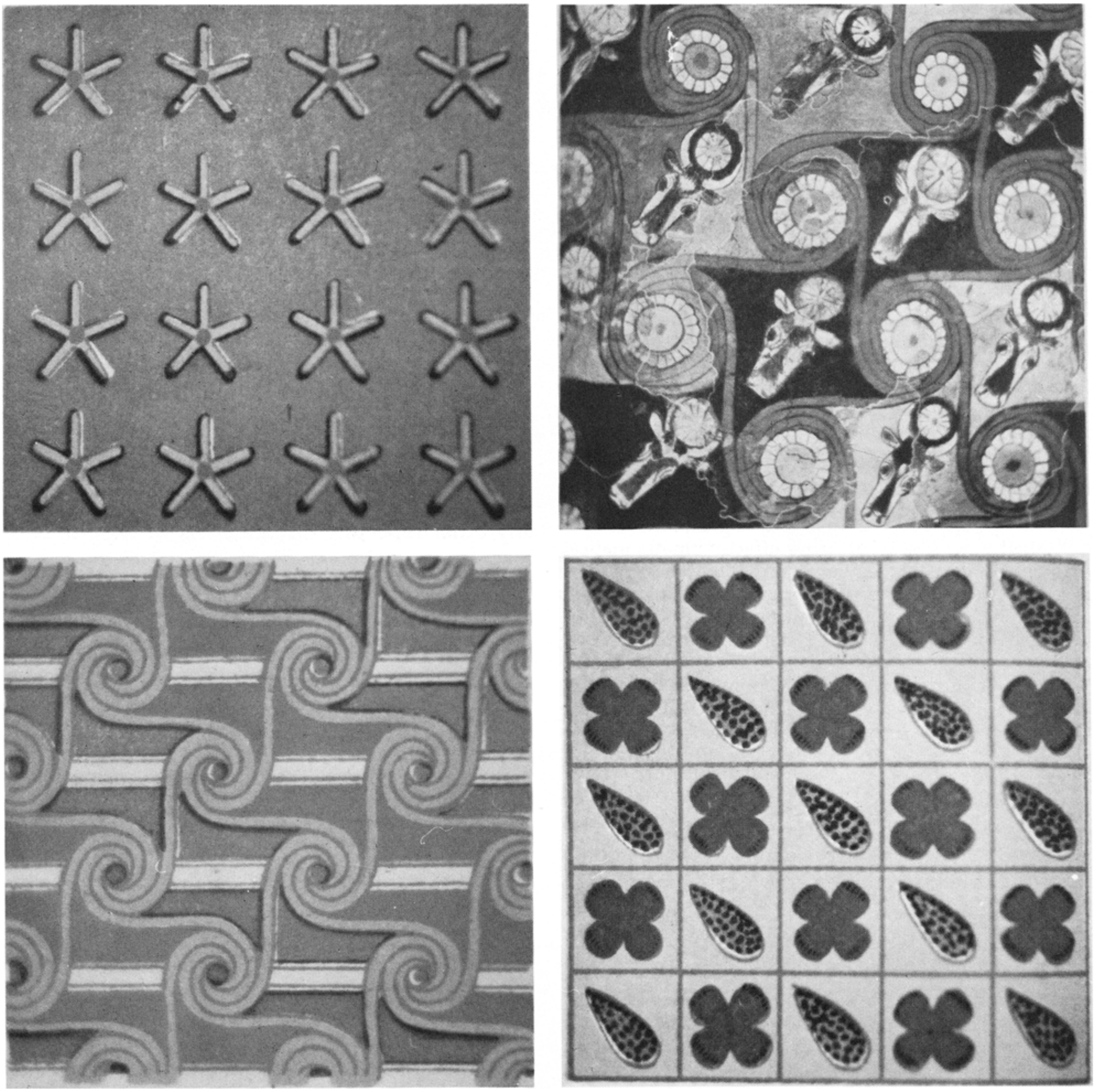

Figure 2. Egyptian ornaments from Owen Jones [7] (similar examples can be found in many other sources): a square lattice arrangement of five-pointed stars; stripes admitting only 180° rotations decorating fields that possess 4-fold rotational symmetry.

the workers by saying that the decorations on the wall of the pyramid corridor should be done so as to coincide with themselves when the wall is turned by 90°, so as to have its length going up. . . . More seriously, even in such simpler situations as those dealing with polygons or polyhedra, up to two centuries ago no artist or craftsman or *mathematician* defined regularity through symmetries. Equal parts—yes; equal position of parts with respect to their neighbors—yes; but equivalence with respect to the whole—never entered the picture.

In reality, the approach to orderliness by having each part be in the same relation to its neighbors as every other part is very well adapted to the practical design of ornaments: the artist is creating the parts one after the other, and that kind of orderliness does not even require an apology at an edge of the ornament—there simply is no neighbor! We have become accus-

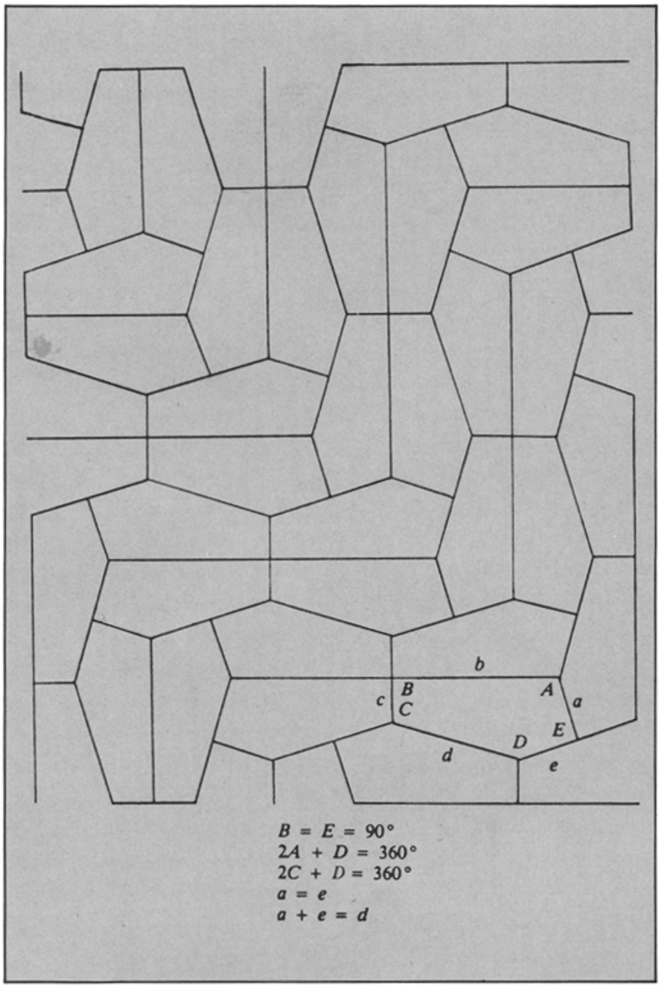

Figure 3. A tiling by congruent pentagons; no tiling with tiles congruent to these has a symmetry group which acts transitively on the tiles. This tiling was discovered by an amateur mathematician, see the account in [17].

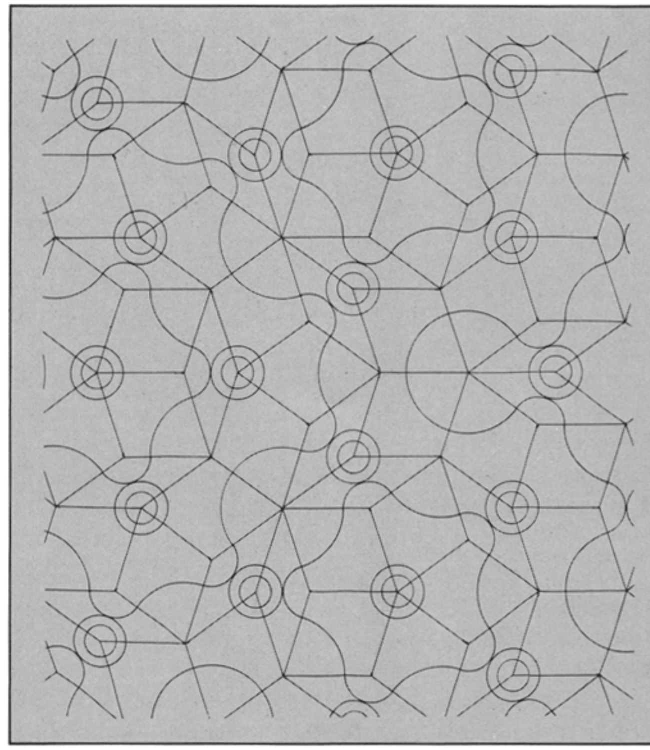

Figure 4. A patch of a tiling by one variant of "Penrose tiles"—rhombi of two shapes, each decorated by circular arcs. In a tiling by these tiles only such positions are allowed in which the circular arcs continue smoothly across the boundaries of the tiles. Each finite patch of a tiling by these tiles (like the one shown) can be extended to a tiling of the plane in uncountably many distinct ways, but none of the resulting tilings has any translational symmetries.

tomed (conditioned?) to think in terms of symmetries of the infinite pattern and we find them convenient in a wide variety of circumstances. There is nothing wrong with that, or more generally, in the utilization of any tools (groups, differentiability, categories, . . .) in the explanation of those phenomena which they are suited to explain. But if we start putting the cart before the horse by insisting that only those ornaments (or whatever else) are interesting or proper which have the "right" symmetries—then we have become addicted and need help.

Let me end by mentioning just a few mathematical topics which certainly fall under the heading of "orderly ornaments" but for which the symmetry groups are not the proper tools; indeed, it could easily be argued that the excessive weight given to group-theoretic considerations delayed the investigation of these attractive phenomena.

First problem—which closed, simply connected regions admit tilings of the plane by congruent copies of themselves? If you prefer, assume them to be polygonal regions, or even convex polygons—the problem is still open. Contrary to what was for a long time believed to be an established fact, it turned out that there exist convex pentagons with the property that the plane can be tiled by mutually congruent copies, but in no such tiling does the group of symmetries of the tiling act transitively on the tiles. Despite appreciable efforts over considerable time—the full characterization is not in sight even for convex pentagonal tiles. (See Figure 3 for an example of a tiling by such a pentagon, and [16], [17] for accounts of the known results.)

A second group of problems and results concerns the "aperiodic" tilings of Robinson [15], Wang [19], Penrose [12] and others. They proved that it is possible to devise tiles of suitable shapes and with appropriate conditions regarding adjacent tiles, which will admit tilings of the plane but no such tiling will possess any translational symmetry. Thus there is a very high degree of order—but no infinite symmetry group (see Figure 4). These kinds of tilings have just begun to be investigated, but already they have been found interesting in various contexts. It is clear that better understanding of such phenomena will have repercussions from logic to physics and beyond.

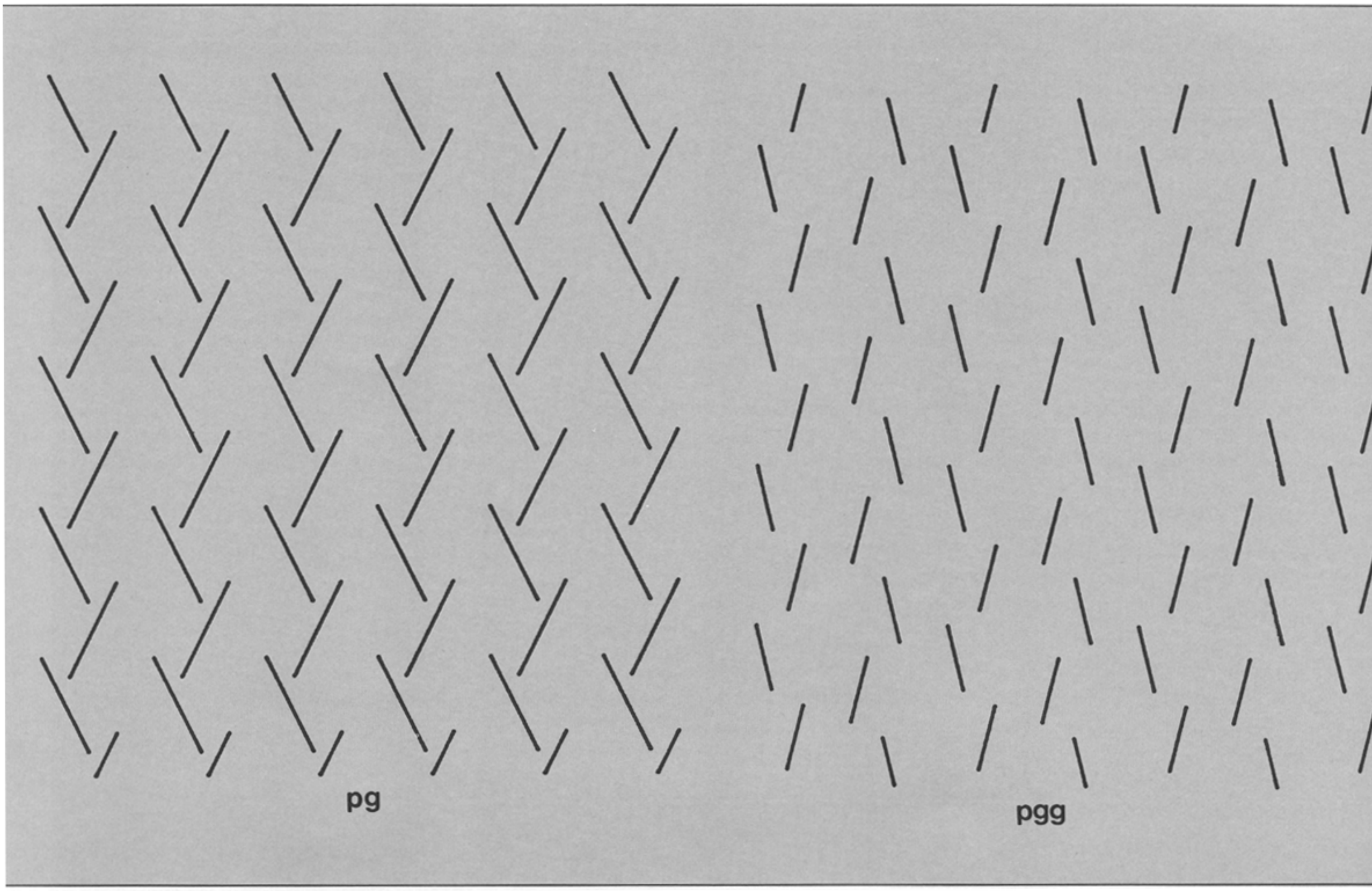

Figure 5. Patterns formed by congruent segments and having symmetry groups pg and pgg, the two groups that the present author has failed to find in any Islamic ornament.

Concerning the variety of Moorish ornaments one often finds exaggerations as blatant as those discussed in the text. In particular, it has been claimed in many publications that all 17 classes of symmetry groups are present among the ornaments in the Alhambra in Granada. A systematic study of these decorations by Edith Müller found only 11 groups (see [8]). The present author, during a visit to the Alhambra in 1982, found two additional symmetry groups represented there (pm and p31m); hence it seems that four groups (pg, p2, pgg and p3m1) are missing in the Alhambra. Possibly more interesting is the fact that despite intensive searching in all publications on Islamic art, I have found not a single instance of decoration with group pg or pgg (see Figure 5 for examples of patterns with these groups of symmetries). Could it be that they are really absent? It is quite remarkable that Wilhelm Ostwald, in his largely forgotten attempt to develop a theory of ornaments in [9] and [10], also missed these classes of symmetry groups.

A third direction is in a less developed state, but deserves at least a clear formulation: what is the "orderliness" that can be expressed through "adjacency relations", and what phenomena happen under this description? In many respects all that can be handled by symmetry groups can be handled by adjacency relations at least as well, but the converse does not hold (see [4], [5], [6]).

To conclude: The Emperor's scepter and crown are safe and unblemished, as is his standing as one of the leading mathematicians of the century. But the imposition of his and the Prime Minister's brand of the "Whig interpretation of history" (see [3], in particular pages 11, 12) on the mathematical explanation of ornaments was misguided; the sooner it is laid to rest—the better. I hope that the outcome here will be different than the ending of the tale [1]:

> . . . And he drew himself up still more proudly, while his chamberlains walked after him carrying the train that wasn't there.

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