GEOMETRY: The Unity of Theory and Practice

Jean Pedersen, Photographs by Chris Pedersen

The ways in which people have used symmetry, for both practical and aesthetic purposes, are far too various and numerous for us to discuss here comprehensively. Therefore we will focus attention on some work in just two (related) areas in which craftsmen have used the mathematics of symmetry, at least implicitly. First, we discuss the covering of a plane surface with ornaments, or, in more familiar language, wallpaper patterns and patterns suitable for tiling. Second, we look at the craft of weaving together strands of materials such as cotton, linen, wool, and straw to make cloth, baskets, or balls.

Flowers, Pentagrams, and Leonardo da Vinci (the Cyclic and Dihedral Groups)

How did people first become aware of symmetry? It's not too far-fetched to imagine a cave dweller distinguishing between a flower with three petals and a flower with five petals—or the same person observing that a perfect circle appeared in the sky at regular intervals. These are observations about symmetry. Even the most casual observer, faced with the abundance of symmetry in nature, would have to make a deliberate effort to avoid it—and would even then certainly fail. We ourselves are bilaterally symmetric, and the consequences of that symmetry affect almost everything we do: our method of locomotion, our vision, how we eat, and how we communicate. (See [2], [8], [15, 16], [18], or [20].)

It is no wonder that when craftsmen and artists began to make objects their constructions possessed symmetries similar to those they had already observed in Nature; and, in fact, some sociologists believe that these symmetrical constructions can be explained as a conscious effort to imitate Nature. There certainly must have been such an effort, but as a complete explanation it is far too simple. Many craftsmen have actually used mathematics in a very real if intuitive sense, usually without the appropriate mathematical tools. This has resulted in a proliferation of symbols and nomenclatures, and one frequently detects a feeling on the part of the craftsmen that, although they have a good deal of conviction about certain things being possible, they are not sure that certain other things cannot be done. What they are lacking is the *real certainty* that mathematics can provide.

Let's return to the flowers. Figure 1a shows a blossom from a common California buttercup (*Ranunculus californicus*) which has the symmetry of the dihedral group D_5, and Figure 1b (reproduced from [20], p. 66) shows the *Vinca herbacea* blossom which has the more restricted symmetry of the cyclic group C_5 (because the individual petals lack bilateral symmetry). (Perhaps the intense appreciation many feel for the beauty of orchids has something to do with the fact that they have only one plane of symmetry.) The bare bones of these symmetries are represented in Figure 2, which typifies three-petaled flowers; flags have been placed on the arms of the tripos at the left to produce the triquetrum on the right. The triquetrum is a mystical symbol placed by the Greeks on the center of Medusa's head to represent the three-cornered island of Sicily.

These and other relatively simple plane figures ap-

Jean Pedersen

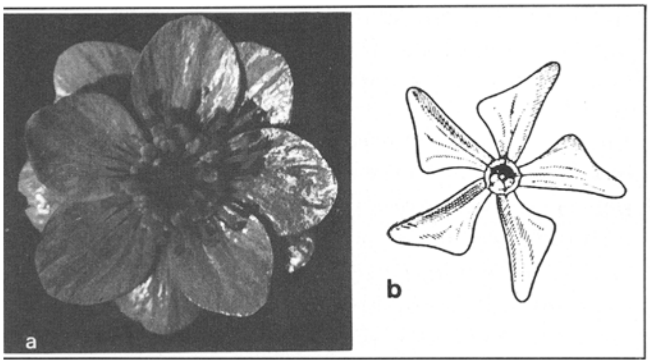


Figure 1

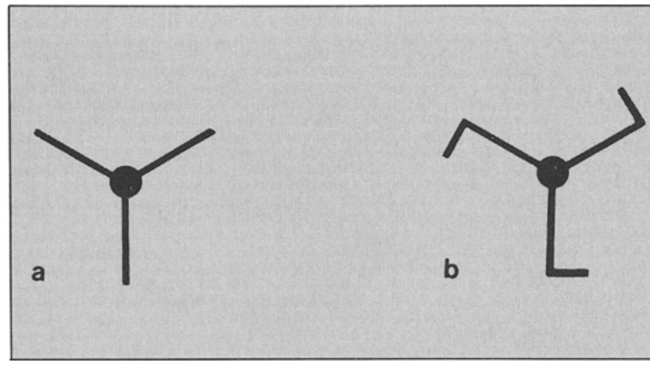


Figure 2

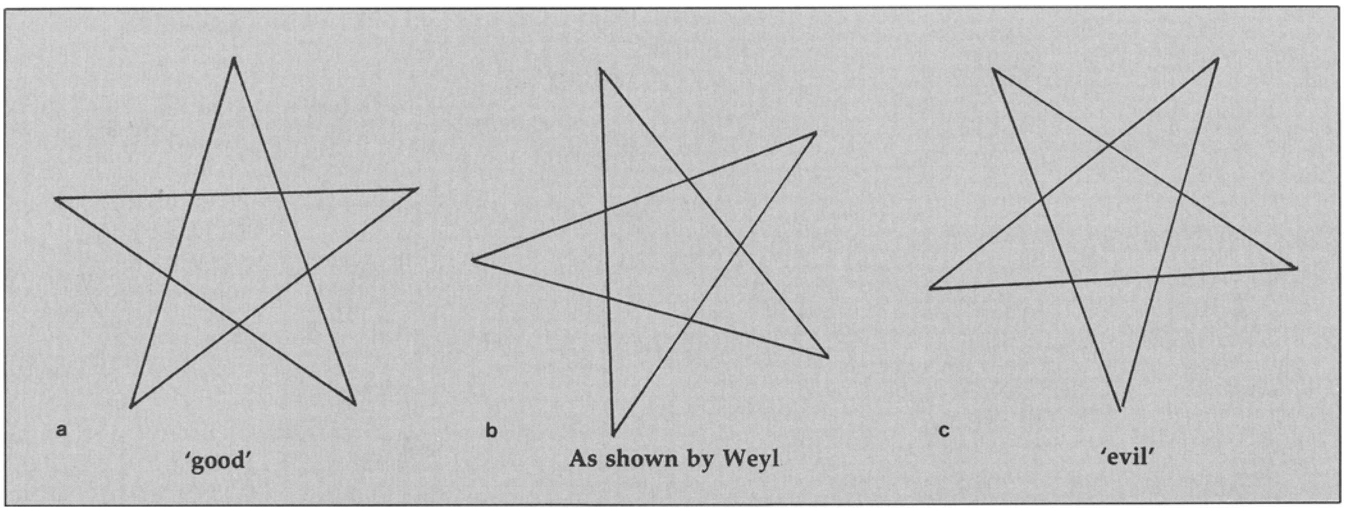


Figure 3

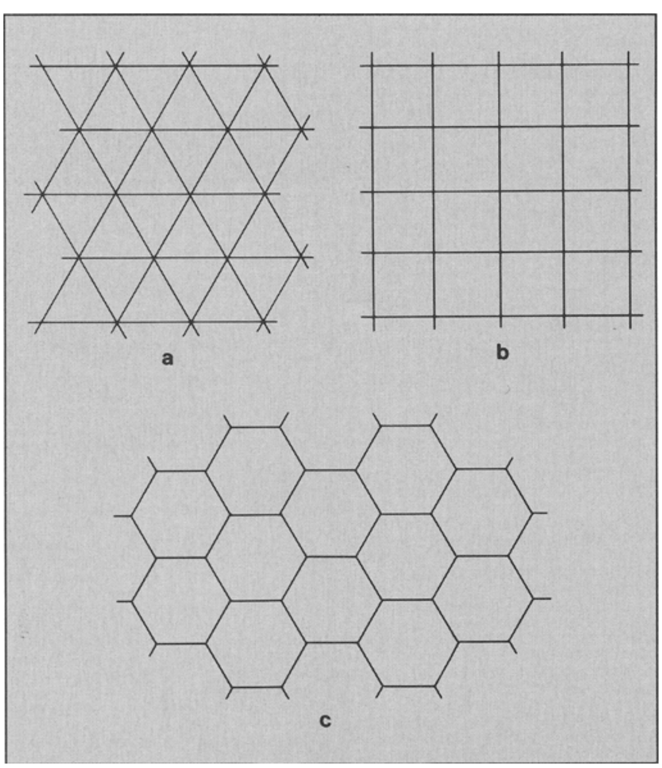


Figure 4

pear frequently in folklore. Hermann Weyl pointed out in [20] that Figure 2b may be modified to have four arms instead of three, thereby yielding the infamous swastika (Weyl made a point *not* to display it in his book). In his account of a lecture he gave on symmetry in Vienna in the fall of 1937 he stated, "I added concerning the swastika; 'In our days it has become the symbol of a terror far more terrible than the snake-girdled Medusa's head'—and a pandemonium of applause and booing broke loose in the audience."

Less current, but still fascinating, is the pentagram of Figure 3 (possessing D_5 symmetry). According to Rouse Ball [1] it was used by the Pythagoreans as a sign by which they could recognize each other, and Weyl says it is "the famous pentagram by which Dr. Faust banned Mephistopheles the devil." There is even a legend saying that, if the pentagram is oriented so that one vertex points directly down it represents evil, but if one vertex points directly up it represents good. Leonardo da Vinci's perfect man can be neatly inscribed inside the "good" pentagram. I find it intriguing that Weyl shows it with one side vertical with respect to the bottom of the page, suspended exactly midway between good and evil!

Man imitates Nature. Although appearing at first sight to be unrelated to the study of flowers, Leonardo da Vinci's work on the possible symmetries of a central building surrounded by chapels and niches consisted, in fact, of an attempt to list all the cyclic and dihedral groups. For da Vinci stipulated in his work [21] that the attachment of the chapels and niches should not diminish the symmetry of the central core.

Wallpaper, Pottery, and Baskets (Classification by Symmetry)

As is well known (see [3, 4], [10], or [19]), it is possible to tile the plane with regular triangles, squares, or hexagons (Figure 4). Long ago creative tilemakers sought to make symmetrical designs of greater interest. They looked for, and found, all the distinct pattern types that were possible based on their criteria. And, as Weyl said,

One can hardly overestimate the depth of geometric imagination and inventiveness reflected in these patterns. Their construction is far from being mathematically trivial. The art of ornament contains in implicit form the oldest piece of higher mathematics known to us. To be sure, the conceptual means for a complete abstract formulation of the underlying problem, namely the mathematical notion of a group of transformations, was not provided before the nineteenth century; and only on this basis is one able to prove that the 17 symmetries already implicitly known to the Egyptian craftsmen exhaust all possibilities. Strangely enough the proof was carried out only as late as 1924 by George Pólya, now teaching at Stanford.

Weyl's book was published in 1952. I am very happy to report that George Pólya and his wife, Stella, still live near Stanford, close enough for the author to visit them fairly often.

Figure 5, taken from [14], shows Pólya's examples of the 17 different types of symmetry with double infinite rapport (that is, doubly periodic). The international symbol for the ornamental class had been added in parentheses below each pattern. If you don't already know what the notations mean you may enjoy trying to decipher them! (If you have trouble see [5].)

A fascinating application of mathematical classification by symmetry type is currently being carried out by Dorothy K. Washburn (an anthropologist with the California Academy of Sciences) and Donald W. Crowe (a mathematician at the University of Wisconsin). They believe it may be useful to classify shards and baskets by symmetry type. In order to make their approach acceptable and useful to anthropologists they have devised a scheme [5] which involves posing and answering a sequence of "yes-no" questions regarding symmetry, so that the completed questionnaire (in flowchart form) enables one to identify the symmetry type for singly periodic and doubly periodic designs. Since these symmetry types occur in many crafts, such as pottery and basket weaving, found in various cultures, they believe it may be useful to look at the underlying "bare bones" symmetry, rather than to concentrate on the less definitive trims the artist attaches to those symmetries. It appears that such considerations of symmetry are largely neglected in current research in this area.

On the basis of the results of preliminary investigations (see [5] and [17]) their assumptions seem to be justified, and they hope that their scheme may help anthropologists to solve more of the mysteries concerning the migrations and trading activities of ancient civilizations. It is surprising to a mathematician that no one had previously thought of examining these more basic features of the designs. (It is not so surprising that Washburn and Crowe are having some difficulty in convincing anthropologists of the merit of their more mathematical method.)

The geometry connected with woven materials like those pictured in Figures 6 and 7 is readily apparent, yet new mathematics relating to the simplest and most regular weaves in the plane appeared only three years ago [6], and Grünbaum and Shephard's definitive paper on weavings in the plane, "Isonemal Fabrics" [7] is still awaiting publication. The main definitions and the principal theorem from that paper are given here so

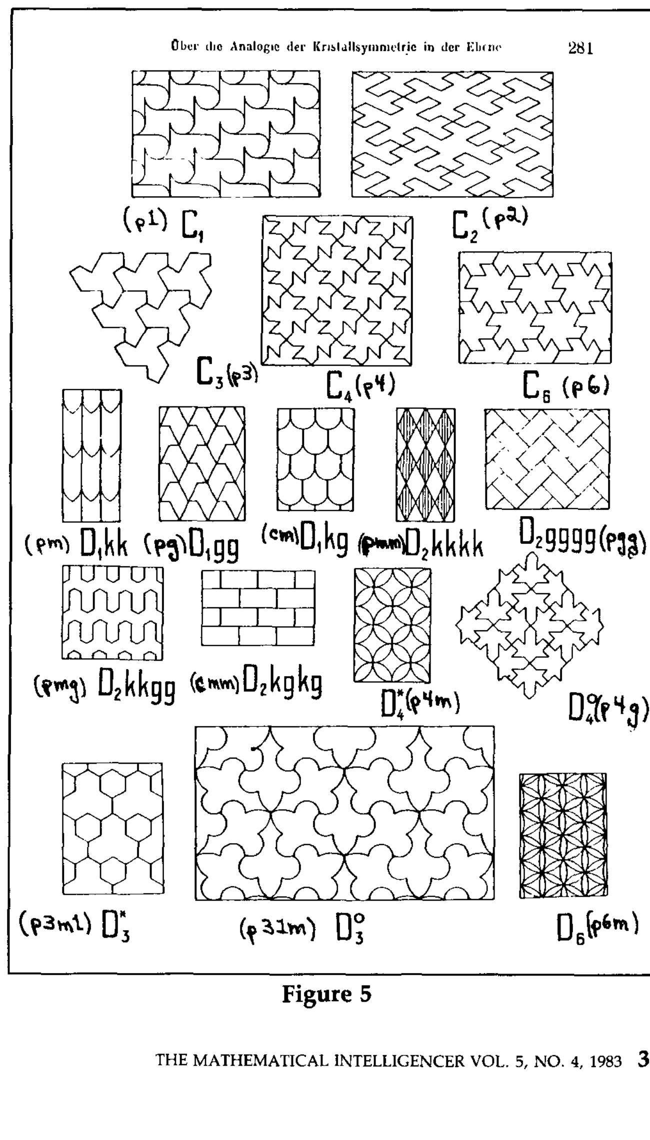


Figure 5

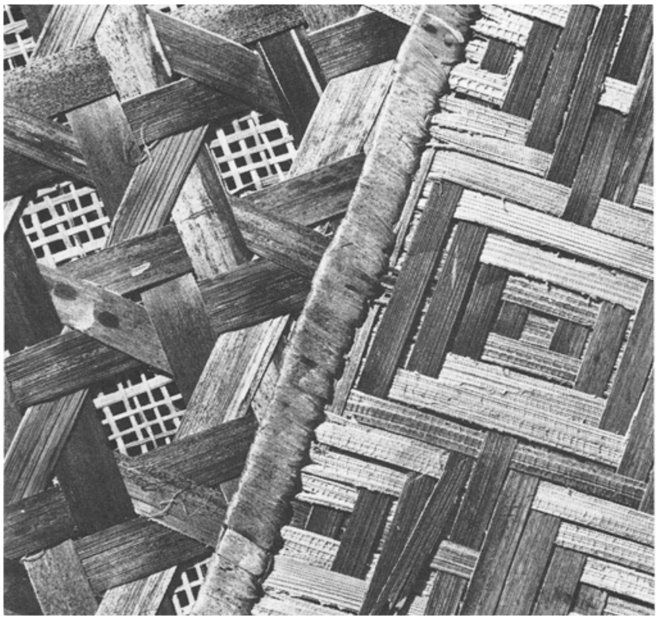


Figure 6. Left: a four-way (large) weave over a two-way isonemal (small) weave. Right: a two-way weave.

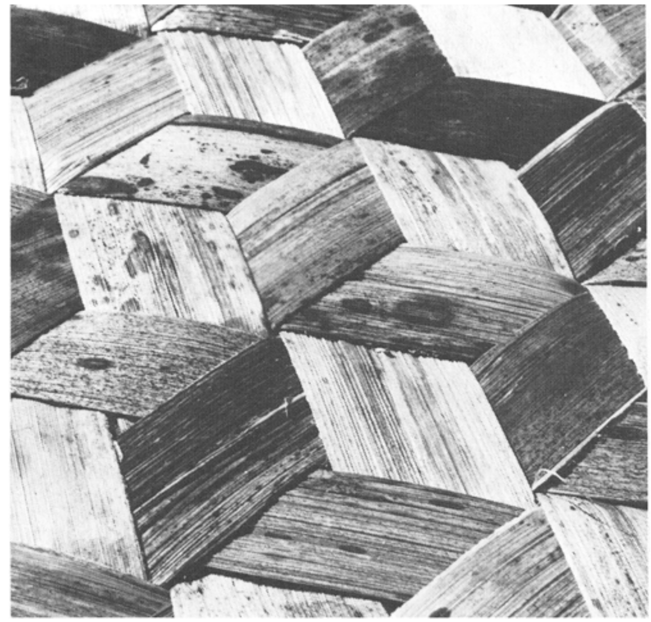


Figure 7. Example of a three-way, threefold isonemal weave commonly found in baskets.

that they can be used (with appropriate adjustment) to obtain related results in three-dimensional space.

DEFINITIONS A *strand* (see Figure 8a) is a doubly infinite open strip of constant width (think of a strip of paper having zero thickness). In these illustrations it is shaded in accordance with its direction.

A *layer* is a collection of disjoint (parallel) strands such that each point of the plane either belongs to one of the strands or is on the boundary of two adjacent strands (see Figure 8b).

A *fabric* is, roughly speaking, two or more layers of connected strands in the same plane E such that the strands of different layers are nonparallel and they "weave" over and under each other in a specified way (the *ranking*) so that the fabric "hangs together." To be

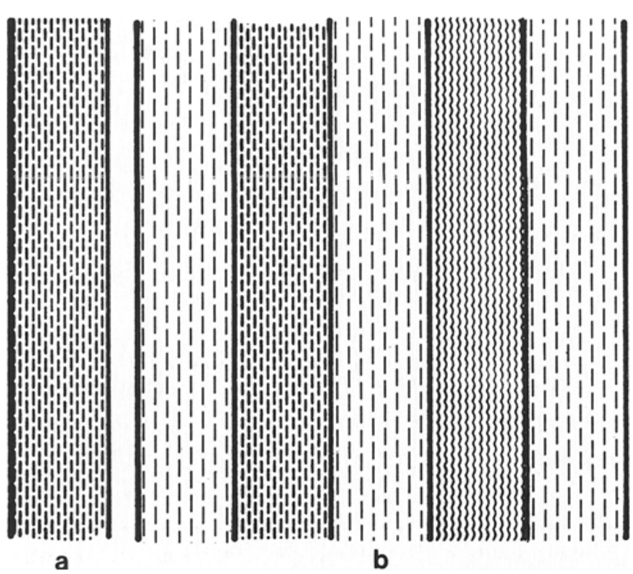


Figure 8.

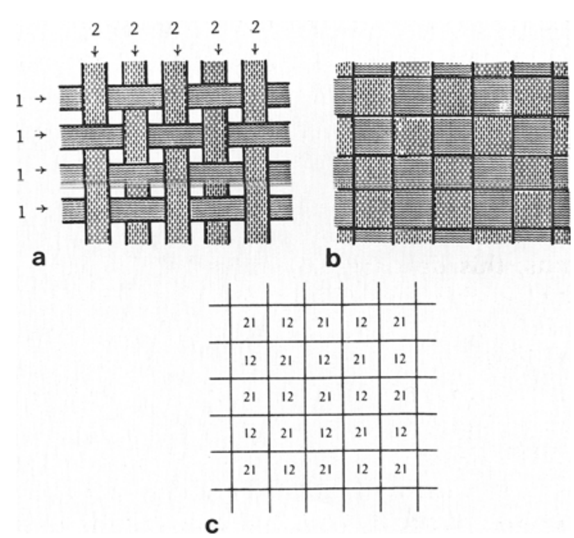


Figure 9. Horizontal strands are labeled "1" and vertical strands are labeled "2". The ranking "21" means that the vertical strand goes over the horizontal strand.

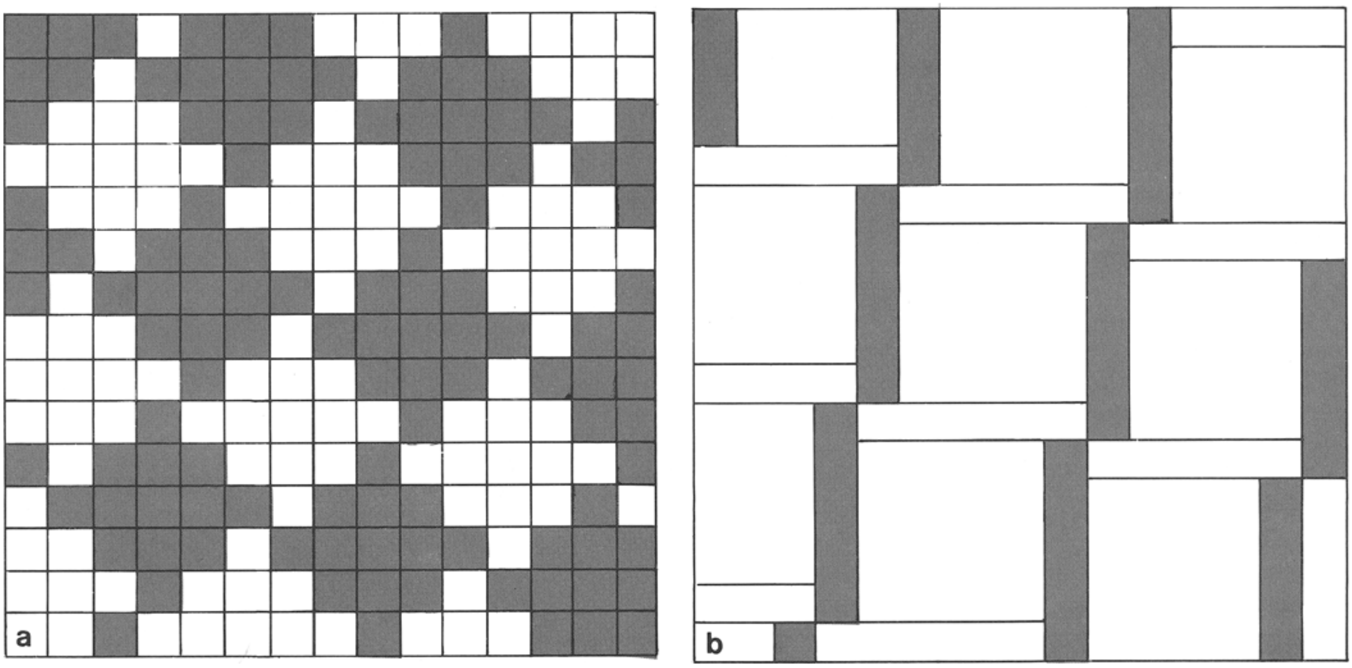


Figure 10

more precise, *weaving* means that, at any point Q of E which does not lie on the boundary of a strand, the strands containing Q have a stated ranking, which is the same for each point Q contained in those strands.[1] The ranking indicates that one strand is *higher than* or *passes over* the other. Saying that the fabric hangs together means that it is impossible to partition the set of all strands into two nonempty subsets so that each strand of the first subset passes over every strand of the second subset.

If a fabric consists of n layers, it is called an *n-fold* fabric. Figures 9a and b represent the same two-fold fabric [in (a) the strands have been separated for clarity—Figure 9a may be regarded as representing the real fabric corresponding to the idealized fabric of (b)]. Figure 9 illustrates the most common and familiar of all fabrics, known variously as the over-and-under, plain, calico, or tabby weave. This weave forms the small mesh on the left of the picture in Figure 6.

A systematic way of representing fabrics graphically is to draw all the straight lines that bound the various strands; this determines a tiling of the plane. The strands are then labeled according to the layer to which they belong, and their ranking in each tile is indicated by the order in which the labels are written (the top layer first, the second layer next, etc.). This is illustrated in Figure 9c for a plain weave.

A fabric F is said to be *k-way* provided the layers that form F are parallel to k directions. Figure 9 illustrates a two-way, two-fold fabric. If the strands of this figure were all "doubled up," the resulting fabric would be two-way and four-fold, and if only the horizontal strands were doubled up, it would be a two-way, three-fold fabric (which one senses, intuitively, is not as symmetric as the other two-way fabrics mentioned). A three-way, three-fold fabric is pictured in Figure 7.

A symmetry of fabric F is any isometry of the plane of F onto itself which maps each strand of F onto a strand of F. We include, as a matter of convention, the symmetries which reverse all rankings ("turning the fabric over"). The group of symmetries of F is denoted by $S(F)$. The subgroup consisting of those symmetries of F that preserve the rankings of the strands is denoted by $S_0(F)$—those are the symmetries that preserve the sides of F (so that rotations and translations of the fabric in its plane are permitted; turning the fabric over is not).

Finally, a fabric F is called *isonemal*[2] if its group of symmetries $S(F)$ acts transitively on its strands—that means that for any two strands there is some element of $S(F)$ taking one strand onto the other. The two-way, three-fold variation of Figure 9 and the large weave on the right in Figure 6 are *not* isonemal.

Theorem [7] *If* F *is a* k-*way*, n-*fold isonemal fabric, then the pair* (k, n) *is one of the following six:* (2, 2), (2, 4), (3, 3), (3, 6), (4, 4), *or* (6, 6). *Conversely for each of these six pairs* (k, n) *there exist infinitely many distinct* k-*way*, n-*fold periodic isonemal fabrics.*

[1] Thus, because of the "holes" and different number of layers, the large weave on the left in Figure 6 is *not* considered to be a fabric.

[2] Isonemal is a term derived from the Greek word ισοσ (equal) and γημα (thread or yarn). This term was introduced by Grünbaum and Shephard [7].

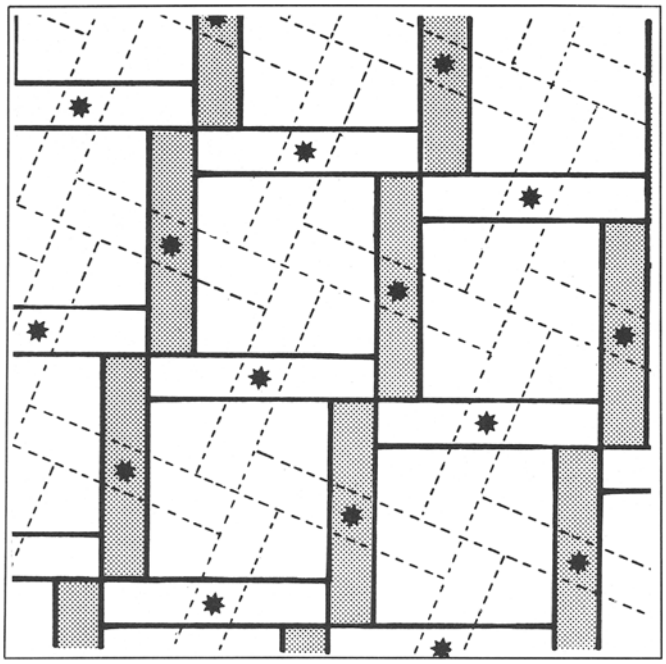


Figure 11

Examples of the first three have already been illustrated. To obtain an example of a three-way, six-fold isonemal fabric you simply double up all the strands in the fabric pictured in Figure 7. The four-way, four-fold isonemal fabric is obtained by weaving together two copies of a two-way, two-fold isonemal fabric in such a way that each strand of that fabric "floats" at regular intervals over several other strands. Figure 10 shows what is known as a "sponge weave," and it is one of the many fabrics that can be woven this way.

If you think of the black squares as being the visible portions of the vertical strands and the white squares as being the visible portions of the horizontal strands, then you can verify from the illustration that each strand on this fabric repeats the following sequence, with regard to the number of strands it goes over (O) and under (U) in succession:

$$\underbrace{O5,\ U1,\ O3,\ U3,\ O1,\ U5,\ O1,\ U3,\ O3,\ U1,}_{\text{one complete period}}\ O5,\ \ldots$$

The "O5" portions of the strands in this fabric constitute an array of floating parts. If you take two copies of this fabric, they can be oriented so that those floating portions can be interwoven, producing a four-way, four-fold isonemal fabric. Figure 11 shows only the floating strands of the two copies which are interwoven at the places marked by the stars.

The six-way, six-fold fabric is realizable by an analogous technique, that is, by interweaving the floating strands from two copies of a suitable three-way, three-fold fabric (see [7]).

Some of the infinitely many distinct kinds of two-way, two-fold isonemal fabrics are discussed and illustrated in [6].

So far as we have been able to determine, no craftsman, working empirically, has discovered either a four-way, four-fold or six-way, six-fold isonemal fabric.

Grünbaum and Shephard's original version of "Isonemal Fabrics" [7] contained a question at the end asking if it is possible to find analogous weavings that completely cover the surface of a polyhedron. In other words: What is the nature of fabrics woven on topological spheres where you use, for strands, cylinderlike

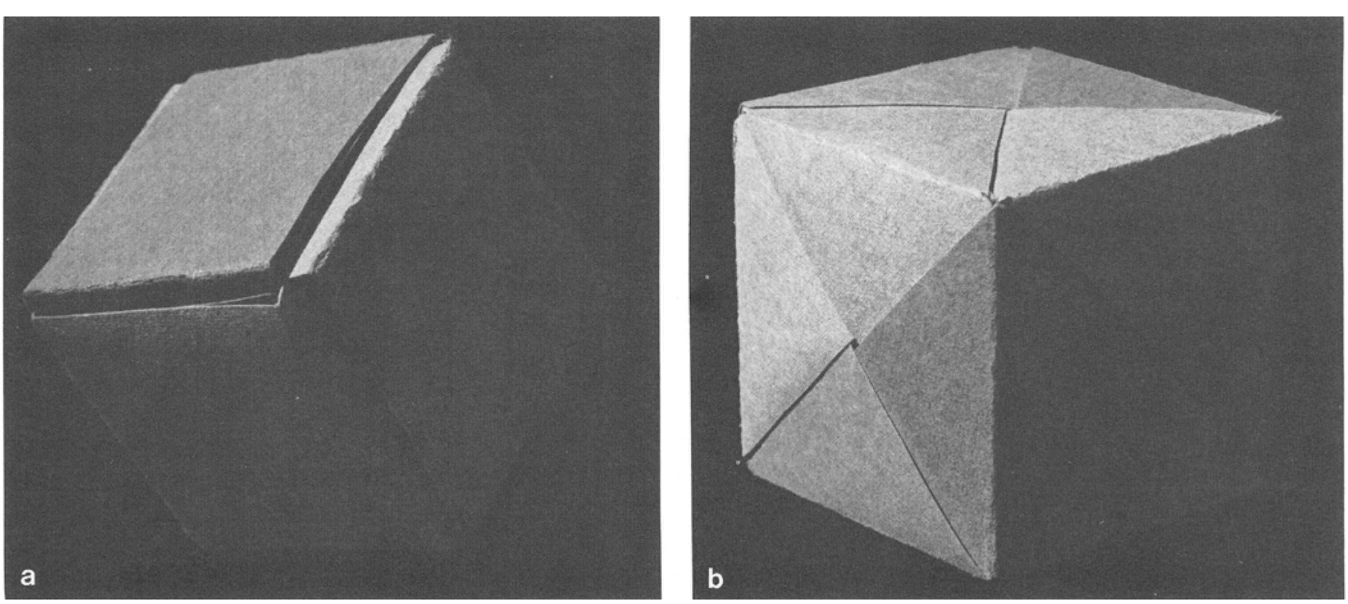


Figure 12. (a) A cube woven from three strands. Each crossing area is a square. An example of a three-way, two-fold isonemal covering of a cube. **(b)** A "diagonal cube" woven from four strands. Each crossing area is a square. An example of a four-way, two-fold covering of the cube.

rings instead of straight strips (as were used in the plane) and then require that the closures of the strands completely cover the sphere in some uniform and symmetric way? The resulting fabrics would be referred to as "woven polyhedra." The author was fortunate to see the original version of [7] and was thus able to communicate to Grünbaum and Shephard her construction of such polyhedra.

Isonemal Weaves in Space

A year or so before reading Grünbaum and Shephard's paper I had been concerned with constructing models that would enable one simply to *count* the number of unbounded regions produced in space by extending the face planes of each of the platonic solids. Suitably interpreted, the models of Figures 12a, 12b, 13a, and 13d, constructed, respectively, with 3, 4, 6 and 10 strips, answer this question for the cube, octahedron, dodecahedron, and icosahedron (for more detail see [11] and [13]).[3]

[3] The Sepak Tackraw ball shown in Figure 13b was given to me in 1980 by Dr. Mee-Chooi Cheng, chairman of the mathematics department at the University of Malaya in Kuala Lumpur, after I had given a talk there in which the model in Figure 13a played a prominent role. Martin Gardner has told me in a letter that it is of ancient origin, being mentioned in very old books concerned with leather braiding.

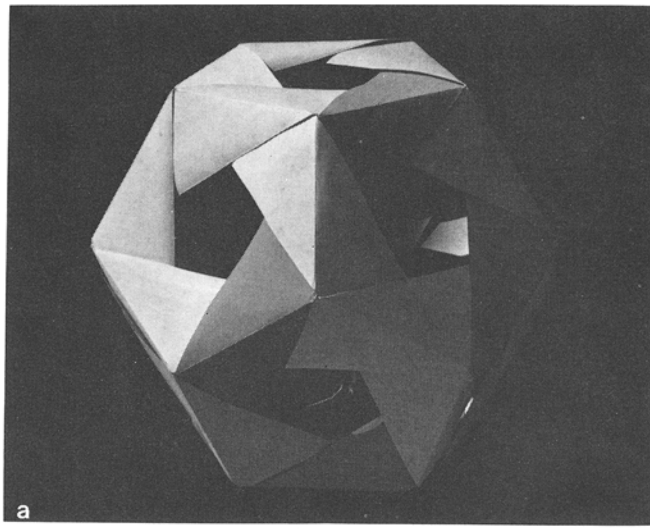
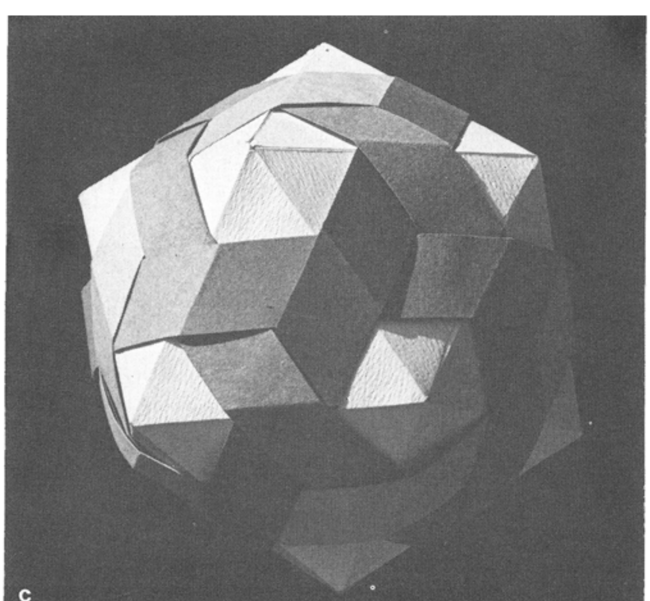
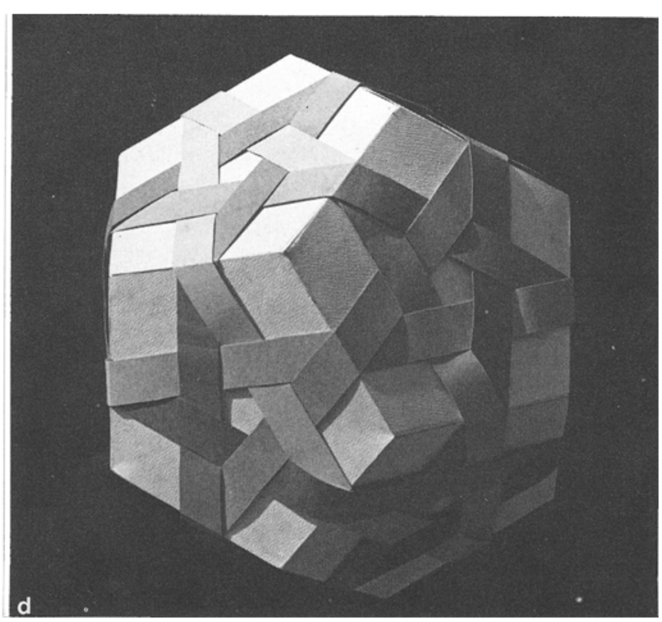


Figure 13 (a) A "golden dodecahedron" woven from six strands. Each crossing area is a rhomb. **(b)** A Sepak Tackraw ball from Malaysia, used to play a type of football. **(c)** Six strands woven about the ghost of an icosahedron. **(d)** Ten strands woven about the ghost of a dodecahedron.





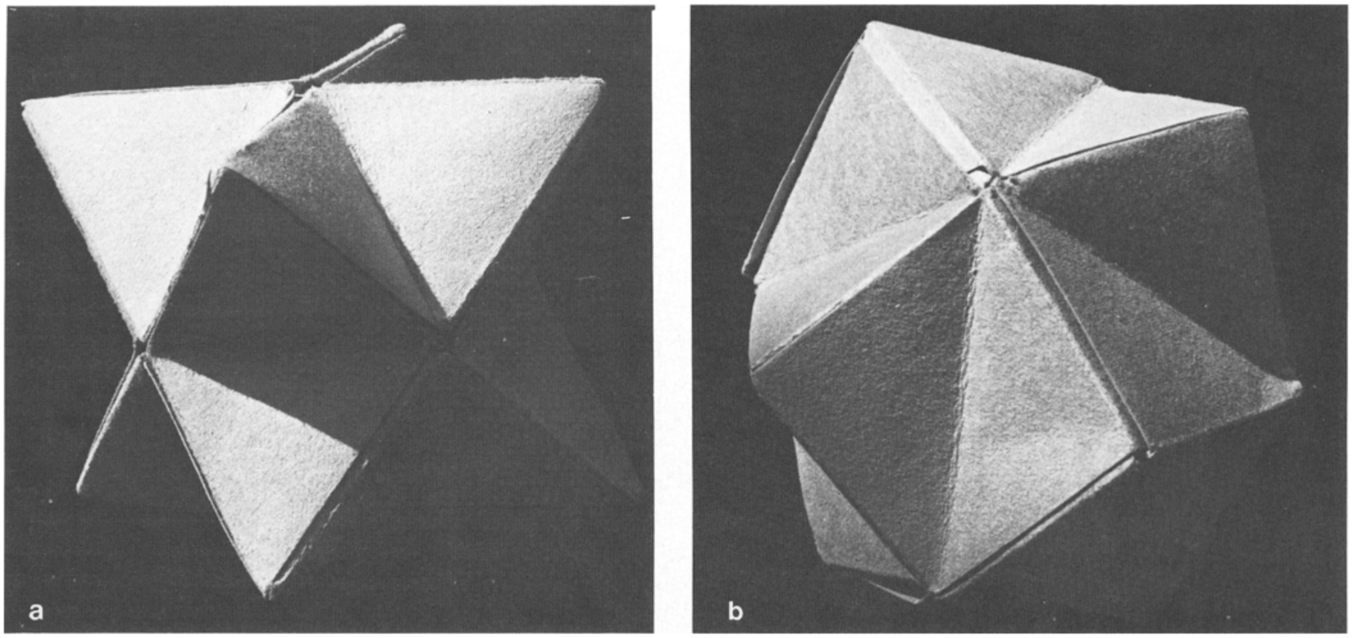


Figure 14 (a) Four-way, three-fold isonemal covering of a stella octangula. Each crossing is a 60°–120°–60°–120° rhomb. **(b)** Four-way, two-fold isonemal covering of a "subdued" stella octangula. Each crossing is a square.

It is not surprising that knowledge of these constructions suggested that there might be a result in space corresponding to Grünbaum and Shephard's theorem about isonemal weavings in the plane. Of course there were difficulties. For example: (1) How could one get rid of the holes (as in Figures 13a and d); (2) what might be a reasonable three-dimensional analogy to their theorem, and (3) how would one go about constructing the possibilities once it was known what those possibilities should be?

The first question seemed relatively simple (in principle). It looks as if all one has to do is to tighten up each strand until the holes disappear. My intuition said that it could be done—but it was a little difficult to imagine what the resulting polyhedron would look like. I felt that the answer to this question for the model of Figure 13a would indicate the general procedure for eliminating holes.

The clue to solving the problem is to look at the stella octangula (pictured in Figure 14a). This model can be woven with four strips of equilateral triangles. Each portion of its surface is covered by exactly three strips (like the portion of the basket shown in Figure 7). More important, from our point of view, is the fact that the surface of this model can be thought of as the surface that results when you glue a regular tetrahedron to each face of a regular octahedron. A little experimenting shows that, if one were to glue a "shorter" pyramid—that is, one whose slant faces are all right-angled isosceles triangles—to each face of the octahedron, the model pictured in Figure 14b would result. It is easy to see, with the model in hand, that its surface can be completely covered with four strips and, as in Figure 14a, the strips can be woven over and under each other in such a way as to preserve octahedral symmetry.

As will be shown, the idea of gluing either regular tetrahedra or "cube corners" onto the faces of polyhedra having triangular faces makes it possible to obtain *all* the different types of isonemal coverings of polyhedra that are analogous (in one certain way) to the isonemal coverings of the plane. Before looking at these details we need to discuss precisely what the analogy is and determine exactly what all the different types should be.

We begin by making the appropriate changes in the previously mentioned definitions, so that the weaving becomes a linkage of strands on the surface of a polyhedron. A *strand* (or ring) is isometric to the curved surface of a (short) cylinder. It may be scored to produce flat polygonal faces, as in Figure 13a, c, and d. We say that a *k*-way, *n*-fold polyhedral fabric is a set of *k* strands on a topological sphere (or polyhedron) such that every point on the surface not on the boundary of a strand belongs to exactly *n* strands. And we further require that the fabric hang together in the same way as for the fabrics in the plane. Thus for any point *Q* on the surface of the polyhedron and not on the boundary of a strand, the strands containing *Q* have a stated ranking—this can be denoted on the net (or netlike) diagrams exactly as in the plane. A symmetry of the polyhedral weaving is any isometry of the polyhedron onto itself which maps each strand of the polyhedron onto a strand of the polyhedron. If *P* de-

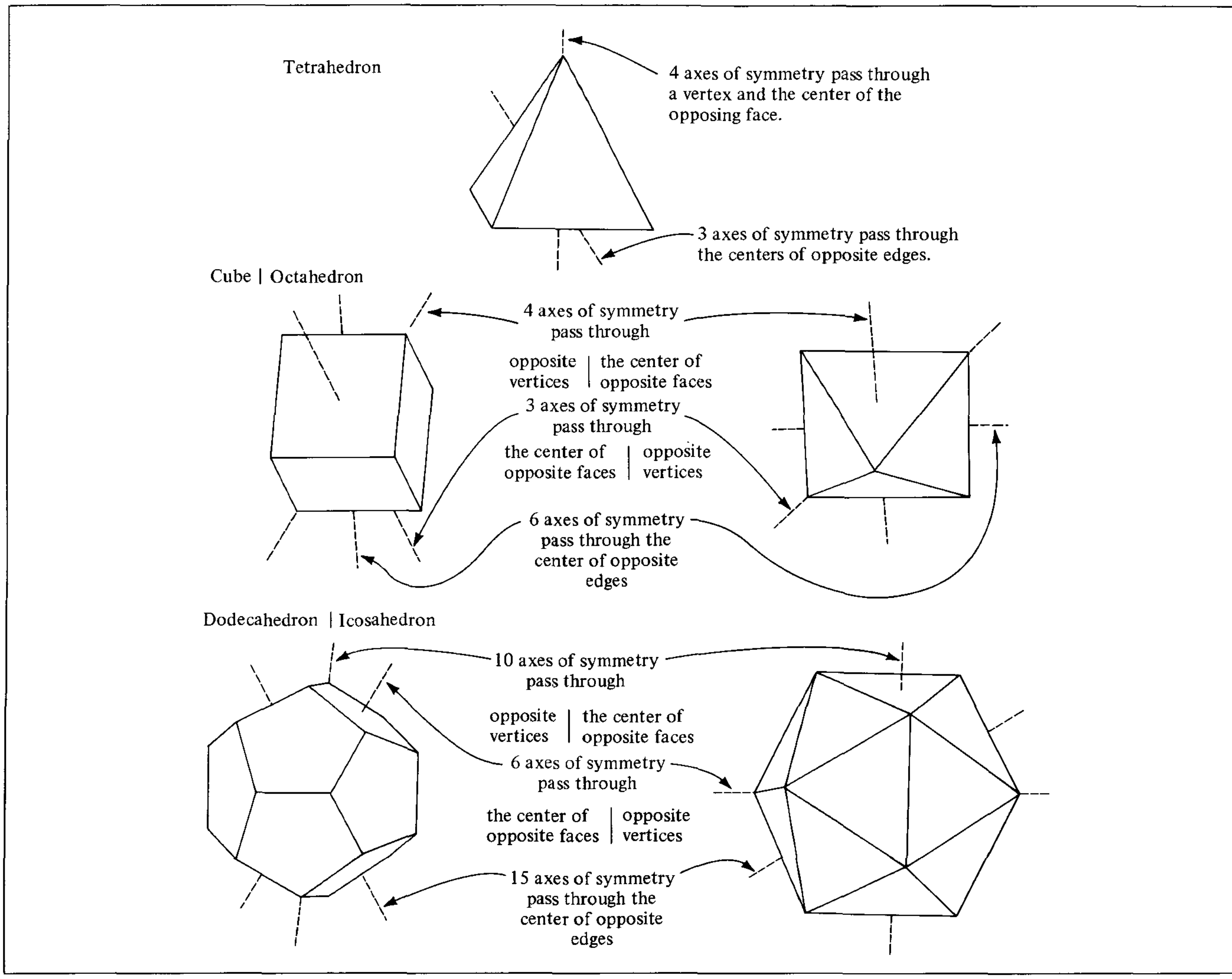


Figure 15

notes the polyhedral weaving, then all symmetries of P form a group under composition; call it $S(P)$, the *group of symmetries* of P. However, more significant for us is the subgroup $S_0(P)$ that consists of those symmetries of P that preserve the rankings of the strands, since such symmetries may actually be achieved. (In the case of a plane fabric you can actually turn the fabric over—you cannot actually turn a polyhedron inside out.) This time we call a polyhedral weaving *isonemal* if the group $S_0(P)$ [rather than $S(P)$] acts transitively on its strands.

Now we wish to investigate the permissible values of (k, n) in an isonemal polyhedral weaving. It is natural to look first at the most symmetrical arrangements possible (many other arrangements exist, as will be pointed out later), so we observe that the fabrics discussed in the theorem in the plane are related to symmetries of polygons that form tesselations in the plane. For example, the two-way, two-fold isonemal fabric, with strands crossing at right angles, has all its strands perpendicular to one of two axes of symmetry that join opposite sides of some square in the plane of the fabric. Alternatively, we could view these strands as being perpendicular to one of the two axes of symmetry that join opposite vertices of some square in the plane of the fabric. Of course, because of the way the two-way, four-fold fabrics are constructed, their strands will relate to the axes of symmetry for some square in the same way. The four-way, four-fold isonemal fabric in Figure 11 has strands that can be partitioned into exactly four sets, so that the strands of each set are perpendicular to either one of the axes of symmetry joining opposite sides, or opposite vertices, of some *parallelogram*. Corresponding statements hold for the other isonemal fabrics.

This suggests that an analogous situation might exist for symmetric isonemal weavings on surfaces of polyhedra. Since Platonic solids may be viewed as three-dimensional analogs of the regular tesselations in the plane, it seems plausible that we could find isonemal weavings for polyhedra such that the rings "go around" each of the axes in the various sets of axes for Platonic solids.

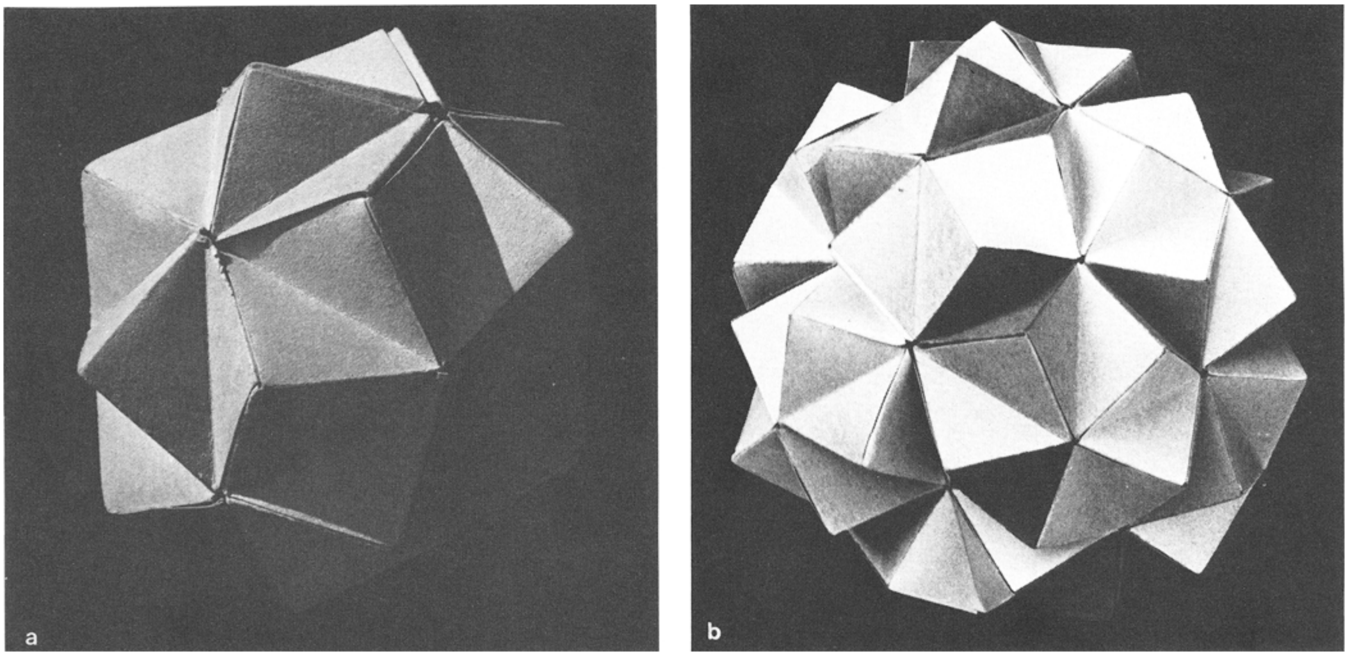


Figure 16 **(a)** Six-way, two-fold isonemal weave. **(b)** Ten-way, two-fold isonemal weave.

An enumeration of all the various sets of axes of symmetry related to Platonic solids is illustrated in Figure 15. Consequently we know that we must look for polyhedra that can be woven with 3, 4, 6, 10, or 15 rings.

The details for the construction of each of these possibilities are carried out in [11], but we show here pictures of each case for the two-fold models. Examples of three-way, two-fold and four-way, two-fold models appear in Figure 12. In both cases the symmetry group can be reduced to A_4 (instead of S_4) by simply drawing the edges of one of the inscribed tetrahedra on the surface of the model. The six-way, two-fold model is shown in Figure 16a. The "parent" polyhedron is the regular icosahedron; that is, you may think of the surface of this model as being constructed by adding a triangular pyramid (whose slant faces are right-angled isosceles triangles) to each face of a regular icosa-

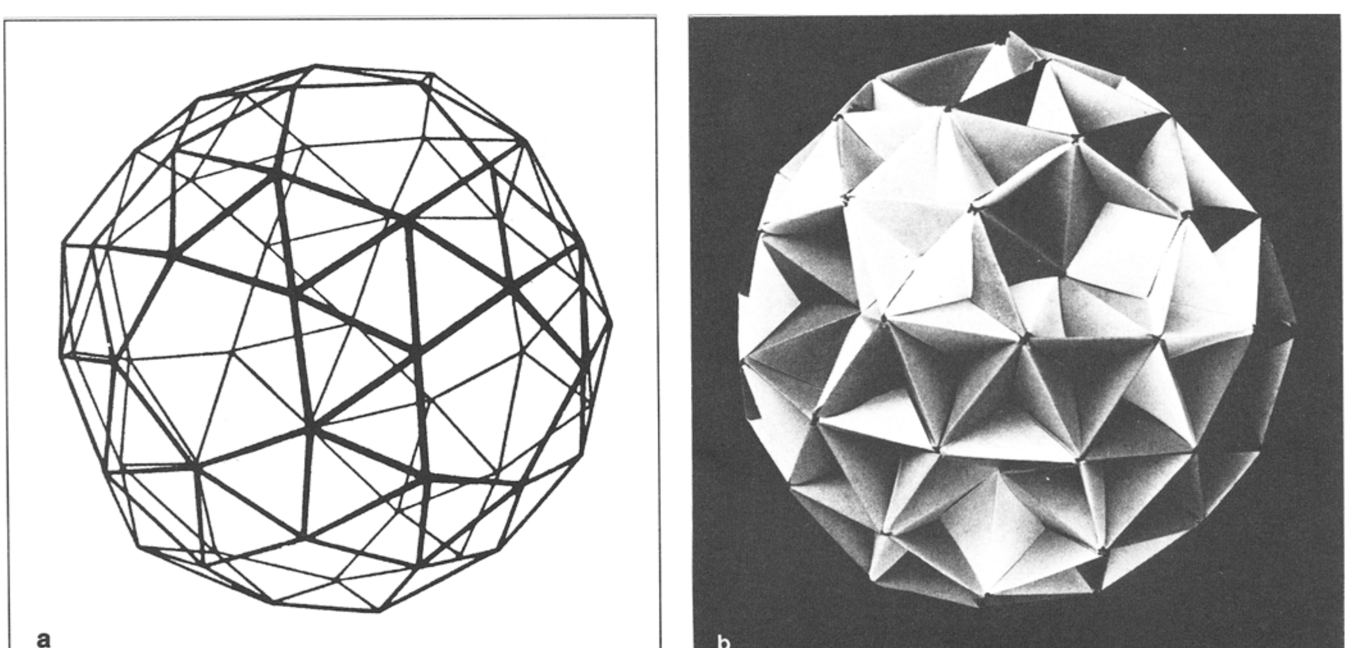


Figure 17 **(a)** Snub dodecahedron (reproduced from [10]). **(b)** Fifteen-way, two-fold isonemal weave.

hedron. The 10-way, two-fold model shown in Figure 16b was constructed by first adding pentagonal pyramids (consisting of five equilateral triangles) to the faces of the regular dodecahedron. This model now becomes the parent polyhedron in the sense explained above.

A 15-way, two-fold isonemal covering may be obtained by taking a snub dodecahedron (as shown in Figure 17a) and replacing the pentagons with pentagonal pyramids (consisting of five equilateral triangles) that point "in"; and, as before, this model becomes the parent polyhedron for the final version which is pictured in Figure 17b.

The surfaces on all of the two-fold models can be converted to models that permit three-fold coverings by replacing all the right-angle isosceles triangles with equilateral triangles. Moreover, it is a fascinating fact that, although the layer cycles on a two-way weave (which go "over, under, . . .") are quite different from the layer cycles on a three-way weave (which go "over, over, middle, under, under, middle. . . .") it turns out that the coloring arrangement is preserved; that is, if you orient the strips so that the colors around one vertex on the two-fold model match the colors around a vertex on the three-fold model, then the entire coloring of both models will match!

Producing these models raises many more questions:

- Are there any other examples of these particular isonemal weaves? (Yes. See [11] for some of them.)
- How many others are there and how do you find them?
- Are there weavings on polyhedra that are, in some sense, equivalent to the satins and twills in the plane?
- Are there models that admit "semiregular" woven coverings? (Yes, see Figure 18c).
- Is it possible to have an isonemal weaving on a polyhedron so that the polygon formed when the strips cross is some polygon other than a square or a 60°-120°-60°-120° rhombus? (I don't think so, but I can't prove it.)
- Is it possible to construct an isonemal weave (covering a polyhedron) with *one* strip? (Yes!)
- Can you find a polyhedron, whose surface is a torus, that will admit a woven isonemal covering? (Yes, see [12].)
- What about other genera?

Epilogue

I would like to close by relating a delightful personal encounter I had while preparing this article. A picture appeared in our local newspaper of a collection of Temari balls[4] made by Kazuko Yamamoto (of Saratoga, California). I was instantly attracted to their beauty and then to their startling similarity to my own woven models. I called Mrs. Yamamoto and we decided to meet, so that I could learn more about her craft and so that my son Chris could photograph her Temari balls for this article. During that first meeting I discussed Temari balls, art, photography, and dramatic performances with Mrs. Yamamoto, a woman of remarkable talent and versatility who is a painter, sculptor, and pantomimist.

Kazuko Yamamoto

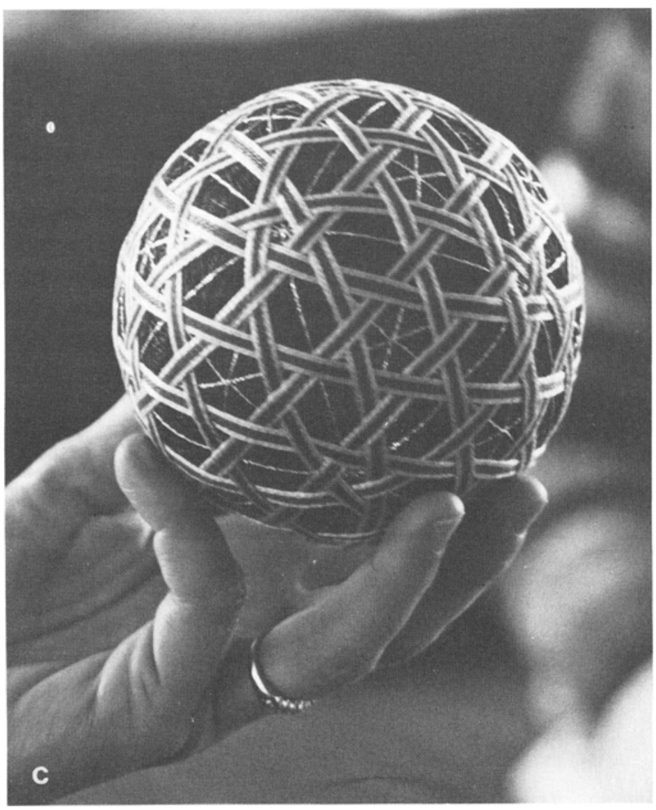
Mrs. Yamamoto explained that certain Temari balls are the *only* possibilities if you want the entire ball to be "completely symmetric". Kazuko, as I quickly came to address her, then pointed out that another ball had a single axis of symmetry through the north and south poles and that you can exchange the poles and the symmetry will remain the same. She was completely unaware that she had constructed objects from all the proper rotation groups described by Felix Klein [9], but she told me that it is not possible to make a ball with any other kind of symmetry. She was not in the least surprised to discover that this had been proved; *her* "proof" was that "basically *people* are the same everywhere"!

Kazuko and I have now met several times to discuss Temari balls and mathematics. The result is that Kazuko is now able to make Temari balls with somewhat different designs, and she is finding other "reason-

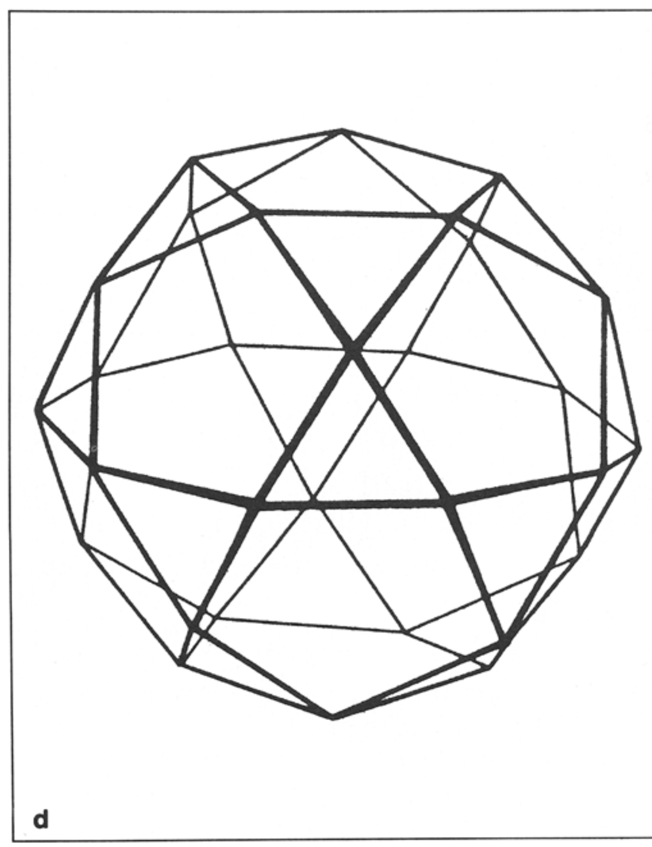
[4] Temari balls were originally made in China, but they have been known in Japan for over a thousand years. The center of the ball consists of some reasonably resilient material like rice hulls. This material is held together by cloth (or, in modern times, plastic), and yarn or thread is then wound around it until the whole forms a spherical shape. The surface is then decorated with yarn or thread to form beautiful and imaginative symmetrical designs.

Figure 18 (a) A collection of Temari balls.

(b) Left: The ball before decoration is added. Center: The initial markings of a "10 divider." Right: The initial markings of an "8 divider."



(c) A "semiregular" weave, involving 18 strands. Note that there are 2 kinds of strands, 6 about great circles and 12 (in pairs) on either side of those 6.



(d) Icosadodecahedron (reproduced from [10]).

able" ways to color and design the "open spaces"; and I have begun to investigate "semiregular" weaves and the possibilities of twill or satin weaves that seem to be suggested by the designs Kuzuko had created on her Temari balls before we met. We are learning each other's language. She now knows what Platonic solids are; and I have learned that a "10 divider" means a finished Temari ball with icosahedral symmetry (A_5), while an "8 divider" is one with octahedral symmetry (S_4) (see Figure 18).

At our next meeting we will discuss the relation of Archimedean solids to the semiregular weavings on her Temari balls (Figure 18c and d); and we are planning a joint exhibition of our models next year.

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Department of Mathematics
University of Santa Clara
Santa Clara, California