METALS AND MATERIALS, Vol. 4, No. 4 (1998), pp. 680~684

Large Deflections of a Clamped Circular Plate Pressed by a Hemispherical-Headed Punch

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A new method of analysis based on the consideration of equilibrium and a physically acceptable displacement field is proposed in this paper to investigate the fully plastic behaviour of a clamped circular plate which is loaded axisymmetrically by a rigid hemispherical-headed punch. The attention is confined to the range of loads for which the central deflection of the plate exceeds the plate thickness, and the effect of the induced membrane forces is duly allowed for in the theoretical framework to obtain a realistic expression for the load-deflection relation in the plastic range. When the central deflection becomes sufficiently large, the deformation of the plate occurs essentially under membrane stresses alone, and the analysis then becomes similar to the one presented earlier by the author for a material that work-hardens isotropically according to the Ludwik power law. Since the considered range of deflections is sufficiently large, the material is assumed to be rigid/plastic, and the work-hardening of the material is disregarded as a necessary first step towards a more general solution. The complete load-deflection relation is presented in a graphical form for the situation where the punch radius is equal to the radius of the plate.

Key words : circular plate, hemispherical head, membrance forces

1. INTRODUCTION

When a transversely loaded circular plate is deformed beyond the elastic limit, the membrane forces developed during the bending become increasingly more significant as the deflection is increased. For a realistic estimation of the load-deflection relation, it is necessary to consider the effect of the membrane forces whenever the maximum deflection is more than about 2 to 3 times that at the initial yielding of the plate. A useful approximate method of analysis for the large deflection of circular plates has been proposed by calladine [1], who employed an energy principle based on an assumed shape of the bent plate. Since the associated work equation is completely equivalent to the condition of equilibrium, the analysis may also be carried out by solving the equation of equilibrium under appropriate boundary conditions. Calladine's method has been used by Yu *et al* [2] to treat the problem of pressing of a simply supported circular plate with a hemispherical-headed punch. These authors have also examined the interesting problem of springback that occurs on removal of the punch load.

When the deformation is so large that the bending stresses are negligible compared to the membrane stresses, a closed form solution for the stretching of a clamped circular plate, made of a work-hardening material, over a hemispherical punch head has been presented by Chakrabarty [3]. The solution to the clamped plate problem over the range of deflections for which the bending and membrane stresses are simultaneously important is considered in the present paper using the equilibrium approach based on a conical mode of deformation. Assuming the punch pressure to be uniformly distributed over the region of contact, a new load-deflection formula is established to describe the post-yield behaviour of the clamped circular plate. Since the strains are considered as large, the elastic deformation of the plate is neglected, the material being regarded as rigid/plastic in the theoretical framework. The rigid/plastic solution would be sufficiently realistic when the central deflection of the plate exceeds the initial plate thickness.

2. LARGE BENDING OF A CLAMPED PLATE

A circular plate of radius a and initial thickness h_o, is fully clamped round its periphery, and is centrally loaded by a rigid punch having a hemispherical head of ra-

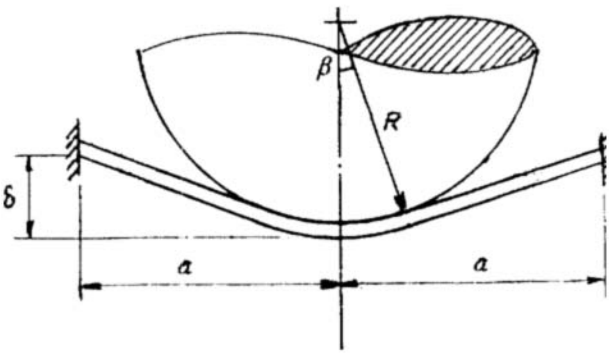

Fig. 1. Deformed configuration of a circular plate pressed by a spherical punch.

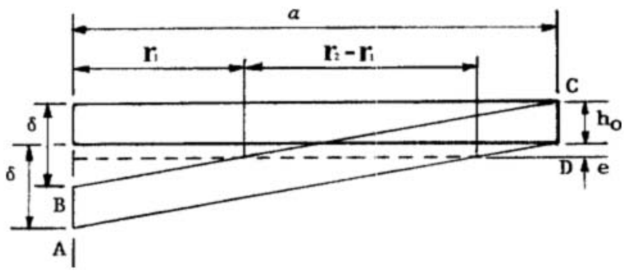

Fig. 2. Conical mode of deformation for a finitely bent circular plate.

dius R as shown in Fig. 1. The material is assumed to be ideally plastic with a constant uniaxial yield stress Y. Since we are concerned here with the range of deformation for which the central deflection δ exceeds the initial plate thickness h_o, the elastic strains will be completely disregarded. Following calladine [1], the bending moments will be referred to the unstretched surface, which may be shown to coincide with the bottom face of the plate. The work done by the membrane forces is then identically zero, and the work equation involves only the bending moments M_r and M_θ. The punch is assumed to be well lubricated, so that the influence of friction is negligible.

There is sufficient theoretical evidence to suggest that the ratio of the applied load to the initial collapse load is virtually unaffected by the choice of the yield criterion. It is convenient therefore to employ a simplified yield criterion, which in this particular case may be written as

$$\sigma_\theta = \pm Y, \quad -Y < \sigma_r < Y$$

Let the unstretched plane be situated at a distance e below the original bottom face of the plate as shown in Fig. 2, where the shape of the deflected plate is assumed conical, the effect of curvature of the bent plate over the region of contact with the punch head being disregarded for the estimation of the bending moment distribution. The broken line, representing the horizontal plane of zero extension, intersects the upper and lower faces of the bent plate at $r=r_1$ and $r=r_2$ respectively. It follows from simple geometry that

$$\frac{r_1}{a} = 1 - \frac{h+e}{\delta}, \quad \frac{r_2}{a} = 1 - \frac{e}{\delta} \tag{1}$$

The circumferential bending moment transmitted across each segment of the radial section ABCD can be easily found from the fact that the hoop stress σ_θ is equal to Y everywhere below the broken line, and is equal to -Y everywhere above the broken line. Thus

$$\int_0^{r_1} M_\theta dr = 2M_0 a\left(\frac{\delta-e}{h_0}\right)\left(1-\frac{h_0+e}{\delta}\right)$$

$$\int_{r_1}^{r_2} M_\theta dr = \frac{4}{3}M_0 a\left(\frac{h_0}{\delta}\right)$$

$$\int_{r_2}^{a} M_\theta dr = 2M_0 a\frac{e}{\delta}\left(1+\frac{e}{\delta}\right)$$

where $M_o = Yh_o^2/4$, representing the fully plastic moment of the cross section. Combining the above relations, the resultant circumferential moment is obtained in the dimensionless form

$$\int_0^1\left(\frac{M_\theta}{M_0}\right)d\left(\frac{r}{a}\right) = 2\left\{\left(\frac{\delta}{h_0}+\frac{2h_0}{3\delta}\right)-\left(1+\frac{2e}{h_0}\right)+\frac{2e}{\delta}\left(1+\frac{e}{h_0}\right)\right\} \tag{2}$$

The analysis is most conveniently carried out by integrating the equation of moment equilibrium, using Eq. 2, instead of considering the associated work equation.

In the absence of friction, the normal pressure p acting over the punch head may be assumed as uniform for simplicity. Then the equation of equilibrium of the bending moments M_r and M_θ over the region of contact may be written as

$$\frac{d}{dr}(rM_r) = M_\theta - \int_0^r prdr = M_\theta - \frac{1}{2}pr^2, \quad 0 \le r \le b \tag{3}$$

where b is the radius of contact between the punch head and the deformed plate. Over the remainder of the plate, the equilibrium equation becomes

$$\frac{d}{dr}(rM_r) = M_\theta - \frac{1}{2}pb^2, \quad b \le r \le a \tag{4}$$

Assuming a hinge circle to form at the clamped edge, the boundary condition may be written as M_r=-M_θ at r=a. In view of the continuity of the bending moments across r=b, the integration of Eqs. 3 and 4 results in

$$\frac{pb^2}{2M_o}\left(1-\frac{2b}{3a}\right)=\int_0^1\left(\frac{M_\theta}{M_o}\right)d\left(\frac{r}{a}\right)+\left(\frac{M_\theta}{M_o}\right)_{r=a}$$

since M_θ=2M_o(1+2e/h) at r=a, in view of the hoop stress distribution of Fig. 2, the preceding relation can be combined with Eq. 2 to give an expression for p which has a least value when e=0. Hence

$$\frac{pb^2}{4M_o}\left(1-2\frac{b}{3a}\right)=\left(\frac{\delta}{h_o}+\frac{2h_o}{3\delta}\right),\quad \frac{\delta}{h_o}\geq 1 \tag{5}$$

For a given b, the left-hand side of Eq. 5 is easily shown to be unity at the initial collapse according to the assumed yield criterion. The left-hand of side of Eq. 5 at any stage is therefore equal to P/P_o, where P_o is the initial punch load and P the current punch load. Setting q_o=P_o/2πM_o, we finally obtain the simple formula

$$\frac{P}{2\pi M_o}=q_o\left(\frac{\delta}{h_o}+\frac{2h_o}{3\delta}\right),\quad \frac{\delta}{h_o}\geq 1 \tag{6}$$

which may be assumed to hold with sufficient accuracy for any other form of the yield criterion. If Tresca's yield criterion is adopted, the initial collapse load is given by [4]

$$\frac{3}{2}\left(1-\frac{1}{q_o}\right)\exp\left(-\frac{1}{q_o-1}\right)=\frac{b}{a},\quad q_o\leq 3 \tag{7}$$

where b=R sin β, with β denoting the semi-angle of contact. It follows from simple geometry of the bent plate (Fig. 1) that

$$\frac{\delta}{h_o}=\frac{a}{h_o}\tan\beta-\frac{R}{h_o}(\sec\beta-1) \tag{8}$$

For given values of R/a and h_0/a, the load-deflection curve for a clamped circular plate pressed by a rigid punch is defined by Eqs. 6, 7 and 8, so long as the effect of bending is comparable to that of stretching of the plate.

3. MEMBRANE SOLUTION FOR FINITE STRAINS

When the central deflection of the plate is large compared to the plate thickness, the membrane forces completely dominate the process. The essential features of the process, in the absence of friction, can be brought out by using a simplified theoretical model in which each element is assumed to deform under an equal biaxial tension [3]. Let ϕ denote the angle of inclination of the surface normal at a generic point of the deformed plate with the vertical axis of symmetry. The boundary r =a corresponds to ϕ=α, while the circle of contact r=b corresponds to ϕ=β, as shown in Fig. 3. If the meridional and circumferential radii of curvature of deformed middle surface are denoted by ρ_r and ρ_θ respectively, then the condition of equilibrium of an element in the direction normal to the surface can be written as

$$\frac{\sigma_r}{\rho_r}+\frac{\sigma_\theta}{\rho_\theta}=\frac{p}{h} \tag{9a}$$

where p is the normal pressure acting on the element, and h the current plate thickness that varies with the angle ϕ. The principal radii of curvature ρ_r and ρ_θ are given by simple geometry as

$$\rho_r=\frac{\partial r}{\partial\phi}\sec\phi,\quad \rho_\theta=r\,\mathrm{cosec}\phi \tag{9b}$$

The geometrical constraint requires ρ_r=ρ_θ=R over the region of contact (0≤r≤b), while the static boundary condition requires p=0 over the outer region (b≤r≤a) that is not in contact with the punch head. Since the material is non-hardening, the uniaxial yield stress has a constant value equal to Y throughout the deforming plate.

Consider first the unsupported region, which will be in a state of balanced biaxial tension (σ_r=σ_θ=Y) only if the radii of curvature satisfy the relation

$$\rho_r=-\rho_\theta=-\rho\,(\text{say})$$

in view of Eq. 9a, where ρ is a positive radius. Eq. 9b

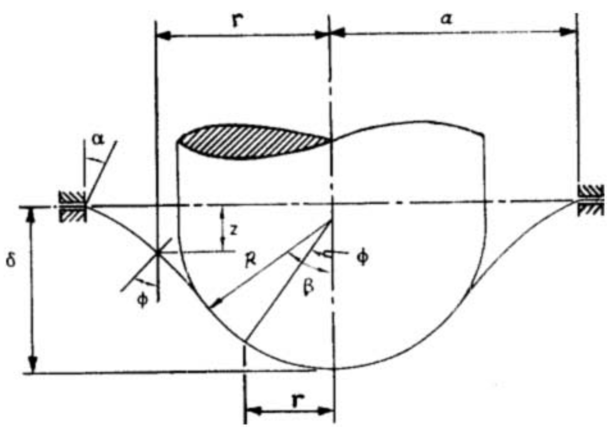

Fig. 3. Geometry of deformation of a circular plate stretched as a membrane.

therefore give

$$\frac{\partial r}{\partial \phi} = -\rho \cos \phi = -r \cot \phi$$

Integrating, and using the boundary condition r=a at $\phi=\alpha$, we get

$$\frac{r}{a} = \frac{\sin \alpha}{\sin \phi}, \quad \frac{\rho}{a} = \frac{\sin \alpha}{\sin^2 \phi} \tag{10}$$

The continuity condition ρ=R at the contact boundary $\phi=\beta$ furnishes the relation

$$\sin \alpha = \frac{R}{a} \sin^2 \beta \tag{11}$$

It may be noted that the meridional radius of curvature changes discontinuously from -R to R as the contact boundary is approached from the outer region. The shape of the unsupported surface is defined by the differential equation

$$\frac{\partial z}{\partial r} = -\tan \phi, \quad \text{or} \quad \frac{\partial z}{\partial \phi} = r = a\left(\frac{\sin \alpha}{\sin \phi}\right)$$

where z is the vertical height of a generic point below the original plane of the plate. In view of the boundary condition z=0 at $\phi=\alpha$, the integration of the above equation results in

$$\frac{z}{a} = \sin \alpha \ln\left\{\frac{\tan(\phi/2)}{\tan(\alpha/2)}\right\} \tag{12}$$

The elimination of ϕ between Eqs. 10 and 12 furnishes r/a as a function of z/a, the result being

$$\frac{r}{a} = \sin \alpha \cosh\left(\frac{z}{a} \operatorname{cosec}\alpha + \ln \tan\frac{\alpha}{2}\right) \tag{13}$$

This part of the deformed middle surface actually forms a minimal surface, since the mean curvature vanishes at each point. Eq. 13 represents a catenoid, which is known to be the only minimal surface of revolution.

The region of contact is a spherical surface of radius R, giving $\rho_r=\rho_\theta$=R and $\sigma_r=\sigma_\theta$=Y, the effect of friction being disregarded. The normal pressure acting over the punch head is p=2Yh/R, which varies over the region of contact. Using the Levy-Mises flow rule, it can be shown [3] that

$$\frac{h_0}{h} = \frac{(1+\cos\phi)^2(1+\cos\alpha)^2}{(1+\cos\beta)^4}, \quad 0 \le \phi \le \beta \tag{14}$$

The punch load P at any stage is given by the condition of overall vertical equilibrium of the spherical cap of radius b. Thus

$$P = 2\pi Y h^* b \sin \beta = 2\pi Y R h^* \sin^2\beta$$

where h* is the thickness at r=b, and is obtained by setting $\phi=\beta$ in Eq. 14. The result may be expressed in the dimensionless form

$$\frac{P}{2\pi M_0} = \frac{4a}{h_0}\left(\frac{1+\cos\beta}{1+\cos\alpha}\right)^2 \sin \alpha \tag{15}$$

in view of Eq. 11. the central deflection δ is obtained from simple geometry as

$$\delta = \delta^* + R(1-\cos \beta)$$

where δ^* is the vertical height of the circle of contact, and is given by Eq. 12 with $\phi=\beta$. Hence

$$\frac{\delta}{h_0} = \frac{R}{h_0}(1-\cos \beta) + \frac{a}{h_0}\sin \alpha \ln\left\{\frac{\tan(\beta/2)}{\tan(\alpha/2)}\right\} \tag{16}$$

Eqs. 15 and 16 define the load-deflection relationship parametrically through α or β, the two angles being related to one another by Eq. 11. The membrane solution would be appropriate for the punch pressing problem when the central deflection exceeds only a few times the initial plate thickness [5].

4. DISCUSSION OF RESULTS

According to the known solution for the velocity distribution at the incipient collapse, a clamped circular plate tends to deform into a surface of negative curvature near the boundary, similar to that predicted by the membrane solution. Consequently, the assumption of a conical mode of deformation for the unsupported part of the bent plate is expected to underestimate the angle of contact and hence the punch load predicted by the bending solution is also underestimated. For a realistic estimation of load-deflection relation in the early part of the process, it seems to be a better approximation to calcute the angle β, corresponding to a given δ/h_0 ratio, by using Eq. 16 rather than Eq. 8 before evaluating q_0 from Eq. 7. The numerical results based on this procedure are plotted in Fig. 4, using the values a/h_0 equal to 30 and 50, and assuming R=a, the membrane solution being found to be appropriate in the two cases for δ/h_0 exceeding about 4.5 and 4.0 respectively. Since the pressure distribution over the punch head resulting from the assumed state of stress becomes increasingly inaccurate for a non-hardening material as the loading is continued, the membrane solution would provide a realistic estimate of

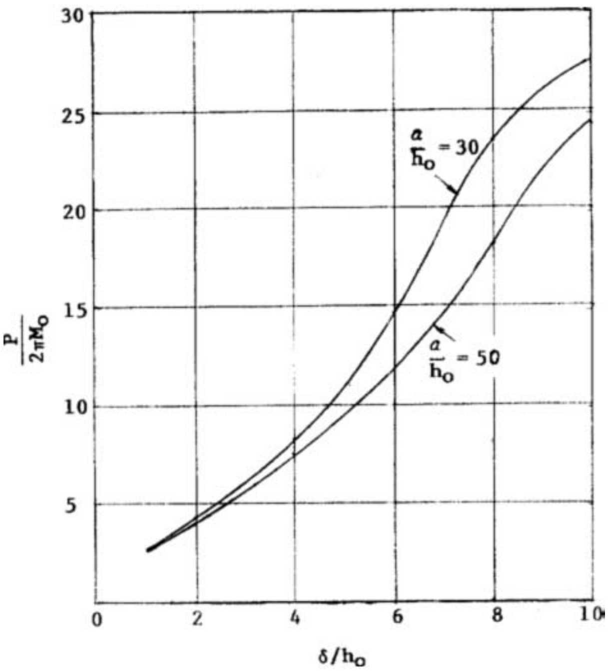

Fig. 4. Load-deflection curves for clamped circular plate pressed by hemisphericl-headed punch.

the load only over the range $\delta/h_0 < 10$. At the other extreme, when $\delta/h_0 < 1$, it would be necessary to modify the results of the rigid/plastic analysis to include the effect of the elastic strains. It is important to note that the equilibrium approach adopted here allows the consideration of other types of deformation mode of the bent plate in order to improve the bending solution.

It would be interesting to have a quantitative assessment of the influence of work-hardening on the load-penetration behaviour of finitely deformed circular plates. When the bending effect is negligible, a useful closed form solution for the punch stretching of clamped circular plates, made of work-hardening materials, is already available [3]. However, a work-hardening solution for the large bending of circular plates, including the effect of the associated membrane forces, does not seem to have been attempted in the past. From the practical point of view, the solution given in this paper may be modified in an approximate manner by replacing the constant yield stress Y with a mean uniaxial yield stress that depends on the amount of central deflection. Such a procedure has been found to be quite useful for a number of important metal forming processes [6], where mathematically rigorous solutions exist when the material is ideally plastic.

5. CONCLUSIONS

This paper demonstrates how Calladine's approach for the inclusion of membrane forces in an approximate manner to deal with large bending deflections of rigid plastic plates can be modified to formulate it in terms of the condition of moment equilibrium, rather than the usual consideration of the work equation. The equilibrium method allows the assumed displacement field to be different from the simple conical field, which is appropriate for a limited range of loading conditions. The load-deflection formula developed here for the large deflection of a partially loaded clamped circular plate appears to be new, and is in complete agreement with the experimental fact that the load rises more rapidly with the deflection in the case of a clamped plate than in the case of a simply supported plate. The predicted load-deflection curves for the punch pressing of a fully clamped circular plate also follows the same trend as that experimentally observed. As in the case of a simply supported plate [2], where the situation is complicated by the possibility of wrinkling as the plate is pressed into a cavity, the punch load is significantly underestimated by the neglect of membrane forces even when the central deflection is only of the order of the plate thickness.

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