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The Tangential Plasticity

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Various constitutive models for describing the inelastic stretching due to the stress rate component tangential to the yield or loading surface, i.e. the non-coaxiality of stress and inelastic stretching have been proposed in the past. However. a pertinent model applicable to a general loading process for materials with an arbitrary yield surface has not been proposed up to the present. In this article, the inelastic constitutive equation extended so as to describe the non-coaxiality is formulated generalizing the Rudnicki and Rice's [1] J_2-deformation theory in rate form and incorporating it into the subloading surface model with a smooth elastic-plastic transition.

Key words : constitutive equation, elastoplasticity, subloading surface model, tangential stress rate, non-coaxiality

1. INTRODUCTION

The plastic stretching is independent of the stress rate component tangential to the yield or loading surface, called the *tangential stress rate*, in the traditional elastoplastic constitutive equation with a single and smooth plastic potential surface. It leads to the *coaxiality*, i.e. principal directions of plastic stretching coincides with those of stress. The extension of the constitutive equation so as to describe the dependency of inelastic stretching on the tangential stress rate, i. e. the *non-coaxiality* would be one of the most fundamental but unsolved problems in elastoplasticity at present. Therefore, various models for this aim have been proposed in the past. A pertinent one applicable to a general loading process for materials with an arbitrary yield surface has not been proposed up to the present, however.

In this article, the elastoplastic constitutive equation is extended so as to describe the *non-coaxiality* of a stress and an inelastic stretching by introducing a novel *para-inelastic stretching* caused by the tangential stress rate into the *subloading surface model* [2~4] with a smooth elastic-plastic transition. It fulfills the mechanical requirements [5~7], i. e. the *continuity condition*, the *smoothness condition*, the *work rate-stiffness relaxation* and the *Masing effect* by keeping the mechanical features of the subloading surface model, and it would be a pertinent inelastic constitutive equation applicable to an arbitrary loading process including unloading, reloading and reverse loading processes for materials with an arbitrary smooth yield surface. It could be regarded as the generalization of Rudnicki and Rice's [1] *J_2-deformation theory* in rate form which is limited to a monotonic loading process in the neighborhood of proportional loading of the isotropic metals with the von Mises yield condition without the kinematic hardening.

2. OUTLINE OF THE SUBLOADING SURFACE MODEL

In this section the subloading surface model is reviewed briefly, since it is the essential one for the formulation of the extended constitutive equation with a tangential stress rate effect.

As usual, let it be assumed that the stretching **D** (symmetric part of velocity gradient) is additively decomposed into the elastic stretching $\boldsymbol{D}^e$ and the plastic stretching $\boldsymbol{D}^p$, i.e.

$$\boldsymbol{D} = \boldsymbol{D}^e + \boldsymbol{D}^p \tag{1}$$

where the elastic stretching is given by

$$\boldsymbol{D}^e = \boldsymbol{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} \tag{2}$$

$\boldsymbol{\sigma}$ is a stress and $(^\circ)$ indicates the proper corotational rate with the objectivity (e. g. cf. Dafalias [8] and Zbib and Aifantis [9] introducing the plastic spin) and the fourth-order tensor $\boldsymbol{E}$ is the elastic modulus given in the Hooke's type as

$$E_{ijkl} = (K - \frac{2}{3}G)\,\delta_{ij}\,\delta_{kl} + G(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}) \tag{3}$$

where K and G are the bulk modulus and the shear modulus, respectively, which are functions of stress and internal state variables in general and δ_{ij} is the Kronecker's delta, i. e. δ_{ij}=1 for i=j and δ_{ij}=0 for i $\neq$ j.

2.1. Normal-yield and subloading surfaces

Consider the following yield condition as the realistic one.

$$f(\hat{\boldsymbol{\sigma}}, \boldsymbol{H}) = F(H) \tag{4}$$

where

$$\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\alpha} \tag{5}$$

$\boldsymbol{\alpha}$ is the specified point on or inside the normal-yield surface, while it plays the role of the kinematic hardening variable if it translates with a plastic deformation. The tensor $\boldsymbol{H}$ and the scalar H denote an anisotropic and an isotropic hardening variable, respectively. Let it be assumed that the function f is homogeneous degree one of the tensor $\hat{\boldsymbol{\sigma}}$ and that $\boldsymbol{H}$ is the dimensionless variable. Then, if $\boldsymbol{H}=\boldsymbol{O}$, the yield surface keeps a similar shape, translating with $\boldsymbol{\alpha}$. An example of $\boldsymbol{H}$ is the rotational hardening variable of the second-order tensor for geomaterials [10]. The yield surface is renamed as the *normal-yield surface*, since its interior is not regarded to be an elastic domain in the present model.

Now, let the *subloading surface* [2~4] be introduced, which always passes through the current stress $\boldsymbol{\sigma}$ and keeps the similarity to the normal-yield surface. By denoting the ratio of the size of the subloading surface to that of the normal-yield surface as R and the *similarity-center* of the normal-yield and the subloading surfaces as $\boldsymbol{s}$, it holds that

$$\boldsymbol{\sigma}_y = \frac{1}{R}\{\boldsymbol{\sigma} - (1-R)\boldsymbol{s}\}\ (\boldsymbol{\sigma} - \boldsymbol{s} = R(\boldsymbol{\sigma}_y - \boldsymbol{s})) \tag{6}$$

where $\boldsymbol{\sigma}_y$ on the normal-yield surface is regarded as the conjugate point of the current stress $\boldsymbol{\sigma}$ on the subloading surface, obeying the similarity (see Fig. 1).

By substituting Eq. 6 into Eq. 4, the subloading surface is described as

$$f(\bar{\boldsymbol{\sigma}}, \boldsymbol{H}) = RF(H) \tag{7}$$

where

$$\bar{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}}\ (= R\hat{\boldsymbol{\sigma}}) \tag{8}$$

$$\bar{\boldsymbol{\alpha}} \equiv \boldsymbol{s} - R(\boldsymbol{s} - \boldsymbol{\alpha})\ (\bar{\boldsymbol{\alpha}} - \boldsymbol{s} = R(\boldsymbol{\alpha} - \boldsymbol{s})) \tag{9}$$

$\bar{\boldsymbol{\alpha}}$ for the subloading surface is the conjugate point of $\boldsymbol{\alpha}$ for the normal-yield surface, obeying the similarity. In calculation first R is determined from Eq. 7 with Eqs. 8 and 9, substituting values of $\boldsymbol{\sigma}$, $\boldsymbol{\alpha}$, $\boldsymbol{s}$, $\boldsymbol{H}$ and H, and thereafter $\bar{\boldsymbol{\alpha}}$ is done from Eq. 9, while four internal variables $\boldsymbol{\alpha}$, $\boldsymbol{s}$, $\boldsymbol{H}$ and H are incorporated in the present model.

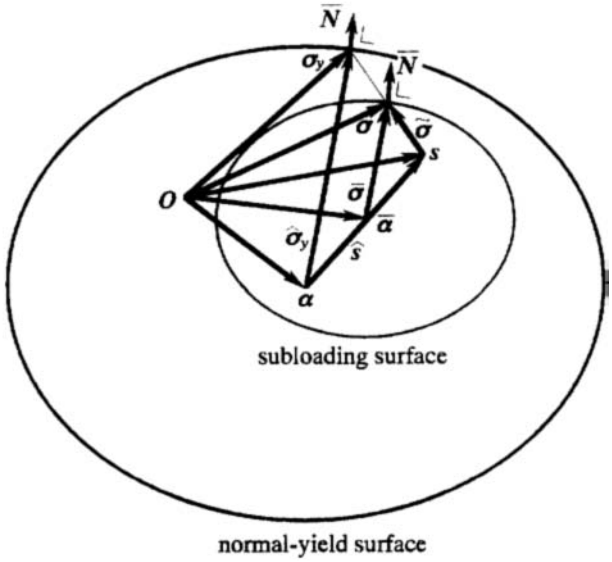

Fig. 1. The normal-yield and the subloading surfaces.

2.2. Translation rule of similarity-center

It is required for the similarity-center $\boldsymbol{s}$ to translate with a plastic deformation in order to describe realistically a cyclic loading behavior with a closed hysteresis loop. The translation rule of $\boldsymbol{s}$ is described below.

The following inequality must hold since the similarity-center has to exist inside the normal-yield surface.

$$f(\hat{\boldsymbol{s}}, \boldsymbol{H}) \le F(H) \tag{10}$$

where

$$\hat{\boldsymbol{s}} = \boldsymbol{s} - \boldsymbol{\alpha} \tag{11}$$

The time-differentiation of (10) is given by

$$\mathrm{tr}\left[\frac{\partial f(\hat{\boldsymbol{s}}, \boldsymbol{H})}{\partial \boldsymbol{s}}\left(\dot{\boldsymbol{s}} - \dot{\boldsymbol{\alpha}} + \frac{1}{F}\{\mathrm{tr}(\frac{\partial f(\hat{\boldsymbol{s}}, \boldsymbol{H})}{\partial \boldsymbol{H}}\dot{\boldsymbol{H}}) - \dot{F}\}\hat{\boldsymbol{s}}\right)\right] \le 0 \quad \text{for } f(\hat{\boldsymbol{s}}, \boldsymbol{H}) = F(H) \tag{12}$$

where $(\dot{\ })$ indicates a material-time derivative. Eq. 10 or Eq. 12 is called the *enclosing condition of similarity-center*.

In the ultimate state $f(\hat{\boldsymbol{s}}, \boldsymbol{H})=F(H)$ in which $\boldsymbol{s}$ exists on the normal-yield surface the vector $\boldsymbol{\sigma}_y$-$\boldsymbol{s}$ makes an obtuse angle with the vector $\partial f(\hat{\boldsymbol{s}}, \boldsymbol{H})/\partial \boldsymbol{s}$ which is outward-normal to the surface $f(\hat{\boldsymbol{s}}, \boldsymbol{H})=F(H)$ coinciding with the normal-yield surface, since $\boldsymbol{\sigma}_y$ exists on the normal-yield surface. Noting this fact, let the following equation be assumed, which fulfills the inequality Eq. 12.

$$\mathring{s} - \mathring{\alpha} + \frac{1}{F}\{\mathrm{tr}(\frac{\partial f(\hat{s},H)}{\partial H}\mathring{H}) - \dot{F}\}\hat{s} = c\,\|D^p\|\;(\sigma_y - s) \quad (13)$$

from which the translation rule of the similarity-center is derived as follows:

$$\mathring{s} = c\,\|D^p\|\,\frac{\tilde{\sigma}}{R} + \mathring{\alpha} + \frac{1}{F}\{\dot{F} - \mathrm{tr}(\frac{\partial f(\hat{s},H)}{\partial H}\mathring{H})\}\hat{s} \quad (14)$$

where c is a material constant influencing the translating rate of the similarity-center and

$$\tilde{\sigma} \equiv \sigma - s \quad (15)$$

Thus, the evolution of the similarity-center s is determined by the plastic stretching D^p and the hardening rates $\mathring{\alpha}$, $\mathring{H}$ and $\dot{H}$.

2.3. Plastic stretching

The time-differentiation of Eq. 7 is given by

$$\mathrm{tr}(\frac{\partial f(\bar{\sigma},H)}{\partial \sigma}\mathring{\sigma}) - \mathrm{tr}(\frac{\partial f(\bar{\sigma},H)}{\partial \sigma}\mathring{\bar{\alpha}}) + \mathrm{tr}(\frac{\partial f(\bar{\sigma},H)}{\partial H}\mathring{H}) = \dot{R}F + RF'\dot{H} \quad (16)$$

where

$$F' \equiv \frac{dF}{dH} \quad (17)$$

Let the evolution rule of R be given by

$$\dot{R} = U\,\|D^p\| \quad \text{for } D^p \neq O \quad (18)$$

where U is the monotonically decreasing function of R, satisfying

$$\left.\begin{aligned} R = 0 &: U = +\infty \\ 0 < R < 1 &: U > 0 \\ R = 1 &: U = 0 \\ R > 1 &: U < 0 \end{aligned}\right\} \quad (19)$$

$\|\;\|$ stands for the magnitude. Let the function U satisfying Eq. 19 be simply given by

$$U = -u\ln R \quad (20)$$

where u is a material constant.

By substituting Eq. 18 into Eq. 16 one has the *extended consistency condition* for the subloading surface:

$$\mathrm{tr}(\frac{\partial f(\bar{\sigma},H)}{\partial \sigma}\mathring{\sigma}) - \mathrm{tr}(\frac{\partial f(\bar{\sigma},H)}{\partial \sigma}\mathring{\bar{\alpha}}) + \mathrm{tr}(\frac{\partial f(\bar{\sigma},H)}{\partial H}\mathring{H}) = U\,\|D^p\|\,F + RF'\dot{H} \quad (21)$$

Assume the associated flow rule

$$D^p = \lambda \bar{N} \quad (22)$$

where λ is the positive proportionality factor, and the second-order tensor $\bar{N}$ is the normalized outward-normal of the subloading surface, i. e.

$$\bar{N} \equiv \frac{\partial f(\bar{\sigma},H)}{\partial \sigma} \Big/ \left\| \frac{\partial f(\bar{\sigma},H)}{\partial \sigma} \right\| \quad (\|\bar{N}\| = 1) \quad (23)$$

The substitution of Eq. 22 into the extended consistency condition Eq. 21 leads to

$$\lambda = \frac{\mathrm{tr}(\bar{N}\mathring{\sigma})}{\bar{M}_p} \quad (24)$$

where

$$\bar{M}_p \equiv \mathrm{tr}\left[\bar{N}\left(\bar{a} + \{\frac{F'}{F}h - \frac{1}{FR}\mathrm{tr}(\frac{\partial f(\bar{\sigma},H)}{\partial H}\boldsymbol{h}) + \frac{U}{R}\}\bar{\sigma}\right)\right] \quad (25)$$

h, $\boldsymbol{h}$ and $\bar{a}$ are functions of the stress, plastic internal state variables and $\bar{N}$ in degree one, which are related to $\dot{H}$, $\mathring{H}$ as $\mathring{\bar{\alpha}}$ as

$$h \equiv \frac{\dot{H}}{\lambda}, \quad \boldsymbol{h} \equiv \frac{\mathring{H}}{\lambda} \quad (26)$$

$$\bar{a} \equiv \frac{\mathring{\bar{\alpha}}}{\lambda} = z - U(s - \alpha) - R(z - a) \quad (27)$$

$$a \equiv \frac{\mathring{\alpha}}{\lambda} \quad (28)$$

$$z \equiv \frac{\mathring{s}}{\lambda} = c\frac{\tilde{\sigma}}{R} + a + \frac{1}{F}\left\{F'h - \left(\frac{\partial f(\hat{s},H)}{\partial H}\boldsymbol{h}\right)\right\}\hat{s} \quad (29)$$

since these rate variables include λ in degree one.

The plastic stretching is given from Eqs. 22 and 24 as

$$D^p = \frac{\mathrm{tr}(\bar{N}\mathring{\sigma})}{\bar{M}_p}\bar{N} \quad (30)$$

which reduces to the following simple form for isotropic hardening materials with $\alpha = s = O$ ($\bar{\alpha} = O$) and $H = O$.

$$\left.\begin{aligned} D^p &= \frac{\mathrm{tr}(N\mathring{\sigma})}{M_p}N \\ N &\equiv \frac{\partial f(\sigma)}{\partial \sigma} \Big/ \left\|\frac{\partial f(\sigma)}{\partial \sigma}\right\| \\ M_p &= (\frac{F'}{F}h + \frac{U}{R})\,\mathrm{tr}(N\sigma) \end{aligned}\right\} \quad (31)$$

2.4. Loading criterion

The loading criterion is given as follows [11,12]:

$$\left.\begin{aligned} \boldsymbol{D}^p \neq \boldsymbol{O} &: \mathrm{tr}(\bar{\boldsymbol{N}}\boldsymbol{E}\boldsymbol{D}) > 0 \\ \boldsymbol{D}^p = \boldsymbol{O} &: \mathrm{tr}(\bar{\boldsymbol{N}}\boldsymbol{E}\boldsymbol{D}) \leq 0 \end{aligned}\right\} \tag{32}$$

which is applicable not only to a hardening state but also to a perfectly-plastic and a softening state. The mechanical background of the loading criterion Eq. 32 is examined by the author [13].

3. EXTENSION TO THE TANGENTIAL STRESS RATE EFFECT

The inelastic stretching predicted by traditional elastoplastic constitutive equations including the subloading surface model is independent of the stress rate tangential to the loading surface. In what follows, let the subloading surface model be extended so as to describe the stretching due to the stress rate tangential to the subloading surface.

First, let it be assumed that the stretching is additively composed of the elastic stretching $\boldsymbol{D}^e$, the plastic stretching $\boldsymbol{D}^p$ and the *tangential stretching* $\boldsymbol{D}^t$ induced by the tangential stress rate, i.e.

$$\boldsymbol{D} = \boldsymbol{D}^e + \boldsymbol{D}^p + \boldsymbol{D}^t \tag{33}$$

Let the tangential stretching $\boldsymbol{D}^t$ be formulated as

$$\boldsymbol{D}^t = \frac{1}{T}\overset{\circ}{\boldsymbol{\sigma}}{}_t^* \tag{34}$$

where T is a monotonically decreasing function of R satisfying the condition

$$\left.\begin{aligned} T &= \infty \ \text{ for } R = 0 \\ T &= a \ \text{ for } R = 1 \end{aligned}\right\} \tag{35}$$

where a is the material constant. Let the function T satisfying Eq. 35 be simply given by

$$T(R) = aR^{-b} \tag{36}$$

b (> 1) being material constants. The second-order tensor $\overset{\circ}{\boldsymbol{\sigma}}{}_t^*$ is given as follows:

$$\boldsymbol{\sigma}^* \equiv \boldsymbol{\sigma} - \sigma_m \boldsymbol{I}, \quad \sigma_m \equiv \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma} \tag{37}$$

$$\left.\begin{aligned} \overset{\circ}{\boldsymbol{\sigma}}{}^* &= \overset{\circ}{\boldsymbol{\sigma}}{}_n^* + \overset{\circ}{\boldsymbol{\sigma}}{}_t^* \\ \overset{\circ}{\boldsymbol{\sigma}}{}_n^* &\equiv \mathrm{tr}(\bar{\boldsymbol{n}}^* \overset{\circ}{\boldsymbol{\sigma}}{}^*)\bar{\boldsymbol{n}}^* \\ \overset{\circ}{\boldsymbol{\sigma}}{}_t^* &\equiv \overset{\circ}{\boldsymbol{\sigma}}{}^* - \overset{\circ}{\boldsymbol{\sigma}}{}_n^* \end{aligned}\right\} \tag{38}$$

$$\bar{\boldsymbol{n}}^* \equiv \left(\frac{\partial f(\bar{\boldsymbol{\sigma}}, H)}{\partial \boldsymbol{\sigma}}\right)^* \Big/ \left\| \left(\frac{\partial f(\bar{\boldsymbol{\sigma}}, H)}{\partial \boldsymbol{\sigma}}\right)^* \right\| = \frac{\bar{\boldsymbol{N}}^*}{\|\bar{\boldsymbol{N}}^*\|} \quad (\|\bar{\boldsymbol{n}}^*\| = 1) \tag{39}$$

$$\bar{\boldsymbol{N}}^* \equiv \left(\frac{\partial f(\bar{\boldsymbol{\sigma}}, H)}{\partial \boldsymbol{\sigma}}\right)^* \Big/ \left\| \frac{\partial f(\bar{\boldsymbol{\sigma}}, H)}{\partial \boldsymbol{\sigma}} \right\| \quad (\|\bar{\boldsymbol{N}}^*\| \neq 1) \tag{40}$$

Let $\overset{\circ}{\boldsymbol{\sigma}}{}_t^*$ be called the *deviatoric-tangential stress rate* which fulfills

$$\mathrm{tr}(\bar{\boldsymbol{N}}\overset{\circ}{\boldsymbol{\sigma}}{}_t^*) = 0, \quad \mathrm{tr}\,\overset{\circ}{\boldsymbol{\sigma}}{}_t^* = 0 \tag{41}$$

Hereinafter, let the plastic modulus $\bar{M}_p$ be renamed the *normal-plastic modulus* and let the function T be named the *tangential-plastic modulus*.

The stretching $\boldsymbol{D}$ is given from Eqs. 2, 30, 33 and 34 as

$$\boldsymbol{D} = \boldsymbol{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{\mathrm{tr}(\bar{\boldsymbol{N}}\overset{\circ}{\boldsymbol{\sigma}})}{\bar{M}_p}\bar{\boldsymbol{N}} + \frac{1}{T}\overset{\circ}{\boldsymbol{\sigma}}{}_t^* \tag{42}$$

which fulfills the extended work rate-stiffness relaxation as follows:

$$\mathrm{tr}(\boldsymbol{D}^i\boldsymbol{E}\boldsymbol{D}) \geq \mathrm{tr}(\boldsymbol{D}^t\boldsymbol{E}\boldsymbol{D}) \geq \frac{1}{T}\{tr(\overset{\circ}{\boldsymbol{\sigma}}{}_t^*\overset{\circ}{\boldsymbol{\sigma}}) + \lambda\,\mathrm{tr}(\overset{\circ}{\boldsymbol{\sigma}}{}_t^*\boldsymbol{E}\bar{\boldsymbol{N}})$$
$$\geq \frac{1}{T}\mathrm{tr}(\overset{\circ}{\boldsymbol{\sigma}}{}_t^*\overset{\circ}{\boldsymbol{\sigma}}) \geq 0 \tag{43}$$

Eq. 42 fulfills the continuity condition and the smoothness condition as well as the original subloading surface model. Besides, $\boldsymbol{D}^t$ is linearly related to the stress rate. Inversely, the stress rate is inversely expressed in terms of the stretching as follows:

$$\overset{\circ}{\boldsymbol{\sigma}} = \frac{1}{1+2G/T}\left\{\boldsymbol{E}\boldsymbol{D} - \frac{\mathrm{tr}(\bar{\boldsymbol{N}}\boldsymbol{E}\boldsymbol{D})}{\bar{M}_p + \mathrm{tr}(\bar{\boldsymbol{N}}\boldsymbol{E}\bar{\boldsymbol{N}})}[\boldsymbol{E}\bar{\boldsymbol{N}} \right.$$
$$+ \frac{2G}{T}\{\frac{1}{3}\mathrm{tr}(\boldsymbol{E}\bar{\boldsymbol{N}})\boldsymbol{I} - (\bar{M}_p + \frac{1}{3}\mathrm{tr}\bar{\boldsymbol{N}}\,\mathrm{tr}(\boldsymbol{E}\bar{\boldsymbol{N}}))\frac{\bar{\boldsymbol{n}}^*}{\|\bar{\boldsymbol{N}}^*\|}\}]$$
$$\left. + \frac{2G}{T}\mathrm{tr}(\boldsymbol{E}\boldsymbol{D})(\frac{1}{3}\boldsymbol{I} - \frac{1}{3}(\mathrm{tr}\bar{\boldsymbol{N}})\frac{\bar{\boldsymbol{n}}^*}{\|\bar{\boldsymbol{N}}^*\|})\right\} \tag{44}$$

The tangential stretching $\boldsymbol{D}^t$ in Eq. 34 is formulated to be induced as R approaches closely to 1, i.e. as the stress approaches closely to the normal-yield surface so that the tangential stretching is hardly generated inside the normal-yield surface. In other words, the elastic property is not disturbed substantially by the tangential stretching. Needless to say, the plastic stretching formulated in the preceding section holds as it is. Thus, the constitutive Eq. 42 or Eq. 44 would not be required to be limited to the neighborhood of proportional loading process but would be applicable to a general loading process. Here, note that the tangential stretching $\boldsymbol{D}^t$ is linearly related to the stress rate so that it diminishes during an infinitesimal stress cycle, i. e. it is the so-called *work-less stretching* during that cycle. Thus, the tangential stretching has the intermediate property between the elastic stretching and the plastic one. Hence, let it be called the *para-inelastic stretching*.

The following stress rate is adopted in the J_2-deformation theory of Rudnicki and Rice [1], i. e. the rate form of the Henky's [14] deformation theory reassessed by Budiansky [15].

$$ {}^{R}\overset{\circ}{\boldsymbol{\sigma}} \equiv \overset{\circ}{\boldsymbol{\sigma}}^{*} - \mathrm{tr}\left(\frac{\boldsymbol{\sigma}^{*}}{\|\boldsymbol{\sigma}^{*}\|}\overset{\circ}{\boldsymbol{\sigma}}^{*}\right)\frac{\boldsymbol{\sigma}^{*}}{\|\boldsymbol{\sigma}^{*}\|} \tag{45} $$

The J_2-deformation theory has been applied to the analysis of plastic instability problems by many workers. However, this theory would not be pertinent generally in the following points.

1. The stress rate tensor ${}^{R}\overset{\circ}{\boldsymbol{\sigma}}^{*}$ generally results in

$$ \mathrm{tr}(\overline{\boldsymbol{N}}\,{}^{R}\overset{\circ}{\boldsymbol{\sigma}}^{*}) \neq 0, \quad \mathrm{tr}\,{}^{R}\overset{\circ}{\boldsymbol{\sigma}}^{*} = 0 \tag{46} $$

That is, ${}^{R}\overset{\circ}{\boldsymbol{\sigma}}^{*}$ is not tangential to the loading surface and thus an inverse expression, i. e. the analytical expression of a stress rate in terms of a stretching cannot be derived except for materials with a yield surface of axisymmetric shape centering on the hydrostatic axis in the principal stress space. For instance ${}^{R}\overset{\circ}{\boldsymbol{\sigma}}^{*}$ would not be applicable to the kinematic hardening materials ($\overline{\boldsymbol{n}}^{*} \neq \boldsymbol{\sigma}^{*}/\|\boldsymbol{\sigma}^{*}\|$). It would not be also adequate for frictional materials, i. e. plastically pressure-dependent materials the shape of yield surface of which is not axisymmetric in the principal stress space although it has been applied to frictional materials by the advocators Rudnicki and Rice [1] themselves, Dorris and Nemat-Nasser [16], Yatomi *et al.* [17], Vermeer [18], etc.

2. The J_2-deformation theory falls within the framework of the conventional plasticity with the conventional yield surface enclosing a purely elastic domain. Therefore, if the stretching due to the stress rate ${}^{R}\overset{\circ}{\boldsymbol{\sigma}}^{*}$ is assumed to occur always, the deformation behavior in the elastic domain is disturbed exhibiting unrealistic response, while that stretching is assumed to be significantly large compared with the elastic stretching. Instead, if that stretching is assumed to occur only in the yield state, it occurs discontinuously depending on whether or not a stress exist on the yield surface so that the continuity condition and the smoothness condition are violated.

Eventually, the application of the J_2-deformation theory has to be limited to the monotonic loading process in the neighborhood of proportional loading for materials with a yield surface of axisymmetric shape centering on the hydrostatic axis in the principal stress space.

4. CONCLUDING REMARKS

The elastoplastic constitutive Eq. 42 or Eq. 44 was proposed in this article, which describes the non-coaxiality of the stress and the inelastic stretching. In this equation the novel *para-inelastic stretching* induced by the tangential stress rate is introduced keeping a single smooth (regular) yield surface without incorporating plural yield surfaces or a corner of the yield surface. It could be regarded as the generalization of the Rudnicki and Rice's [1] J_2-deformation theory. It has the rather simple form and its application to boundary value problems would be easy compared with the phenomenological corner theory [19] which has been applied to analyses of plastic instability problems. Furthermore, the constitutive equation may be applicable to the general loading process including unloading, reloading and reverse loading.

REFERENCES

1. J. W. Rudnicki and J. R. Rice, *J. Mech. Phys. Solids* **23**, 371 (1975).
2. K. Hashiguchi and M. Ueno, in *Constitutive Equations of Soils (Proc. 9th Int. Conf. Soil Mech. Found. Eng., Spec. Session 9)*(eds., S. Murayama, S. and A. N. Schofield), p. 73, JSSMFE, Tokyo (1977).
3. K. Hashiguchi, *J. Appl. Mech. (ASME)* **47**, 266 (1980).
4. K. Hashiguchi, *Int. J. Solids Structures* **25**, 917 (1989).
5. K. Hashiguchi, *Int. J. Plasticity* **9**, 525 (1993).
6. K. Hashiguchi, *Int. J. Plasticity* **9**, 721 (1993).
7. K. Hashiguchi, *Int. J. Plasticity* **13**, 37 (1997).
8. Y. F. Dafalias, *J. Appl. Mech. (ASME)* **52**, 865 (1985).
9. H. M. Zbib and E. C. Aifantis, *Acta Mech.* **75**, 15 (1988).
10. K. Hashiguchi, and Z.-P. Chen, *Int. J. Numer. Anal. Meth. Geomech.* **22**, 197 (1998).
11. R. Hill, *J. Mech. Phys. Solids* **6**, 236 (1958).
12. R. Hill, in *Recent Progress in Appl. Mech. (The Folke Odquivist Volume)*, p. 241, John-Wiley & Sons, Chichester, (1967).
13. K. Hashiguchi, *Int. J. Plasticity* **10**, 871 (1994).
14. H. Hencky, *Z.A.M.M.* **4**, 323 (1924).
15. B. Budiansky, *J. Appl. Mech. (ASME)* **26**, 259 (1959).
16. S. Nemat-Nasser, *J. Appl. Mech. (ASME)* **50**, 1114 (1983).
17. P. Vermeer, in *Modern Approaches to Plasticity* (ed., D. Kolymbas), p. 71, Elsevier, Amsterdam, (1993).
18. C. Yatomi, A. Yashima, A. Iizuka and I. Sano, *Soils and Foundations* **29**, 41 (1989).
19. J. Christoffersen and J. W. Hutchinson, *J. Mech. Phys. Solids* **27**, 465 (1979).