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Harmonic maps into round cones and singularities of nematic liquid crystals

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1 Introduction

Here we study the behavior of energy minimizing harmonic maps into round cones, specifically, maps from a region $\Omega \subset \mathbb{R}^m$ into $\mathbb{C}_\kappa$ where, for $\kappa > 0$,

$$\mathbb{C}_\kappa = \mathbb{C}_\kappa^p = \{(x, y) \in \mathbb{R}^p \times \mathbb{R} : y = |\kappa - 1|^{\frac{1}{2}} |x|\},$$

with the induced metric from either the Euclidean metric of $\mathbb{R}^p \times \mathbb{R} \cong \mathbb{R}^{p+1}$ in case $\kappa \geqq 1$ or from the Minkowski metric of $\mathbb{R}^p \times \mathbb{R} \cong \mathbb{R}^{p,1}$ in case $0 < \kappa < 1$. Thus [L2] $\mathbb{C}_\kappa$ is a singular (for $\kappa \neq 1$) Riemannian submanifold which is positively curved for $\kappa > 1$ and negatively curved for $\kappa < 1$. For a discussion of harmonic maps into singular spaces, in particular, negatively curved spaces, see the important recent work [G-S] of Gromov and Schoen.

Our motivation for studying harmonic maps into such cones comes from an Ericksen model for nematic liquid crystals [E2, L1, H, L2, L4, V] where $m = 3$ and $p = 3$. Several results on partial regularity of energy minimizers have been discussed in [L1, H, Am, L2] with [L2] giving the most general results. Briefly, a minimizer u is Hölder continuous everywhere for all positive κ and Lipschitz for $0 < \kappa < 1$. It is moreover real analytic away from the set $u^{-1}\{0\}$ which has (except for the degenerate case $u \equiv 0$) Hausdorff dimension $\leqq m-2$ for $\kappa > 1$ and dimension $\leqq m-1$ for $\kappa \leqq 1$. Here we give one improvement (3.4) in showing that

$$\text{if } \kappa > 1 \text{ and } p \geqq 3,$$
$$\text{then } u^{-1}\{0\} \text{ has dimension } \leqq m-3 \text{ and is isolated for } m = 3.$$

We also give some discussion concerning the asymptotic behavior of a minimizer u near a point of $u^{-1}\{0\}$. As in [L3], and [Al], the monotonicity of frequency allows useful consideration of possible *homogeneous blow-ups* of v at points of Ω. Let $\mathbb{B}^m$ denote the unit ball in $\mathbb{R}^m$. In 3.2 we obtain a full classification of the possible homogeneous harmonic maps from $\mathbb{B}^2$ to $\mathbb{C}_\kappa$. By dimension

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reduction [L2] this is relevant for understanding the codimension one and codimension two singularities of energy-minimizing harmonic maps from $\mathbb{B}^m$ to $\mathbb{C}_\kappa$.

Note that, a mapping depending on only one variable given by a constant speed geodesic passing through the vertex of $\mathbb{C}_\kappa^2$ is minimizing if and only if $\kappa \leqq 1$. The nonminimality when $\kappa > 1$ may be established by a variation which peels the geodesic off the vertex. Here the first variation of energy is $-\infty$. We generalize this construction in 2.2 to show that,

for $\kappa > 1$, there is no nonconstant energy minimizing map from $\mathbb{B}^2$ to $\mathbb{C}_\kappa^3$ whose image hits 0 and is contained in a subcone $\mathbb{C}_\kappa^2$.

Combining this with the classification result 3.2 and standard dimension reduction arguments leads to the above improved dimension estimate.

In § 4 we discuss the use of cones in the Ericksen model of nematic liquid crystals. As described in [L1, H], one considers, for a region $\Omega \subset \mathbb{R}^3$, a map

$$u = (v, w) \colon \Omega \to \mathbb{C}_\kappa^3.$$

One may recover from this map the two relevant physical quantities of the Ericksen model, the *director field* $n = \frac{v}{|v|}$ (which is defined wherever $v \neq 0$) and the *orientation order parameter* $s = |v| = |\kappa - 1|^{-\frac{1}{2}} |w|$. In terms of these, the energy of u is given by

$$\int_\Omega |\nabla u|^2 \, dx + \int_\Omega W_0(\kappa^{-\frac{1}{2}} |u|) \, dx = \int_\Omega [s^2 |\nabla n|^2 + \kappa |\nabla s|^2] \, dx + \int_\Omega W_0(s) \, dx$$

where W_0 (see [E2, § 5–7]) is a positive smooth function which has a unique minimum in the interval (0, 1), has derivative 0 at 0, and has limit ∞ on approach to either endpoint $-\frac{1}{2}$ or 1. As discussed in [L2], the zero order term W_0 does not affect the estimates on the singularity set or the existence or structure of the homogeneous blowups. In particular, by [L2] and the present paper,

the zero set $u^{-1}\{0\}$ has dimension $\leqq 2$ for $0 < \kappa \leqq 1$ and is isolated for $\kappa > 1$.

Specific examples with $\kappa < 1$ of 2 dimensional singular sets have been obtained in [A-V]. Many physical examples of singularities of dimensions 0, 1, and 2 in liquid crystals and other mechanical systems are described in [K].

In § 4 we consider additional cones that are particularly useful for liquid crystal applications. Since it is usually not physically natural to distinguish the director n from $-n$, it is useful to consider the director lying in the real projective plane $\mathbb{RP}^2$. When the orientation order s is assumed constant, as in [H-K-L], there is, as noted in 4.1, little change in results. For the general Ericksen model, we consider a target $\tilde{V}_\kappa$ that is essentially a cone over real projective space. Fortunately, most of our present work, including the classification of homogeneous minimizers of 2 variables, carries over, with some adjustments. However, because $\mathbb{RP}^1$ is not contractible in $\mathbb{RP}^2$, the construction of 2.2 is not always applicable. In contrast to maps to $\mathbb{C}_\kappa^3$, we now obtain the weaker result that

> the singular set of an energy-minimizing map from $\mathbb{B}^3$ to $\tilde{V}_\kappa$ has dimension ≤ 1 for $\kappa > 1$.

This singularity estimate is nevertheless optimal, as seen in 4.2, and is in accord with most observed nematic liquid crystals. We appreciate the remarks of R. Meyers that one should expect line singularities in nematic liquid crystals even for $\kappa > 1$. We also thank J. Ericksen for his continual encouragement.

As a final remark, we recall that the mathematical derivation of the order parameter [E2, § 2] leads to the range $-\frac{1}{2} \leq s \leq 1$ for the order parameter. Thus it makes sense to allow negative s and consider maps into corresponding truncated double-cones X_κ and $\tilde{X}_\kappa$. With these cones, one has a homogeneouse degree 1 minimizer induced by a straight line passing through 0 from the upper cone to the lower cone. So the new singular set estimate

$$\dim u^{-1}\{0\} \leq 2,$$

which we now obtain for a nonconstant minimizing map u into X_κ or $\tilde{X}_\kappa$, is optimal for all p and κ. We also observe that use of the cone $\tilde{X}_\kappa$ overcomes the sign ambiguity, pointed out by Ericksen [E2, § 2], that arises with an $\mathbb{R}^3$-valued model based on the function $s \cdot n$.

A serious restriction on the applicability of the present work to nematic liquid crystals is the absence of a treatment of the general (non-equal-constant) Oseen-Frank energy functional. It is an open problem whether our estimates of the singular set continue to hold in such a general context.

2 Harmonic maps into subcones

In this section, we show that, for $\kappa > 1$, a nonsmooth energy minimizing map into $\mathbb{C}_\kappa^3$ *cannot* have image in the lower dimensional subcone $\mathbb{C}_\kappa^3 \cap (\mathbb{R}^2 \times \{0\} \times \mathbb{R}) \cong \mathbb{C}_\kappa^2$. To motivate this, consider the simple case of a piecewise constant-speed geodesic map u of the interval $[-1, 1]$ into $\mathbb{C}_\kappa^2$ with $u(0) = 0$ and $|u(-1)| = |u(+1)| \neq 0$. Here $u(t)$ is simply $-tu(-1)$ for $t < 0$ and $tu(+1)$ for $t \geq 0$. To be minimizing, even harmonic, the (minimum) angle between v_{-1} and v_{+1}, with respect to the $\mathbb{C}_\kappa^2$ metric must be at least π. (To see this geometrically simply unroll $\mathbb{C}_\kappa^2$ isometrically to a sector in the plane.) This means the Euclidean angle must be at least $\pi\kappa^{-\frac{1}{2}}$. Thus, for $\kappa > 1$, the curve fails to be minimizing.

In the following we use spherical coordinates (ρ, ω), $\rho = |x|$, $\omega = \dfrac{x}{|x|}$ for a point $x \in \mathbb{R}^m$.

2.1. Theorem. *Suppose $\kappa > 1$ and h is a finite energy map from $\mathbb{B}^m$ into the subcone $\mathbb{C}_\kappa^p \cap (\mathbb{R}^{p-1} \times \{0\} \times \mathbb{R}) \cong \mathbb{C}_\kappa^{p-1}$. Suppose also that h is homogeneous so that*

$$h(\rho, \omega) = \rho^\alpha(\psi(\omega), 0, \sqrt{\kappa - 1}\,|\psi(\omega)|)$$

for some $\alpha > 0$ and mapping $\psi : \mathbb{S}^{m-1} \to \mathbb{R}^{p-1}$ with $\psi \not\equiv 0$. Then under the variation $h^\lambda : \mathbb{B}^m \to \mathbb{C}_\kappa^p$,

$$h^\lambda(\rho, \omega) = (\rho^\alpha \psi, \lambda(1 - \rho^\alpha), \sqrt{\kappa - 1}\sqrt{\rho^{2\alpha}|\psi|^2 + \lambda^2(1 - \rho^\alpha)^2}),$$

the energy $E(\lambda)=\int_{\mathbb{B}}|\nabla h^\lambda|^2\,dx$ *is continuous. Moreover, if* $m\leqq 2$ *or if*

$$\frac{\kappa}{\kappa-1}<\frac{2\alpha+m-2}{m-2},$$

then E *has a local maximum at* 0; *in fact,*

$$\lim_{\lambda\to 0}\frac{[E(\lambda)-E(0)]}{\lambda^2}<0.$$

Proof. We compute, at almost every $x\in\mathbb{B}$, the partial derivatives

$$h^\lambda_{\omega_i}=\left(\rho^\alpha\psi_{\omega_i},0,\frac{\sqrt{\kappa-1}\,\rho^{2\alpha}\psi\cdot\psi_{\omega_i}}{\sqrt{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}}\right),$$

$$h^\lambda_\rho=\left(\alpha\rho^{\alpha-1}\psi,\ -\alpha\rho^{\alpha-1}\lambda,\frac{\sqrt{\kappa-1}\,(\alpha\rho^{2\alpha-1}|\psi|^2-\lambda^2(1-\rho^\alpha)\,\alpha\rho^{\alpha-1})}{\sqrt{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}}\right)$$

$$=\alpha\rho^{\alpha-1}\left(\psi,\ -\lambda,\frac{\sqrt{\kappa-1}\,(\rho^\alpha|\psi|^2-\lambda^2(1-\rho^\alpha))}{\sqrt{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}}\right)$$

where $\omega_1,\omega_2,\ldots,\omega_{m-1}$ are orthonormal vectors perpendicular to $\omega=\frac{x}{|x|}$. Then the energy integrand is

$$|\nabla(h^\lambda)|^2=\rho^{-2}\sum_i|h^\lambda_{\omega_i}|^2+(h^\lambda_\rho)^2$$

$$=\rho^{2\alpha-2}\Bigg[\sum_i|\psi_{\omega_i}|^2+\frac{(\kappa-1)\,\rho^{2\alpha}\sum_i|\psi\cdot\psi_{\omega_i}|^2}{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}$$

$$+\alpha^2\left(|\psi|^2+\lambda^2+\frac{(\kappa-1)(\rho^{2\alpha}|\psi|^4-2\lambda^2\rho^\alpha(1-\rho^\alpha)|\psi|^2+\lambda^4(1-\rho^\alpha)^2)}{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}\right)\Bigg].$$

Thus,

$$\frac{(|\nabla(h^\lambda)|^2-|\nabla(h^0)|^2)}{(\kappa-1)\,\alpha^2\rho^{2\alpha-2}}=\frac{\alpha^{-2}\rho^{2\alpha}\sum_i|\psi\cdot\psi_{\omega_i}|^2}{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}-\frac{\alpha^{-2}\sum_i|\psi\cdot\psi_{\omega_i}|^2}{|\psi|^2}+\frac{\lambda^2}{(\kappa-1)}$$

$$+\frac{(\rho^{2\alpha}|\psi|^4-2\lambda^2\rho^\alpha(1-\rho^\alpha)|\psi|^2+\lambda^4(1-\rho^\alpha)^2)}{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}-|\psi|^2$$

$$\leqq 0+\frac{\lambda^2}{(\kappa-1)}$$

$$+\frac{(-\lambda^2(1-\rho^\alpha)^2|\psi|^2-2\lambda^2\rho^\alpha(1-\rho^\alpha)|\psi|^2+\lambda^4(1-\rho^\alpha)^2)}{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}$$

$$=\frac{\lambda^4(1-\rho^\alpha)^2\left(\frac{\kappa}{(\kappa-1)}\right)+\lambda^2|\psi|^2\left(\frac{\kappa}{\kappa-1}\right)\rho^{2\alpha}-\lambda^2|\psi|^2}{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}$$

$$=\lambda^2(F+G)\quad\text{where}$$

$$F=\frac{\lambda^2(1-\rho^\alpha)^2\left(\frac{\kappa}{(\kappa-1)}\right)}{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2},\qquad G=\frac{|\psi|^2\left[\left(\frac{\kappa}{(\kappa-1)}\right)\rho^{2\alpha}-1\right]}{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}.$$

First note that by simply dropping the term $\rho^{2\alpha}|\psi|^2$ in F, we obtain the pointwise bound

$$F\cdot(\kappa-1)\,\alpha^2\rho^{2\alpha-2}\leqq\kappa\alpha^2\rho^{2\alpha-2}.$$

Decomposing G into its positive and negative parts, $G=G_+-G_-$ and dropping the term $\lambda^2(1-\rho^\alpha)^2$ in G_+, we similarly estimate

$$G_+\cdot(\kappa-1)\,\alpha^2\rho^{2\alpha-2}\leqq\kappa\alpha^2\rho^{2\alpha-2}.$$

Note that the function $\rho^{2\alpha-2}$ is integrable on $\mathbb{B}^m$ even for $m=1$ because in that case the finiteness of the energy of h requires that $\alpha>\frac{1}{2}$. From the inequality

$$\begin{aligned}|\nabla(h^\lambda)|^2&\leqq|\nabla(h^0)|^2+\lambda^2(F+G_+)\cdot(\kappa-1)\,\alpha^2\rho^{2\alpha-2}\\&\leqq|\nabla(h^0)|^2+2\lambda^2\kappa\alpha^2\rho^{2\alpha-2}\end{aligned}$$

and the dominated convergence theorem, we now deduce the continuity in λ of the energy $E(\lambda)$.

To estimate the quotient $\dfrac{[E(\lambda)-E(0)]}{\lambda^2}$, we treat separately two cases:

Case $m\leqq 2$. Here we estimate the G_- term from below by chosing ρ_0 so that $\left(\dfrac{\kappa}{(\kappa-1)}\right)\rho_0^{2\alpha}=\dfrac{1}{2}$ and checking that

$$\begin{aligned}&\int_{\mathbb{S}^{m-1}}\int_0^1 G_-\cdot(\kappa-1)\,\alpha^2\rho^{2\alpha-2}\rho^{m-1}\,d\rho\,d\theta\\&\geqq(\kappa-1)\,\alpha^2\int_{\mathbb{S}^{m-1}}\int_0^{\rho_0}\frac{|\psi|^2\cdot(\frac{1}{2})}{\rho^{2\alpha}|\psi|^2+\lambda^2(1-\rho^\alpha)^2}\,\rho^{2\alpha-2}\rho^{m-1}\,d\rho\,d\theta\end{aligned}$$

which approaches

$$\pi(\kappa-1)\,\alpha^2\int_0^{\rho_0}\rho^{m-3}\,d\rho=\infty\quad\text{as }\lambda\to 0$$

by monotone convergence. Thus for $m\leqq 2$,

$$\begin{aligned}\lim_{\lambda\to 0}\frac{[E(\lambda)-E(0)]}{\lambda^2}&=\lim_{\lambda\to 0}\lambda^{-2}\int_{\mathbb{S}^{m-1}}\int_0^1(|\nabla(h^\lambda)|^2-|\nabla(h^0)|^2)\,\rho^{m-1}\,d\rho\,d\theta\\&=\lim_{\lambda\to 0}\int_{\mathbb{S}^{m-1}}\int_0^1(F+G_+-G_-)\cdot(\kappa-1)\,\alpha^2\rho^{2\alpha-2}\rho^{m-1}\,d\rho\,d\theta=-\infty.\end{aligned}$$

Case $m\geqq 3$. Here, by dropping the term $\lambda^2(1-\rho^\alpha)^2$, we obtain another pointwise bound

$$|G|\cdot(\kappa-1)\,\alpha^2\rho^{2\alpha-2}\leqq\kappa\alpha^2\rho^{2\alpha-2}+(\kappa-1)\,\alpha^2\rho^{-2},$$

and now

$$\int_{\mathbb{S}^{m-1}} \int_0^1 (\kappa\alpha^2 \rho^{2\alpha-2} + (\kappa-1)\,\alpha^2 \rho^{-2})\,\rho^{m-1}\, d\rho\, d\omega < \infty$$

because $m \geqq 3$. We conclude, using dominated convergence and the hypothesis,

$$\begin{aligned}
\lim_{\lambda\to 0} \frac{[E(\lambda)-E(0)]}{\lambda^2} &= \lim_{\lambda\to 0} \int_{\mathbb{S}^{m-1}} \int_0^1 (F+G)\cdot(\kappa-1)\,\alpha^2 \rho^{2\alpha-2} \rho^{m-1}\, d\rho\, d\theta \\
&= \int_{\mathbb{S}^{m-1}} \int_0^1 \lim_{\lambda\to 0} F\cdot(\kappa-1)\,\alpha^2\rho^{2\alpha-2}\rho^{m-1}\,\mathrm{d}\rho\,\mathrm{d}\omega \\
&\quad + \int_{\mathbb{S}^{m-1}} \int_0^1 \lim_{\lambda\to 0} G\cdot(\kappa-1)\,\alpha^2\rho^{2\alpha-2}\rho^{m-1}\, d\rho\, d\omega \\
&= 0 + \alpha^2 \int_{\mathbb{S}^{m-1}} \int_0^1 (\kappa\rho^{2\alpha+m-3} - (\kappa-1)\,\rho^{m-3})\, d\rho\, d\omega \\
&= \alpha^2 \mathscr{H}^{m-1}(\mathbb{S}^{m-1}) \left(\frac{\kappa}{(2\alpha+m-2)} - \frac{(\kappa-1)}{(m-2)}\right) < 0. \quad \square
\end{aligned}$$

2.2. Corollary. *For $\kappa > 1$ and $p \geqq 2$, there is no nonsmooth energy minimizing map $u\colon \mathbb{B}^2 \to \mathbb{C}^p_\kappa$ with image in the subcone*

$$\mathbb{C}^p_\kappa \cap (\mathbb{R}^{p-1} \times \{0\} \times \mathbb{R}) \cong \mathbb{C}^{p-1}_\kappa.$$

Proof. For such a map u with singular point $\alpha \in \mathbb{B}^2$, one sees from [L2] that $u(a) = 0$ and any homogeneous blow-up h of u at a is a nonconstant homogeneous energy minimizing map from $\mathbb{B}^2$ to $\mathbb{C}^p_\kappa$ with image in the same subcone $\mathbb{C}^p_\kappa \cap (\mathbb{R}^{p-1} \times \{0\} \times \mathbb{R})$. But then one may use the variation h^λ of 2.1 to contradict the energy minimality of h. $\square$

3 Homogeneous energy minimizing maps

The dimension reduction argument for estimating the size of the singular set of energy minimizing maps gives interest to the problem of classifying all nonconstant homogeneous energy minimizing maps from $\mathbb{B}^m$ to $\mathbb{C}^p_\kappa$. As discussed above in § 2, the answer is simple for $m = 1$. Here the order α of homogeneity is necessarily one, and the map is given by two constant equal-speed geodesic rays through 0 whose angle, with respect to the $\mathbb{C}^p_\kappa$ metric must be at least π. So necessarily $\kappa \leqq 1$ with equality only if the rays form a straight line.

The case $m = 2$ offers many new possibilities described in 3.2 below. A useful tool for the study of energy minimizing maps (or more generally stationary [S]) harmonic maps is the holomorphic quadratic differential.

3.1. Lemma. *Suppose $h\colon \mathbb{B}^2 \to \mathbb{C}^p_\kappa$ is an energy minimizing map. Then the local coordinate expression*

$$[(|h_x||^2 - |h_y||^2) - 2i(h_x \cdot h_y)]\, dx \otimes dy$$

defines a holomorphic quadratic differential ω^h *on* $\mathbb{B}^2$.

Proof. One may follow the proof of the first part of [J], and note that only smooth variations of the parameter domain are used. The vanishing of the first variation of energy with respect to such variations proves the holomorphicity. □

3.2. Theorem (Classification of homogeneous minimizers from $\mathbb{R}^2$ to $\mathbb{C}_\kappa$). *Suppose $h(r,\theta)=r^\alpha\varphi(\theta)$ is a nonconstant homogeneous energy minimizing map from $\mathbb{R}^2$ to $\mathbb{C}_\kappa=\mathbb{C}_\kappa^p$ where $\alpha>0$ and $\varphi:\mathbb{R}\to\mathbb{C}_\kappa$ is absolutely continuous and 2π periodic. Then*

$$\varphi=(\psi,|\kappa-1|^{\frac{1}{2}}|\psi|) \quad \text{where } \psi\colon \mathbb{R}\to\mathbb{R}^p$$

has, after suitable orthogonal changes of coordinates in domain and range, one of the following forms:

(1) *(constant length)*

With $\kappa>0$, $\alpha=m\kappa^{-\frac{1}{2}}$ for some $m\in\{1,2,\ldots)$ and $\lambda>0$ such that $\kappa\leqq 1$ for $p\geqq 3$.

$$\psi(\theta)=\lambda(\cos m\theta, \sin m\theta, 0, \ldots, 0).$$

(2) *(piece-wise constant direction)*

With $0<\kappa\leqq 1$, $\alpha=\frac{1}{2}m$ for some $m\in\{2,3,\ldots\}$, and $\lambda>0$,

$$\psi(\theta)=\lambda(\sin\tfrac{1}{2}m(\theta-\theta_\ell))\,v_\ell \quad \text{on} \quad [\theta_\ell,\theta_{\ell+1}] \quad \text{for } \ell=1,2,\ldots,m,$$

where $\theta_\ell=\dfrac{2\pi(\ell-1)}{m}$ and the v_ℓ's are unit vectors in $\mathbb{R}^p$ such that the (Euclidean) angles between consecutive vectors $v_1, v_2, v_3, \ldots, v_m, v_1$ are all at least $\pi\kappa^{\frac{1}{2}}$.

(3) *(varying lenght, varying direction)*

With $\kappa=\dfrac{2\ell}{m}$, $\alpha=\frac{1}{2}m$ for some $\ell\in\{1,2,\ldots\}$, $m\in\{2,3,\ldots\}$, and a, b, $c\in\mathbb{R}$ such that $\kappa\leqq 1$ for $p\geqq 3$,

$$P\circ\psi(\theta)=(a\cos\tfrac{1}{2}m\theta, b\cos\tfrac{1}{2}m\theta+c\sin\tfrac{1}{2}m\theta, 0, \ldots, 0)$$

where $P(r\cos\theta, r\sin\theta, x_3, \ldots, x_p)=(r\cos\kappa^{-1}\theta, r\sin\kappa^{-1}\theta, x_3, \ldots, x_p)$ for $r>0$, $\theta\in(-\pi,\pi)$, and $(x_3,\ldots,x_p)\in\mathbb{R}^{p-3}$.

Proof. First observe that the condition $\kappa\leqq 1$ for $p\geqq 3$ in conclusions (1) and (3) is required by 2.2. Next we derive some differential equations. Since h is smooth away from $h^{-1}\{0\}$ and $|\varphi|^2=\kappa|\psi|^2$, we may differentiate φ near every point θ_0 with $\varphi(\theta_0)\neq 0$ to find the relations

$$(*) \qquad \varphi_\theta=\left(\psi_\theta, |\kappa-1|^{\frac{1}{2}}\frac{\psi\cdot\psi_\theta}{|\psi|}\right), \qquad \varphi\cdot\varphi_\theta=\kappa\psi\cdot\psi_\theta$$

where we use subscripts to denote partial differentiation. Near such a point we may derive the Euler-Lagrange equation for ψ by considering a variation

$$h^t=r^\alpha(\psi^t,|\kappa-1|^{\frac{1}{2}}|\psi^t|)$$

with $\psi^t(r,\theta)=\psi(\theta)+t\zeta(r,\theta)$ where ζ is smooth, 2π periodic in θ,

$$R=\sup\{r:\zeta(r,\theta)\neq 0 \text{ for some } \theta\}<\infty,$$

and $|t|$ and spt ζ are small enough so that ψ_t does not vanish on spt ζ.

Then

$$h_r^t=\left(\alpha r^{\alpha-1}\psi^t+r^\alpha t\zeta_r,\ \alpha r^{\alpha-1}|\kappa-1|^{\frac12}|\psi^t|+r^\alpha|\kappa-1|^{\frac12}\psi^t\cdot\frac{t\zeta_r}{|\psi^t|}\right),$$

$$h_\theta^t=r^\alpha(\psi_\theta^t,|\kappa-1|^{\frac12}(\psi^t\cdot\psi_\theta^t)/|\psi^t|),$$

$$\left(\frac{|h_\theta^t|}{r}\right)^2=r^{2\alpha-2}\left(|\psi_\theta^t|^2+|\kappa-1|\frac{(\psi^t\cdot\psi_\theta^t)^2}{|\psi^t|^2}\right),$$

$$\begin{aligned}|h_r^t|^2=&\alpha^2r^{2\alpha-2}|\psi^t|^2+r^{2\alpha}|t\zeta_r|^2+2\alpha r^{2\alpha-1}(\psi^t\cdot t\zeta_r)\\&+\alpha^2r^{2\alpha-2}|\kappa-1||\psi^t|^2+r^{2\alpha}|\kappa-1|\frac{(\psi^t\cdot t\zeta_r)^2}{|\psi^t|^2}\\&+2\alpha r^{2\alpha-1}|\kappa-1|(\psi^t\cdot t\zeta_r).\end{aligned}$$

By the energy minimality of $h=h^0$, we find that

$$\begin{aligned}0=&\frac{d}{dt}\bigg|_{t=0}\frac12\int_{\mathbb{B}_R}|\nabla h^t|^2\,dx=\frac{d}{dt}\bigg|_{t=0}\frac12\int_{\mathbb{B}_R}\left[(h_r^t)^2+\left(\frac{h_\theta^t}{r}\right)^2\right]dx\\
=&\int_{\mathbb{B}_R}\bigg[\alpha^2r^{2\alpha-2}\psi\cdot\zeta+\alpha r^{2\alpha-1}\psi\cdot\zeta_r+\alpha^2r^{2\alpha-2}|\kappa-1|\,\psi\cdot\zeta+\alpha r^{2\alpha-1}|\kappa-1|\,\psi\cdot\zeta_r\\
&+r^{2\alpha-2}\psi_\theta\cdot\zeta_\theta+r^{2\alpha-2}|\kappa-1|\frac{(\psi\cdot\psi_\theta)(\psi\theta\cdot\zeta+\psi\cdot\zeta_\theta)}{|\psi|^2}\\
&-r^{2\alpha-2}|\kappa-1|\left(\frac{(\psi\cdot\psi_\theta)}{|\psi|^2}\right)^2\psi\cdot\zeta\bigg]dx\\
=&\int_0^{2\pi}\int_0^R\bigg[\tilde\kappa\alpha^2r^{2\alpha-1}\psi\cdot\zeta+\tilde\kappa\alpha r^{2\alpha}\psi\cdot\zeta_r\\
&+r^{2\alpha-1}\psi_\theta\cdot\zeta_\theta+r^{2\alpha-1}|\kappa-1|\frac{(\psi\cdot\psi_\theta)(\psi_\theta\cdot\zeta)}{|\psi|^2}\\
&+r^{2\alpha-1}|\kappa-1|\frac{(\psi\cdot\psi_\theta)(\psi\cdot\zeta_\theta)}{|\psi|^2}\\
&-r^{2\alpha-1}|\kappa-1|\left(\frac{(\psi\cdot\psi_\theta)^2}{|\psi|^4}\right)\psi\cdot\zeta\bigg]dr\,d\theta\end{aligned}$$

where $\tilde\kappa=1+|\kappa-1|$. Noting that $\psi_r\equiv 0$ and letting $Q=\frac{\psi\cdot\psi_\theta}{|\psi|^2}$, we deduce that the weak equation

$$(**)\qquad \psi_{\theta\theta}+[\tilde\kappa\alpha^2+|\kappa-1|\,Q_\theta+|\kappa-1|\,Q^2]\,\psi=0$$

holds on any interval on which ψ does not vanish.

Fortunately we may simplify this ODE by considering the holomorphic quadratic differential ω^h associated with h as in the 3.1. Here ω^h equals $f(z)\,dz^2$ for some entire function f. Clearly f has the form $r^{2\alpha-2}\Phi(\theta)$. In particular,

either $\omega^h \equiv 0$
or $\alpha = \frac{1}{2}m$ and $f(z) = Cz^{2\alpha-2}$ for some $m \in \{2, 3, \ldots\}$ and $C \in \mathbb{C} \sim \{0\}$.

First we consider the case $\omega^h \equiv 0$. Here we differentiate to find that

$$h_r = \alpha r^{\alpha-1}\varphi, \qquad h_\theta = r^\alpha \varphi_\theta,$$

and

$$\begin{aligned}(\alpha r^{2\alpha-2})\tfrac{1}{2}(|\varphi|^2)_\theta &= \alpha r^{2\alpha-2}(\varphi\cdot\varphi_\theta)\\ &= h_r\cdot\left(\frac{h_\theta}{r}\right) = (h_x \cos\theta + h_y \sin\theta)\cdot(-h_x \sin\theta + h_y\cos\theta)\\ &= (-|h_x|^2 + |h_y|^2)\cos\theta\sin\theta + (h_x\cdot h_y)(\cos^2\theta - \sin^2\theta) \equiv 0.\end{aligned}$$

Thus, $\frac{1}{2}(|\psi|^2)_\theta = \psi\cdot\psi_\theta = \kappa^{-1}(\varphi\cdot\varphi_\theta) \equiv 0$, $|\psi|$ is a positive constant λ, $Q \equiv 0$, and the ODE(**) becomes simply

$$\psi_{\theta\theta} + \tilde{\kappa}\alpha^2 = 0$$

which is now valid for all θ. Since ψ is periodic of period 2π,

$$\tilde{\kappa}\alpha^2 = m^2$$

for some integer m. Moreover, the vector ψ maps into a 2 dimensional subspace. To check this, we may choose a nonzero mutually orthogonal vectors $v_1, v_2, \ldots, v_{p-2} \in \mathbb{R}^p$ with $v_i\cdot\psi(0) = 0 = v_i\cdot\psi_\theta(0)$, observe that each scalar functions $y = v_i\cdot\psi$ is identically zero because

$$y_{\theta\theta} + m^2 y = 0 \quad \text{with } y(0) = 0 = y_\theta(0),$$

and conclude that ψ has image in the 2 plane

$$V = \{x \in \mathbb{R}^p : x\cdot v_i = 0 \text{ for } i = 1, 2, \ldots, p-2\}.$$

After rotating $\mathbb{R}^p$ to make $V = \mathbb{R}^2 \times \{0\}$, we infer from the simplified ODE that

$$\psi = (\alpha \cos m(\theta - \gamma), b\cos m(\theta - \delta), 0, \ldots, 0)$$

for some constants a, b, γ, δ. The condition $|\psi| \equiv \lambda$ gives the form

$$\psi = \lambda(\cos m(\theta - \gamma), \sin m(\theta - \gamma), 0, \ldots, 0),$$

and we may rotate the domain to make $\gamma = 0$. Thus we obtain (1).

Now we turn to the case where

$$\omega^h \not\equiv 0$$

so that

$$\omega^h = r^{2\alpha-2}\Phi(\theta)\,dz^2 = \lambda e^{i\beta} z^{2\alpha-2}\,dz^2$$

where $\lambda>0$, $\beta\in\mathbb{R}$, and $\alpha=\frac{1}{2}m$ for some $m\in\{2,3,\dots\}$; hence,

$$\Phi(\theta)=\lambda(\cos(\beta+2\alpha\theta-2\theta)+i\sin(\beta+2\alpha\theta-2\theta)).$$

We may rotate the domain to make $\beta=0$. Returning to the definition of ω^h, we compute $\Phi(\theta)$ in terms of α and $\varphi(\theta)$. Since $r^2=x^2+y^2$ and $\tan\theta=\frac{y}{x}$, we see that

$$\begin{aligned}
h_x&=h_r\cdot r_x+h_\theta\cdot\theta_x=h_r\cdot\left(\frac{x}{r}\right)+h_\theta\cdot\left(\frac{-y}{r^2}\right),\\
h_y&=h_r\cdot r_y+h_\theta\cdot\theta_y=h_r\cdot\left(\frac{y}{r}\right)+h_\theta\cdot\left(\frac{x}{r^2}\right),\\
f&=(|h_x|^2-|h_y|^2)-2i(h_x\cdot h_y)\\
&=\left[h_r^2-\left(\frac{h_\theta}{r}\right)^2\right]\left[\frac{(x^2-y^2)}{r^2}\right]-4h_r\left(\frac{h_\theta}{r}\right)\left(\frac{xy}{r^2}\right)\\
&\quad-2i\left(\left[h_r^2-\left(\frac{h_\theta}{r}\right)^2\right]\left(\frac{xy}{r^2}\right)+h_r\left(\frac{h_\theta}{r}\right)\left[\frac{(x^2-y^2)}{r^2}\right]\right)\\
&=\left[h_r^2-\left(\frac{h_\theta}{r}\right)^2\right]\cos 2\theta-2h_r\left(\frac{h_\theta}{r}\right)\sin 2\theta\\
&\quad-i\left(\left[h_r^2-\left(\frac{h_\theta}{r}\right)^2\right]\sin 2\theta+2h_r\left(\frac{h_\theta}{r}\right)\cos 2\theta\right).
\end{aligned}$$

On the other hand, the sum formulas show that

$$\begin{aligned}
f=\lambda(&r^{2\alpha-2}\cos 2\alpha\theta\cos 2\theta+r^{2\alpha-2}\sin 2\alpha\theta\sin 2\theta)\\
&-i(r^{2\alpha-2}\cos 2\alpha\theta\sin 2\theta-r^{2\alpha-2}\sin 2\alpha\theta\cos 2\theta).
\end{aligned}$$

Equating the real and imaginary parts, we conclude that the two vectors

$$\left(h_r^2-\left(\frac{h_\theta}{r}\right)^2,\ 2h_r\left(\frac{h_\theta}{r}\right)\right)\quad\text{and}\quad(\lambda r^{2\alpha-2}\cos 2\alpha\theta,\lambda r^{2\alpha-2}\sin 2\alpha\theta)$$

are equal because they coincide after rotation by 2θ. We obtain the two formulas

$$\alpha^2|\varphi|^2-|\varphi_\theta|^2=r^{2-2\alpha}\left[h_r^2-\left(\frac{h_\theta}{r}\right)^2\right]=\lambda\cos 2\alpha\theta,$$

$$2\alpha\varphi\cdot\varphi_\theta=2r^{2-2\alpha}h_r\left(\frac{h_\theta}{r}\right)=-\lambda\sin 2\alpha\theta.$$

Integrating the second one gives

$$|\varphi|^2=\tfrac{1}{2}\alpha^{-2}\lambda(c+\cos 2\alpha\theta)$$

for some constant $c\geqq 1$, and substituting into the first gives

$$|\varphi_\theta|^2=\tfrac{1}{2}\lambda(c-\cos 2\alpha\theta).$$

We now use $(*)$ to compute

$$Q=\frac{(-\alpha\sin 2\alpha\theta)}{(c+\cos 2\alpha\theta)},$$
$$Q_\theta=\frac{(-2\alpha^2)(1+c\cos 2\alpha\theta)}{(c+\cos 2\alpha\theta)^2}.$$

Substituting into $(**)$ we find that

$$(***)\qquad \psi_{\theta\theta}+\alpha^2\left[\tilde{\kappa}-\frac{|\kappa-1|(-\sin^2 2\alpha\theta+2+2c\cos 2\alpha\theta)}{(c+\cos 2\alpha\theta)^2}\right]\psi=0$$

which is valid on $\mathbb{R}\sim\psi^{-1}\{0\}$; that is, whenever $c+\cos 2\alpha\theta$ is positive.

Rather than try to find suitable solutions of $(***)$ for various c, we first argue by the linearity of equation $(***)$ as before to see that,

for each component I of $\mathbb{R}\sim\psi^{-1}\{0\}$, $\psi(I)$ always lies in some two dimensional subspace V_I of $\mathbb{R}^p$.

We now consider the two possible cases:

Case 1 $\psi\wedge\psi'\equiv 0$ on every component I of $\mathbb{R}\sim\psi^{-1}\{0\}$.

Here, we let $v_I=\dfrac{\psi(t_I)}{|\psi(t_I)|}$ for some fixed point $t_I\in I$. Then $\psi\wedge v_I\equiv 0$ on I because $\psi\wedge v_I$ is a solution of $(***)$ with the initial conditions

$$(\psi\wedge v_I)(t_I)=\psi(t_I)\wedge\psi(t_I)=0 \quad\text{and}\quad (\psi\wedge v_I)'(t_I)=\psi'(t_I)\wedge\psi(t_I)=0.$$

Thus, on I, $\psi=g_I\,v_I$ for some positive function g_I and

$$h=r^\alpha\varphi(\theta)=r^\alpha g_I(\theta)(v_I,|\kappa-1|^{\frac{1}{2}})$$

is a harmonic mapping into the linear span of the vector $(v_I,|\kappa-1|^{\frac{1}{2}})$.

First, it follows that ψ must vanish somewhere. Otherwise, $I=\mathbb{R}$ and $r^\alpha g_I(\theta)$ would be a real-valued harmonic function on the plane. This implies that the function g_I must be of the form $\lambda_I\sin\alpha(\theta-\theta_I)$ so that g_I, and hence ψ, vanishes somewhere.

We now see that I has the form (θ_I,θ_I+d_I) where $0\leqq\theta_I<\theta_I+d_I\leqq 2\pi$ and $\psi(\theta_I)=0=\psi(\theta_I+d_I)$. Then the rescaled function

$$H_I(r,\theta)=r^{\left(\frac{d_I}{\pi}\right)\alpha}\,g_I\left(\left(\frac{d_I}{\pi}\right)\theta+\theta_I\right)$$

is a positive harmonic function on the upper half plane with boundary values 0 along the X-axis. By odd reflection about the X-axis, H_I extends to a harmonic function on the plane. Reasoning as above,

$$H_I(r,\theta)=\lambda_I\,r^{\left(\frac{d_I}{\pi}\right)\alpha}\sin\left(\frac{d_I}{\pi}\right)\alpha\theta$$

where, by the positivity in the upper half plane, $\lambda_I > 0$ and $\left(\frac{d_I}{\pi}\right)\alpha = 1$. In particular, $d_I = \frac{\pi}{\alpha}$, being independent of I, must exactly divide 2π by the 2π periodicity of ψ. Thus,

$$d_I = \frac{2\pi}{m} \quad \text{and} \quad \alpha = \tfrac{1}{2}m \qquad \text{for some positive integer } m,$$

and we deduce that

$$\psi(\theta) = \lambda_I (\sin \tfrac{1}{2} m(\theta - \theta_I))\, v_I \quad \text{on } I.$$

Since the holomorphic quadratic differential then satisfies

$$|\omega^h| = \left(\frac{m^2}{4}\right) \lambda_I^2\, r^{m-2} \quad \text{on} \quad \{(r,\theta): r > 0 \text{ and } \theta \in I\},$$

we see that $\lambda = \lambda_I$ is also independent of I. These conclusions remain true under the rotation of domain. So we can forget our initial normalizing rotation to achieve $\beta = 0$ and again rotate the domain to get the intervals I to be in the form

$$I_\ell = \left(\frac{2\pi(\ell - 1)}{m}, \frac{2\pi\ell}{m}\right) \qquad \text{for } \ell = 1, 2, \ldots, m.$$

We now need to verify the necessity and sufficiency of the lower bound on the angles between the consecutive $v_\ell = v_{I_\ell}$ as described in (2).

For this we will use the local fact that the *composition of a geodesic curve with a real-valued harmonic function is a harmonic map.* This is easily verified for maps into a smooth Riemannian manifold. For the present context of maps into the cone $\mathbb{C}_\kappa$ we may, for example, use the smooth approximation of [L2]. Since each generating ray of $\mathbb{C}_\kappa$ is a geodesic in $\mathbb{C}_\kappa$, it only remains to consider the harmonicity of $h(r,\theta)$ near every point

$$\left(r, \frac{2\pi\ell}{m}\right) \qquad \text{for } r \geqq 0 \quad \text{and} \quad \ell = 1, 2, \ldots, m.$$

For $r > 0$ this is simply a question of whether each union of two rays generated by consecutive vectors v_ℓ, $v_{\ell+1}$ forms a geodesic in $\mathbb{C}_\kappa$ for $\ell = 1, 2, \ldots, m$ (where we set $v_{m+1} = v_0$ for notational convenience). This is true if and only if the rays meet at an angle $\geqq \pi$, where the angle is measured with respect to the metric induced from the Euclidean metric of $\mathbb{R}^4$ in case $\kappa \geqq 1$ or the Minkowski metric of $\mathbb{R}^3 \times \mathbb{R} \cong \mathbb{R}^{3,1}$ in case $\kappa \leqq 1$. This is the case if and only if the Euclidean angle between the rays is at least $\pi\kappa^{\frac{1}{2}}$. Under these angle conditions $m \geqq 2$ (because the angle between v_1 and itself is 0) and $\kappa \leqq 1$. The case $\kappa = 1$ only occurs when $m = 2$ and $v_2 = -v_1$. To complete the proof of case 1, we need to verify the harmonicity at 0 of an h satisfying the above conditions. We'll show that h actually minimizes energy among maps having the same boundary values on the unit circle. For this we'll use the fact that each geodesic formed from a pair v_ℓ, $v_{\ell+1}$ minimizes energy. Now consider a competing map $g\in H^1(\mathbb{B}^2, \mathbb{C}_\kappa)$ with $g|\partial\mathbb{B}^2 = h|\partial\mathbb{B}^2$. Note that h maps each sector

$$D_\ell = \left\{(r,\theta): 0<r<1, \theta - \frac{\pi}{2m} \in I_\ell\right\} \quad \text{for } \ell = 1, 2, \ldots, m$$

into the geodesic formed by the 2 rays spanned by v_ℓ, $v_{\ell+1}$. Let D denote the unit upper half ball in $\mathbb{R}^2$, and $f_\ell: D \to D_\ell$ be the associated conformal homeomorphism which maps, in polar coordinates,

$$(r,\theta) \quad \text{to} \quad \left(r^{\frac{2}{m}}, \left(\frac{2}{m}\right)\theta + \theta_\ell + \left(\frac{\pi}{2m}\right)\right).$$

Then $(g\circ f_\ell)|\mathbb{S}^1 \cap \partial D = (h\circ f_\ell)|\,\mathbb{S}^1 \cap \partial D$. Moreover, we readily compute that, in rectangular coordinates on D,

$$\begin{aligned} (h\circ f_\ell)(x_1, x_2) &= (\lambda|x_1|)(-v_\ell) && \text{for } x_1 \leqq 0, \\ (h\circ f_\ell)(x_1, x_2) &= (\lambda|x_1|)(v_{\ell+1}) && \text{for } x_1 \geqq 0. \end{aligned}$$

Thus, for each fixed x_2 with $0<x_2<1$, the map $(h\circ f_\ell)(\cdot, x_2)$ is a constant speed energy minimizing geodesic in $\mathbb{C}_\kappa$. Using this minimality along with the conformal invariance of energy, we conclude

$$\begin{aligned} \int_{D_\ell} |\nabla h|^2\,dx &= \int_D |\nabla(h\circ f_\ell)|^2\,dx \\ &= \int_0^1 \int_{-\sqrt{1-x_2^2}}^{+\sqrt{1+x_2^2}} \left|\frac{\partial(h\circ f_\ell)}{\partial x_1}(x_1,x_2)\right|^2 dx_1\,dx_2 \\ &\leqq \int_0^1 \int_{-\sqrt{1-x_2^2}}^{\sqrt{1-x_2^2}} \left|\frac{\partial(g\circ f_\ell)}{\partial x_1}(x_1,x_2)\right|^2 dx_1\,dx_2 \\ &\leqq \int_D |\nabla(g\circ f_\ell)|^2\,dx = \int_{D_\ell} |\nabla g|^2\,dx. \end{aligned}$$

Summing in ℓ gives the desired energy minimality.

Case 2 $(\psi \wedge \psi')(\theta_0) \neq 0$ at some point $\theta_0 \in [0, 2\pi)$.

Here the linear span of $\psi(\theta_0)$ and $(\psi')(\theta_0)$ is the two dimensional space V_I which contains $\psi(I)$ where I is the component of $\mathbb{R} \sim \psi^{-1}\{0\}$ that contains θ_0. Rotating $\mathbb{C}_\kappa$, we assume that $V_I = \mathbb{R}^2 \times \{0\}$. Note that h now has image in the 2-dimensional cone $\mathbb{C}_\kappa^2 \times \{0\}$, for which we have the local isometry into $\mathbb{R}^2$ given by simply rolling along the rays of the cone. That is, near each ray

$$R_0 = \{(r\cos\eta_0, r\sin\eta_0, |\kappa - 1|^{\frac{1}{2}} r, 0, \ldots, 0): r>0\} \subset \mathbb{C}_\kappa^2 \times \{0\},$$

the formula

$$\Pi(r\cos\theta, r\sin\theta, |\kappa-1|^{\frac{1}{2}} r, 0, \ldots, 0) = (\kappa r \cos\kappa^{-1}(\theta - \eta_0), \kappa r \sin \kappa^{-1}(\theta - \eta_0))$$

defines a local isometry of $\mathbb{C}_\kappa^2\times\{0\}$ into $\mathbb{R}^2$. So, near any ray Q_0 of $h^{-1}(R_0)$, $\Pi\circ h$ is a homogeneous-degree-α. $\mathbb{R}^2$-valued harmonic function. Thus $\Pi\circ h$ has the form

$$(Ar^\alpha\cos\alpha(\theta-\theta_1), Br^\alpha\cos\alpha(\theta-\theta_2))$$

for some real A, B and θ_1, $\theta_2\in[0,2\pi)$. Assuming Q_0, has argument in I, we see that A and B are nonzero and that $\theta_1-\theta_2$ is not a multiple of π. The cosine sum formula shows that the image under $\Pi\circ h$ of an arc of $\mathbb{S}^1$ lies in a nondegenerate, noncircular, ellipse centered at the origin. In particular, by analytic continuation, $|\psi|$ never vanishes and $I=\mathbb{R}$. Let ℓ be the abolute value of the winding number of $\psi(\theta)$ about the origin, as θ varies over a single period. Then $\ell\neq 0$ and

$$\kappa^{-1}\ell=\alpha=\tfrac{1}{2}m \qquad \text{for some } m\in\{2,3,\ldots\}.$$

Rotating the domain and using the cosine sum formula gives the form in conclusion (3).

Finally we verify that each of the homogeneous harmonic maps h given in 3.2(1) (2) (3) is, in fact, energy minimizing. In case $\kappa\leq 1$, $\mathbb{C}_\kappa$ has negative curvature, and the argument of [H-K-W] implies that $h|\mathbb{B}_r$ is, for every $r>0$, the unique harmonic map into $\mathbb{C}_\kappa$ with respect to its Dirichlet boundary data.

Suppose now that $\kappa>1$. Then, in accord with 3.2(1) (3) and 2.1, we only have the case $p=2$. That is, $\mathbb{C}_\kappa^2$ is an ordinary "ice-cream cone" which we may, as above, slit open and map isometrically into sector in the plane via the projection

$$\Pi(r\cos\theta, r\sin\theta, (\kappa-1)^{\frac{1}{2}}r)=(\kappa r\cos\kappa^{-1}\theta, \kappa r\sin\kappa^{-1}\theta).$$

In case, κ is rational, say $\kappa=\frac{i}{j}$ for positive integers i, j, we consider $H=\Pi\circ h\circ z^j$, which is defined in the sector $0\leq\theta\leq\frac{2\pi}{j}$. For the second sector $\frac{2\pi}{j}\leq\theta<\frac{4\pi}{j}$, we define $H(r,\theta)$ by rotating $H\left(r,\theta-\frac{2\pi}{j}\right)$ through an angle of $\frac{2\pi i}{j}$. After j such extensions, we close up and obtain a continuous homogeneous harmonic map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Since, for each $r>0$, $H|\mathbb{B}_r$ is unique with respect to its Dirichlet boundary values, so is $h|\mathbb{B}_r$.

Finally suppose that $\kappa>1$ is irrational and, for contradiction, that one of the homogeneous functions $h\colon\mathbb{R}^2\to\mathbb{C}_\kappa^2$ from 3.2(1) or 3.2(3) were not energy minimizing. Then, there would be an $r>0$, and a function $g|\mathbb{B}_r\to\mathbb{C}_\kappa^2$ with $g|\mathbb{B}_r=h|\partial\mathbb{B}_r$ giving the strict inequality

$$\int_{\mathbb{B}_r}|\nabla g|^2\,dx<\int_{\mathbb{B}_r}|\nabla h|^2\,dx.$$

For any rational $\mu>1$, let $h_\mu:\mathbb{R}^2\to\mathbb{C}_\mu^2$ be the corresponding homogeneous function from 3.2(1) or 3.2(3). Then, for μ sufficiently close to κ,

$$\int_{\mathbb{B}_r}|\nabla(\eta_\mu\circ g)|^2\,dx<\int_{\mathbb{B}_r}|\nabla h_\mu|^2\,dx,$$

where $\eta_\mu(x,y,z)=\left(x,y,\left(\frac{|\mu-1|^{\frac{1}{2}}}{|\kappa-1|^{\frac{1}{2}}}\right)z\right)$. This contradicts the previous paragraph and completes the proof. □

3.3. Corollary. *There are no nonconstant homogeneous energy minimizing maps from $\mathbb{B}^2$ to $\mathbb{C}_\kappa^p$ if $\kappa>1$ and $p\geqq 3$.*

Proof. Theorem 3.2 implies that such a map would have image in a subcone $\mathbb{C}_\kappa^2$. But this would contradict 2.2. □

3.4. Corollary. *If $\kappa>1$, $p\geqq 3$, and u is a nonconstant energy minimizing map from $\mathbb{B}^m$ to $\mathbb{C}_\kappa^p$, then $u^{-1}\{0\}$ has Hausdorff dimension $\leqq m-3$ and is isolated for $m=3$.*

Proof. This follows from 3.3 by Federer's dimension reduction argument as in [L2]. □

3.5. Remarks. There remains the interesting question of classifying homogeneous harmonic maps into $\mathbb{C}_\kappa^p$ of 3 or more variables. Of particular interest for liquid crystals is the case of a homogeneous harmonic map

$$u:\ \mathbb{B}^3\to\mathbb{C}_\kappa^3,\qquad u(\rho\omega)=\rho^\alpha\varphi(\omega)\quad\text{where } \alpha>0 \quad\text{and}\quad \varphi:\ \mathbb{S}^2\to\mathbb{C}_\kappa^3.$$

To discuss this, we use the variables

$$s(\rho\omega)=\rho^\alpha|\varphi(\omega)|,\qquad n(\omega)=\frac{\varphi(\omega)}{|\varphi(\omega)|}.$$

As derived in [L2] we obtain the Euler-Lagrange system

$$\begin{aligned}\kappa\Delta s-s|\nabla n|^2&=0,\\ s^2\Delta n+2s\nabla s\cdot\nabla n+n|\nabla n|^2s^2&=0.\end{aligned}$$

In case $|\varphi|$ is a nonzero constant, the second equation becomes the equation for a harmonic map from $\mathbb{S}^2$ to itself:

$$\Delta n+n|\nabla n|^2=0.$$

However, the first equation becomes

$$|\nabla n|^2=\frac{\kappa\alpha(\alpha-1)}{\rho^2}.$$

Referring to the classification of harmonic maps of $\mathbb{S}^2$, we see that the constancy of $|\nabla n|^2$ on $\mathbb{S}^2$ implies that $n(x)=\gamma\left(\frac{x}{|x|}\right)$ for some orthogonal rotation γ of $\mathbb{R}^3$. Moreover, α is a positive solution of $\kappa\alpha(\alpha-1)=2$. Thus,

$$\text{if}\quad |\varphi|\equiv c,\quad\text{then}\quad \varphi=c\gamma\left(\frac{x}{|x|}\right)\quad\text{for some rotation } \gamma \text{ of } \mathbb{R}^3.$$

It would be interesting to find solutions s, φ of the above system with $|\varphi|$ not being a constant.

4 Other cones for liquid crystal models

As noted in [E1], [E2], most discussions of nematic liquid crystals do not involve distinguishing the director field n from $-n$. Thus it is reasonable to consider the director as having values in the real projective space $\mathbb{RP}^2$ rather than in the sphere $\mathbb{S}^2$.

4.1. Remarks. In the simpler model [H-K-L] where one neglects the orientation order s and considers minimizers of the Oseen-Frank free energy $\int_\Omega W(n, \nabla n)\,dx$ among maps $n: \Omega \to \mathbb{S}^2$, the qualitative properties obtained in [H-K-L] change little with the replacement of $\mathbb{S}^2$ by $\mathbb{R}P^2$. In fact one first notes that the Oseen-Frank free energy is well-defined and H^1 weakly lower semicontinuous for such maps. For Dirichlet boundary data of some finite energy map from Ω to $\mathbb{RP}^2$, one may establish the existence of an energy minimizer by the device of [H-K-L, § 1]. As discussed briefly in [H-L1] the blowing-up argument of [H-K-L, § 2] carries over for an arbitrary target manifold. One may use [S-U, 4.3] to replace [H-K-L, 2.3] whose proof does not immediately carry over. Thus the singular set Z is again a closed set of 1 dimensional Hausdorff measure zero. This means that locally one may actually locally orient a minimizing line field to get a corresponding minimizing unit vectorfield. In fact, suppose $u: \mathbb{B}_r(a) \to \mathbb{RP}^2$ minimizes the Oseen-Frank free energy with singular set Z. Since the vanishing of $\mathscr{H}^1(Z)$ implies that $\mathbb{B}_r(a) \sim Z$ is simply connected, we may lift $u|(\mathbb{B}_r(a) \sim Z)$ to a map $v_u: \mathbb{B}_r(a) \sim Z \to \mathbb{S}^2$ with the same Oseen-Frank energy. Then v_u is itself minimizing which gives further information about the singular behavior of u. For example, in the harmonic map case of equal Oseen-Frank constants, the singular points are isolated and the precise behavior at a singularity for u follows from that of v_u (see e.g. [H-L1]).

One does find some differences with maps to $\mathbb{RP}^2$ when one considers boundary singularities. In particular, unlike [H-L1].

> there exists an $H^{\frac{1}{2}}$ mapping $\varphi: \partial \mathbb{B}^3 \to \mathbb{RP}^2$ which admits no extension to a finite energy map from $\mathbb{B}^3$ to $\mathbb{RP}^2$.

For example suppose $\varphi(x)$ is the horizontal line passing through $x=(x_1, x_2, x_3)$ and $(0, 0, x_3)$ for $x \in \partial \mathbb{B}^3 \sim \{(0, 0, -1), (0, 0, -1)\}$. Then, as in [H-L1], $\varphi \in H^{\frac{1}{2}}(\partial \mathbb{B}^3, \mathbb{RP}^2)$. But the existence of a finite energy extension would give an energy minimizing map $u: \mathbb{B}^3 \to \mathbb{RP}^2$ with Dirichlet boundary data φ. The regularity theory would then imply that $u|\mathbb{B}^3 \cap \{x_3=c\}$ is continuous for all but finitely many $c \in (-1, 1)$. But this is impossible because each curve $\varphi[(\partial \mathbb{B}^3) \cap \{x_3=c\}]$ is not contractible in $\mathbb{RP}^2$.

4.2. Ericksen model. To obtain a target manifold more suitable for the Ericksen model, we will essentially work with a cone over $\mathbb{RP}^2$. More precisely, we will say two points $x, x' \in \mathbb{R}^3$ are *sign equivalent* if $x = \pm x'$, and define the projectivised cone

$$\tilde{V}_\kappa = \{(\tilde{x}, y): (x, y) \in V_\kappa \text{ and } \tilde{x} \text{ is the sign equivalence class of } x\}.$$

Here $V_\kappa = \mathbb{C}^3_\kappa$ and $\tilde{V}_\kappa$ has the quotient topology and the quotient metric whereby the quotient map $(x, y) \in V_\kappa \sim \{(0, 0)\} \to (\tilde{x}, y) \in \tilde{V}_\kappa \sim \{(0, 0)\}$ is a locally isometric

2 sheeted covering map. Thus $\tilde{V}_\kappa$ is metrically a cone over $\mathbb{RP}^2$, equipped with the round metric from its 2-sheeted covering $\mathbb{S}^2$.

We will briefly discuss how things carry over to the $\tilde{V}_\kappa$ model. Theorem 2.1 does not carry over because $\mathbb{RP}^1$ is not contractible in $\mathbb{RP}^2$. However, Lemma 3.1, whose proof has no relation to the target, is still true. Most of the proof of 3.2 follows as before. Solving the resulting ODE, one needs only make some simple modifications in the periodicity of the functions $\psi(\theta)$. For example, the constant length solutions of 3.2(a) now come from functions $\tilde{\psi}(\theta)$ where

$$\psi(\theta)=\lambda(\cos m\theta, \sin m\theta, 0, \ldots, 0)$$

with $\lambda>0$ and m being a positive integer or now possibly a positive half-integer. However, for each integer j, the closed curve $\tilde{\psi}\left|\left[0, \frac{2\pi j}{m}\right]\right.$ covers its image with even multiplicity and so is contractible away from the origin. In fact, it clearly lifts to the closed curve $\psi\left|\left[0, \frac{2\pi j}{m}\right]\right.$ in V_κ. Restricting to the wedge defined by $\theta\in\left[0, \frac{2\pi j}{m}\right]$, we can, upstairs in V_κ, apply the peeling-off construction of 2.1 provided that $p\geqq 3$ and $\kappa>1$. We conclude that in 3.2(a) with $p\geqq 3$ and $\kappa>1$, $\tilde{\psi}(\theta)$ is determined by

$$\psi(\theta)=\lambda(\cos(\pm\tfrac{1}{2}\theta), \sin(\pm\tfrac{1}{2}\theta), 0, \ldots, 0).$$

When $p=2$ or $\kappa\leqq 1$, the other integer or half-integer values of m all give suitable $\tilde{\psi}$. The cases corresponding to 3.2(b) and 3.2(c) are similarly modified. Replacing Corollary 3.4, is the estimate that

> the zero set of a nonconstant energy-minimizing map from $\mathbb{B}^3$ to $\tilde{V}_\kappa$ has dimension $\leqq 2$ for $0<\kappa\leqq 1$ and dimension $\leqq 1$ for $\kappa>1$.

This is optimal. For $0<\kappa\leqq 1$ one has the rank-one maps of 3.2(2). For $\kappa>1$, one may chose boundary data $\varphi: \partial\mathbb{B}^3\to\tilde{V}_\kappa$ precisely as in 4.1. Let u be an energy minimizer with boundary data φ. Since u is continuous away from $u^{-1}\{0\}$, we conclude, as in 4.1, that $u(\mathbb{B}^3\cap\{x_3=a\})$ must contain 0 for each $a\in(-1,1)$. In particular, $\dim u^{-1}\{0\}\geqq 1$.

Finally, we recall that the mathematical derivation [E2, § 2] of the director field n and the orientation order function s leads to the range $-\frac{1}{2}\leqq s\leqq 1$. Thus we also consider maps into the truncated double-cone

$$X_\kappa=\{(x,y)\in V_\kappa: -\tfrac{1}{2}|\kappa-1|^{\frac{1}{2}}\leqq y\leqq|\kappa-1|^{\frac{1}{2}}\}$$

or, better yet, the projectivised truncated double-cone

$$\begin{aligned}\tilde{X}_\kappa&=\{(\tilde{x},y):(x,y)\in X_\kappa\}\\&=\{(\tilde{x},y):(\tilde{x},|y|)\in\tilde{V}_\kappa \text{ and } -\tfrac{1}{2}|\kappa-1|^{\frac{1}{2}}\leqq y\leqq|\kappa-1|^{\frac{1}{2}}\}.\end{aligned}$$

To handle simultaneously n and s, it is geometrically and analytically tempting to let n be a unit vector and work with the vector function $s\cdot n$. However, as J. Ericksen has pointed out [E2, § 2], one cannot unambiguously extract the orientation order and director field from the vector $s\cdot n$. Here use of the cone $\tilde{X}_\kappa$ overcomes this sign ambiguity. Thus, associated to a map $u=(\tilde{v}, w)$: $\Omega \to \tilde{X}_\kappa$, we have the well-defined order parameter

$$s=|\kappa-1|^{\frac{1}{2}} w: \ \Omega \to [-\tfrac{1}{2}, 1]$$

and director

$$n=\left(\frac{v}{|v|}\right): \ \Omega \sim s^{-1}\{0\} \ \to \ \mathbb{RP}^2.$$

One may modify the classification of 3.2 for either X_κ and $\tilde{X}_\kappa$. The most significant difference is the possibility of nontrivial rank one maps in the new version of 3.2(b) even when $\kappa>1$. Then arise, for example, by using two rays from 0, one in the upper cone and one in the lower cone. The resulting minimizing map on $\mathbb{B}^3$ has a 2 dimensional singular set, and this estimate may be obtained, as in the previous cases, by the dimension reduction argument.

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