Math. Z. 213, 117–131 (1993)

Mathematische Zeitschrift

Stable hypersurfaces with constant scalar curvature

H. Alencar[1], **M. do Carmo**[2], **A.G. Colares**[3]

[1] Departamento de Matemática, Universidade Federal de Alagoas, 57000 Maceió-AL, Brazil
[2] Instituto de Matemática Pura e Aplicada, Estrada D. Castorina 110, Jardim Botânico, 22460 Rio de Janeiro-RJ, Brazil
[3] Departamento de Matemática, Universidade Federal do Ceará, Campus do Pici, 60000-Fortaleza-CE, Brazil

Received 20 December 1991; in final form 30 June 1992

1 Introduction

(1.1) It is well known that hypersurfaces M^n with constant mean curvature in a Riemannian manifold $\bar{M}^{n+1}(c)$ of constant sectional curvature c are solutions to the variational problem of extremizing the area function for volume-preserving variations. In [BdCE] a notion of stability for this situation was considered and it was proved that if M^n is compact and stable, and $\bar{M}^{n+1}(c)$ is complete and simply-connected, then M^n is a geodesic sphere.

Less widely known but equally true is that hypersurfaces M^n of $\bar{M}^{n+1}(c)$ with constant scalar curvature are solutions to a similar variational problem, namely, of extremizing the integral of the mean curvature for volume-preserving variations. This has been known since at least 1973 and follows from a paper of Reilly [R] (see also the references there). The situation is actually more general than that. Let us denote by S_r the r^{th} elementary symmetric function of the principal curvatures of M^n, $r=0, 1, \ldots, n$, and consider the problem of extremizing $\int_M S_r\, dM$ under arbitrary variations. Then Reilly computed [R, p. 470] the formulas for the first and second variations of such a problem; from these formulas the above statement follows (see § 2 of this paper). Thus, in analogy with the case of constant mean curvature, questions of stability can be considered for hypersurfaces with constant scalar curvature.

We want to extend to hypersurfaces with constant scalar curvature the above stability result on constant mean curvature. So far, we have been able to solve the cases where $\bar{M}$ is the euclidean space $\mathbf{R}^{n+1}$ and $\bar{M}$ is the sphere $\mathbf{S}^{n+1}(c)\subset\mathbf{R}^{n+2}$ of constant curvature $c>0$. More precisely, we prove

(1.2) **Theorem.** *Let $\bar{M}^{n+1}(c)$ be a complete, simply-connected Riemannian manifold with constant sectional curvature $c\geqq 0$, and let $x\colon M^n\to\bar{M}^{n+1}(c)$ be a compact orientable hypersurface with constant scalar curvature R. If $c>0$, assume, in addition, that M^n is contained in an open hemisphere of $\bar{M}^{n+1}(c)=S^{n+1}(c)$. Then M is stable if and only if it is a geodesic sphere.*

One interesting fact about the stability question for hypersurfaces of constant scalar curvature is that it involves the study of a second order differential operator related to the second fundamental form which appeared also in a paper of Cheng and Yau [CY] who used it to classify the complete noncompact convex hypersurfaces with constant scalar curvature in $\mathbf{R}^{n+1}$.

(1.3) *Remark.* If x is an embedding, the result of Theorem 1.2 holds without any assumption of stability (see [MR]). By using the methods of equivariant geometry, one should be able to construct examples of nonspherical compact hypersurfaces with constant scalar curvature.

(1.4) *Remark.* It has been pointed out to us by Barbosa and Sa Earp that the operator above mentioned is elliptic if the scalar curvature is constant and greater than that of the ambient space. This should have interesting implications.

(1.5) *Remark.* It the context of surfaces with constant Gaussian curvature in a three-space form, we first heard from the above variational problem from H. Rosenberg. We want to thank him for sharing this information with us.

2 The variational problem for constant scalar curvature

(2.1) In this section, we adapt § 2 of [BdCE] to the present situation.

Let $\bar{M}^{n+1}(c)$ be an oriented Riemannian manifold of constant sectional curvature c and let $x: M^n \to \bar{M}^{n+1}(c)$ be an immersion of a compact, connected, orientable manifold M with boundary ∂M (possibly, $\partial M = \phi$) into $\bar{M}^{n+1}(c)$. Choose an orthonormal frame $\{e_1, \ldots, e_{n+1}\}$ around $x(p)$, $p \in M$, in $\bar{M}$ so that $e_1, \ldots, e_n$ are tangent to $x(M)$ and $d\bar{M}(e_1, \ldots, e_{n+1}) > 0$, where $d\bar{M}$ is the volume form of $\bar{M}$; then $e_{n+1} = N$ is globally defined and gives an *orientation* for M.

A *variation* of x is a differentiable map $X: (-\varepsilon, \varepsilon) \times M \to \bar{M}$, $\varepsilon > 0$, such that, for each $t \in (-\varepsilon, \varepsilon)$, $X_t(p) = X(t, p)$, $p \in M$, is an immersion, $X_o = x$, and $X_t|_{\partial M} = x|_{\partial M}$. We define the *volume function* V: $(-\varepsilon, \varepsilon) \to \mathbf{R}$ of X by

$$V(t) = \int_{[0,t] \times M} X^* d\bar{M}.$$

We will need the first three symmetric elementary functions of the principal curvatures $k_1, \ldots, k_n$ of an immersion x, namely:

$$S_1 = \sum k_i, \quad S_2 = \sum_{i<j} k_i k_j, \quad S_3 = \sum_{i<j<\ell} k_i k_j k_\ell,$$

$i, j, \ell = 1, \ldots, n$. Recall that the *mean curvature* H and the *scalar curvature* R of x are given by:

$$H = \frac{1}{n} S_1, \quad R - c = \frac{2}{n(n-1)} S_2.$$

Let X be a variation of $x: M^n \to \bar{M}^{n+1}(c)$ and $W(p) = \left.\frac{\partial X}{\partial t}\right|_{t=0}$ be the *variational vector* of X. Set $f = \langle W, N \rangle$ where N is the unit normal vector along x. A variation is *normal* if W is parallel to N and *volume-preserving* if $V(t) = V(0)$ for all t.

(2.2) **Lemma.**

(i) $\frac{d}{dt} \int_M nH(t)\, dM_t|_{t=0} = \int_M (-n(n-1)(R-c)+cn) f\, dM,$

(ii) $\left.\frac{dV}{dt}\right|_{t=0} = \int_M f\, dM.$

Proof. (i) is just a translation in our notation of the formula for the first variation in p. 470 of [R]. (ii) has been proved in [BdCE, p. 125]. □

Now set

$$R_o = A^{-1} \int_M R\, dM, \qquad A = \int_M dM,$$

and define $J: (-\varepsilon, \varepsilon) \to \mathbf{R}$ by

$$J(t) = n \int_M H(t)\, dM_t + (n(n-1)(R_o - c) - cn)\, V(t).$$

(2.3) **Lemma.** *Let $x: M^n \to \bar{M}^{n+1}(c)$ be an immersion. The following statements are equivalent*:

(i) *x has constant scalar curvature R_o.*
(ii) *For all volume-preserving variations,*

$$\frac{d}{dt} \int_M nH(t)\, dM_t|_{t=0} = 0.$$

(iii) *For all variations, $J'(0) = 0$.*

Proof. The proof follows the same pattern of the proof of Proposition 2.7 in [BdC] using Lemma 2.2 of [BdCE]. We shall omit it. □

Before presenting the second variation of J we need to introduce the operator mentioned in the Introduction. For that, consider for each $p \in M$ the linear map $T: T_p M \to T_p M$

$$T = nHI - B,$$

where I is the identity map and B is the linear map associated to the second fundamental form of x along N. In an orthonormal frame $\{e_1, \ldots, e_n\}$ around p, the matrix of T is

$$T_{ij} = nH\, \delta_{ij} - h_{ij},$$

where h_{ij} is the matrix of B. Let f be a differentiable function on M and let f_{ij} be the matrix of the hessian of f. We will define an operator □ acting on f by

$$\Box f = \sum_{i,j} T_{ij} f_{ij} = \sum_{i,j} (nH\, \delta_{ij} - h_{ij}) f_{ij}.$$

This operator was considered by Cheng and Yau in [CY]. Since T_{ij} is divergence-free [R, p. 470] it follows [CY] that the operator $\square$ is self-adjoint relative to the L^2 inner product of M, i.e.,

$$\int_M f \square g = \int_M g \square f.$$

We are now in a position to compute the second variation of $J(t)$.

(2.4) **Lemma.** *Let $x: M^n \to \bar{M}^{n+1}(c)$ be a hypersurface with constant scalar curvature R and let X be a variation of x. Then $J''(0)$ depends only on f and it is given by*

$$J''(0)(f) = -2 \int_M (f \square f + f^2 [\tfrac{1}{2} n^2 (n-1)(R-c) H + cn(n-1) H - 3S_3]) \, dM.$$

Proof. We first observe that

$$\frac{dJ}{dt} = \int_M [(-n(n-1)(R_t - c) + cn) + (n(n-1)(R_o - c) - cn)] f_t \, dM_t.$$

Here R_t is the scalar curvature of X_t, dM_t is its volume element, and $f_t = \left\langle \frac{\partial X}{\partial t}, N_t \right\rangle$, where N_t is the unit normal vector of X_t. Thus, setting $n(n-1)(R_t - c) = -A_t$, we can write

$$\frac{DJ}{dt} = \int_M (A_t - A_o) f_t \, dM_t.$$

It follows that

$$\frac{d^2 J}{dt^2} = \int_M A'_t f_t \, dM_t + \int_M A_t f'_t \, dM_t - \int_M A_o f'_t \, dM_t + \int_M (A_t - A_o) f_t \frac{\partial}{\partial t} dM_t$$

which, for $t=0$, gives

$$\left. \frac{d^2 J}{dt^2} \right|_{t=0} = \int_M A'_o f \, dM = - \int_M \left(n(n-1)\left(\frac{\partial R_t}{\partial t}(0)\right)\right) f \, dM.$$

Now, we use the formula (9c) in [R, p. 469] to obtain

$$\tfrac{1}{2} n(n-1) \frac{\partial R_t}{\partial t}(0) = f\{(\tfrac{1}{2} n^2 (n-1)(R-c) H - 3S_3) + cn(n-1) H\} + \square f$$

and this completes the proof. $\square$

We can now define stability.

(2.5) **Definition.** Let $x: M^n \to \bar{M}^{n+1}(c)$ have constant scalar curvature. The immersion x is *stable* if

$$\frac{d^2}{dt^2} \int_M nH_t \, dM_t \Big|_{t=0} \geqq 0,$$

for all volume-preserving variations of x. If M is non compact, we say that x is *stable* if for every compact submanifold $\tilde{M} \subset M$ with boundary, the restriction $x|_{\tilde{M}}$ is stable.

Just as in [BdCE] we can prove the following criterion of stability. Let $\mathscr{F}$ the set differentiable functions $f: M \to \mathbf{R}$ with $f|_{\partial M} = 0$ and $\int_M f\, dM = 0$. *Then* $x: M^n \to \bar{M}^{n+1}(c)$ *with constant scalar curvature is stable if and only if*

$$J''(0)(f) \geqq 0, \qquad \text{for all } f \in \mathscr{F}.$$

We can also introduce a notion of a Jacobi field for the present situation. Since this is quite similar to the case treated in [BdCE], we omit the details.

We will close this section by proving that there exist stable hypersurfaces with constant scalar curvature.

(2.6) **Proposition.** *Let* $\bar{M}^{n+1}(c)$ *be complete and simply-connected and let* $\Sigma^n \subset \bar{M}^{n+1}(c)$ *be a geodesic sphere. Then* Σ *is stable.*

Proof. Choose $f: \Sigma \to \mathbf{R}$ such that $\int_\Sigma f\, dM = 0$. Since Σ is umbilic, we have that $\|B\|^2 = nH^2$ and that

$$\square f = (n-1)\, H \Delta f,$$

where Δf is the Laplacian of f in Σ. From the formula for the second variation of J, we obtain

$$\begin{aligned} J''(0)(f) = & -2(n-1) \int_\Sigma H f \Delta f \\ & -2 \int_\Sigma f^2 \left[\tfrac{1}{2} n^2 (n-1)(R-c)\, H + c n(n-1)\, H - 3 S_3\right]. \end{aligned}$$

Since

$$\operatorname{tr} B^3 = nH \|B\|^2 - \tfrac{1}{2} n^2 (n-1)\, H(R-c) + 3 S_3,$$

we obtain, by umbilicity,

$$-\tfrac{1}{2} n^2 (n-1)\, H(R-c) + 3 S_3 = \operatorname{tr} B^3 - nH \|B\|^2 = -n(n-1)\, H^3.$$

Thus, by Stokes' theorem,

$$\begin{aligned} J''(0)(f) & = 2(n-1)\, H \int_\Sigma (\|\nabla f\|^2 - n(c+H^2) f^2) \\ & \geqq 2(n-1)\, H \int_\Sigma (\mu(\Sigma) - n(c+H^2)) f^2, \end{aligned}$$

where $\mu(\Sigma)$ is the first eigenvalue of the Laplacian Δ in Σ. Since Σ is a sphere, $\mu(\Sigma) = n(c+H^2)$. Hence $J''(0)(f) \geqq 0$, for all f such that $\int_\Sigma f\, dM = 0$, and Σ is stable as we wished. $\square$

3 Preliminary results

(3.1) We use the method of moving frames. Let $x\colon M^n \to \bar{M}^{n+1}(c)$ be an immersion of a smooth manifold M^n into a Riemannian manifold $\bar{M}^{n+1}(c)$ with constant sectional c. Choose $p \in M$ and a local orthonormal frame $\{e_1, \ldots, e_n, e_{n+1} = N\}$ in $\bar{M}^{n+1}(c)$ around $x(p)$, so that $e_1, \ldots, e_n$ are tangent to $x(M)$. Take the corresponding coframe $\{w_1, \ldots, w_n, w_{n+1}\}$ and write the structure equations (indices A, B, C range from 1 to $n+1$):

$$dw_A = \sum_B w_{AB} \wedge w_B, \qquad w_{AB} = -w_{BA}$$

$$dw_{AB} = \sum_C w_{AC} \wedge w_{CB} + \Omega_{AB}, \qquad \Omega_{AB} = -\Omega_{BA},$$

where Ω_{AB} is the curvature matrix of $\bar{M}$, i.e.,

$$\Omega_{AB} = -\frac{1}{2} \sum_{C,D} \bar{R}_{ABCD}\, w_C \wedge w_D.$$

If we denote by the same letters the restrictions of w_A and w_{AB} to $x(M)$, we can separate the tangent part of the above equations (latin indices range from 1 to n):

$$dw_i = \sum_j w_{ij} \wedge w_j,$$

$$dw_{ij} = \sum_k w_{ik} \wedge w_{kj} + \Omega_{ij}, \qquad \Omega_{ij} = -\frac{1}{2} \sum_{k,\ell} R_{ijk\ell}\, w_k \wedge w_\ell,$$

where $R_{ijk\ell}$ is the curvature tensor of the induced metric on M.

Notice that the restriction of $w_{n+1} = 0$. Thus since

$$0 = dw_{n+1} = \sum_j w_{n+1,j} \wedge w_j,$$

we can write

$$w_{j,n+1} = \sum_k k_{jk}\, w_k, \qquad h_{jk} = h_{kj}.$$

The quadratic form $\sum_{j,k} h_{jk}\, w_j w_k$ is the second fundamental form B of the immersion. It relates the curvatures of M and $\bar{M}$ by the Gauss' formula:

(3.2) $$R_{ijk\ell} = \bar{R}_{ijk\ell} - (h_{i\ell}\, h_{jk} - h_{ik}\, h_{j\ell}).$$

Notice that in $\bar{M}^{n+1}(c)$,

(3.3) $$\bar{R}_{ijk\ell} = c(\delta_{ik}\, \delta_{j\ell} - \delta_{i\ell}\, \delta_{jk}).$$

For a smooth function f on M, the gradient and the hessian (f_{ij}) are defined by:

$$df = \sum_i f_i\, w_i, \qquad \sum_j f_{ij}\, w_j = df_i + \sum_j f_j\, w_{ji}.$$

Similarly, the covariant derivative of the second fundamental form is defined by

(3.4) $$\sum_k h_{ijk} w_k = d h_{ij} + \sum_k h_{kj} w_{ki} + \sum_k h_{ik} w_{kj}.$$

The second fundamental form satisfy the Codazzi equation, i.e., $h_{ijk} = h_{ikj}$. It follows that h_{ijk} is symmetric in all indices.

The second covariant derivative of h_{ij} is defined by

$$\sum_\ell h_{ijk\ell} \omega_\ell = d h_{ijk} + \sum_m h_{mjk} \omega_{mi} + \sum_m h_{imk} \omega_{mj} + \sum_m h_{ijm} \omega_{mk}.$$

By exterior differentiation of (3.4), one can show that the following "commutation formula" holds

(3.5) $$h_{j\ell ki} - h_{j\ell ik} = -\sum_m h_{m\ell} R_{mjik} - \sum_m h_{jm} R_{m\ell ik}.$$

Finally, for a smooth function f on M, we recall that

$$\Box f = \sum_{k,\ell} (nH\, \delta_{k\ell} - h_{k\ell}) f_{k\ell},$$

where $nH = \sum_i h_{ii}$.

We start with a simple lemma that is quite general.

(3.6) **Lemma.** *Let $f, g: M^n \to \mathbf{R}$. Then*

$$\Box (f g) = g \Box f + f \Box g + 2nH \sum_k f_k g_k - 2 \sum_{k,\ell} h_{k\ell} f_k g_\ell.$$

Proof. Clearly the hessian of fg is given by

$$(f g)_{k\ell} = g_\ell f_k + f_\ell g_k + g f_{k\ell} + f g_{k\ell}.$$

Thus

$$\begin{aligned} \Box (f g) &= \sum_{k,\ell} (nH\, \delta_{k\ell} - h_{k\ell}) (f g)_{k\ell} \\ &= g \Box f + f \Box g + \sum_{k,\ell} (nH\, \delta_{k\ell} - h_{k\ell}) f_k g_\ell \\ &\quad + \sum_{k,\ell} (nH\, \delta_{k\ell} - h_{k\ell}) f_\ell g_k \end{aligned}$$

and the lemma follows.

(3.7) **Lemma.** *Assume that $R = \text{const}$. Then*

$$\begin{aligned} \Box H = \frac{1}{n} \|\nabla B\|^2 - n \|\nabla H\|^2 + (n-1)(R-c)\, \|B\|^2 \\ - \frac{1}{2} n^2 (n-1) H^2 (R-c) + 3H S_3 + c \|B\|^2 - nH^2 c. \end{aligned}$$

Proof. It is known that

$$n^2 H^2 - \sum_{k,\ell} h_{k\ell}^2 = n(n-1)(R-c).$$

Take the hessian of both sides to obtain

$$n^2 H_i H_j + n^2 H H_{ij} - \sum_{k,\ell} h_{k\ell i} h_{k\ell j} - \sum_{k,\ell} h_{k\ell} h_{k\ell ij} = 0.$$

Setting $i=j$, using Codazzi equation and (3.5), we obtain

$$\begin{aligned} n^2(H_i)^2 - \sum_{k,\ell} h_{k\ell i}^2 &= -nH \sum_k h_{ikki} + \sum_{k,\ell} h_{k\ell} h_{i\ell ki} \\ &= -nH \sum_k (h_{ikik} - \sum_m h_{mk} R_{miik} - \sum_m h_{im} R_{mkik}) \\ &\quad + \sum_{k,\ell} (h_{k\ell} h_{i\ell ik} - h_{k\ell} \sum_m h_{m\ell} R_{miik} - h_{k\ell} \sum_m h_{im} R_{m\ell ik}) \\ &= -nH \sum_{k,\ell} \delta_{k\ell} h_{iik\ell} + \sum_{k,\ell} h_{k\ell} h_{ii\ell k} + nH \sum_{k,m} (h_{mk} R_{miik} + h_{im} R_{mkik}) \\ &\quad - \sum_{k,\ell,m} (h_{k\ell} h_{m\ell} R_{miik} + h_{k\ell} h_{im} R_{m\ell ik}). \end{aligned}$$

Thus

$$\begin{aligned} \square h_{ii} = \sum_{k,\ell} h_{k\ell i}^2 - n^2(H_i)^2 - \sum_{k,\ell,m} (h_{k\ell} h_{m\ell} R_{miik} + h_{k\ell} h_{im} R_{m\ell ik}) \\ + nH \sum_{k,m} (h_{mk} R_{miik} + h_{im} R_{mkik}). \end{aligned}$$

Using the Gauss' Eq. (3.2), we obtain from the above

$$\begin{aligned} \square h_{ii} &= \sum_{k,\ell} h_{k\ell i}^2 - n^2 (H_i)^2 \\ &\quad - \sum_{k,\ell,m} \{h_{k\ell} h_{m\ell} \bar{R}_{miik} + h_{k\ell} h_{m\ell} h_{mi} h_{ik} - h_{k\ell} h_{m\ell} h_{mk} h_{ii} \\ &\quad + h_{k\ell} h_{im} \bar{R}_{m\ell ik} + h_{k\ell} h_{im}^2 h_{k\ell} - h_{k\ell} h_{im} h_{mk} h_{\ell i}\} \\ &\quad + nH \sum_{k,m} h_{mk} \bar{R}_{miik} + nH \sum_{k,m} h_{mk} h_{mi} h_{ik} - nH \sum_{k,m} h_{mk}^2 h_{ii} + nH \sum_{k,m} h_{im} \bar{R}_{mkik} \\ &\quad + nH \sum_{k,m} h_{im}^2 h_{kk} - nH \sum_{k,m} h_{im} h_{mk} h_{ki} \\ &= \sum_{k,\ell} h_{k\ell i}^2 - n^2 (H_i)^2 - \sum_{k,\ell,m} h_{k\ell} h_{m\ell} \bar{R}_{miik} + (\operatorname{tr} B^3) h_{ii} \\ &\quad - \sum_{k,\ell,m} h_{k\ell} h_{im} \bar{R}_{m\ell ik} - \|B\|^2 \sum_m h_{im}^2 + nH \sum_{k,m} h_{mk} \bar{R}_{miik} - nH \|B\|^2 h_{ii} \\ &\quad - nH \sum_{k,m} h_{im} \bar{R}_{mkik} + n^2 H^2 \sum_m h_{im}^2. \end{aligned}$$

By (3.3), we then have

$$\begin{aligned}\square h_{ii} = & -c \sum_{k,\ell,m} h_{k\ell} h_{m\ell}(\delta_{mi}\delta_{ik} - \delta_{mk}\delta_{ii}) - c \sum_{k,\ell,m} h_{k\ell} h_{im}(\delta_{mi}\delta_{\ell k} - \delta_{mk}\delta_{\ell i}) \\ & + nHc \sum_{k,m} h_{mk}(\delta_{mi}\delta_{ik} - \delta_{mk}\delta_{ii}) + nHc \sum_{k,m} h_{im}(\delta_{mi}\delta_{kk} - \delta_{mk}\delta_{ik}) \\ & + (\operatorname{tr} B^3) h_{ii} + n(n-1)(R-c) \sum_m h_{im}^2 - nH \|B\|^2 h_{ii} + \sum_{k,\ell} h_{k\ell i}^2 - n^2 (H_i)^2 \\ = & \sum_{k,\ell} h_{k\ell i}^2 - n^2 (H_i)^2 + c\|B\|^2 - n^2 H^2 c + n^2 H c h_{ii} - nHc h_{ii} \\ & + (\operatorname{tr} B^3) h_{ii} + n(n-1)(R-c) \sum_m h_{im}^2 - nH\|B\|^2 h_{ii}.\end{aligned}$$

Thus

$$\begin{aligned}(3.8) \qquad n\square H = & \|\nabla B\|^2 - n^2 \|\nabla H\|^2 + nc\|B\|^2 - n^2 c H^2 + nH(\operatorname{tr} B^3) \\ & + n(n-1)(R-c)\|B\|^2 - n^2 H^2 \|B\|^2 \\ = & \|\nabla B\|^2 - n^2 \|\nabla H\|^2 + nc\|B\|^2 - n^2 H^2 c + nH(\operatorname{tr} B^3) - \|B\|^4.\end{aligned}$$

Finally, using in (3.8) the fact that

$$(3.9) \qquad \operatorname{tr} B^3 = nH\|B\|^2 - \tfrac{1}{2} n^2 (n-1) H(R-c) + 3S_3,$$

the lemma follows. □

4 Proof of Theorem 1.2

In this section, we will use the notation of § 3. We will need the following lemma.

(4.1) **Lemma.** *Let $x\colon M^n \to \bar{M}^{n+1}(c)$ be an immersion with constant scalar curvature R. Assume that $R-c \geqq 0$. Then*

$$\frac{1}{n}\|\nabla B\|^2 - n\|\nabla H\|^2 \geqq 0.$$

Here B is the second fundamental form and H is the mean curvature of x.

Proof. We know that

$$n^2 H^2 - \sum_{k,\ell} h_{k\ell}^2 = n(n-1)(R-c).$$

Taking the covariant derivative of the above expression, and using the fact that $R = \text{const.}$, we obtain

$$n^2 H H_i = \sum_{k,\ell} h_{k\ell} h_{k\ell i}.$$

It follows that

$$\sum_i n^4 H^2 (H_i)^2 = \sum_i \Big(\sum_{k,\ell} h_{k\ell} h_{k\ell i}\Big)^2 \leqq \Big(\sum_{k,\ell} h_{k\ell}^2\Big)\Big(\sum_{k,\ell,i} h_{k\ell i}^2\Big),$$

that is,

$$n^4 H^2 \|\nabla H\|^2 \leqq \|B\|^2 \|\nabla B\|^2.$$

On the other hand, if $(R-c) \geqq 0$, we have that $n^2 H^2 - \|B\|^2 \geqq 0$. Thus

$$n^2 H^2 \|\nabla H\|^2 \leqq H^2 \|\nabla B\|^2$$

and the lemma follows.

(4.2) *Proof of Theorem* 1.2 *for* $c=0$

Assume that x is stable. We first observe that if this is the case,

$$-J''(0)(f) = 2 \int_M ((\tfrac{1}{2} n^2 (n-1) HR - 3S_3) f^3 + f \square f)\, dM \leqq 0$$

for all functions $f: M \to \mathbf{R}$ with $\int_M f\, dM = 0$. To choose a convenient test-function, observe that the second Minkowski's formula [H, p. 286] gives

$$\int_M (H + gR)\, dM = 0,$$

where $g = \langle x, N \rangle$ is the support function of $x: M^n \to \mathbf{R}^{n+1}$. Thus we can choose $f = H + gR$.

Let us compute the integrand of $J''(0)(f)$ for $f = H + gR$, using $R = \text{const.}$:

$$\begin{aligned} & f \square f + (\tfrac{1}{2} n^2 (n-1) HR - 3S_3) f^2 \\ & \quad = H \square H + HR \square g + gR \square H + gR^2 \square g + \tfrac{1}{2} n^2 (n-1) H^3 R + n^2 (n-1) g H^2 R^2 \\ & \qquad + \tfrac{1}{2} n^2 (n-1) g^2 H R^3 - 3H^2 S_3 - 6gHRS_3 - 3g^2 R^2 S_3. \end{aligned}$$

On the other hand, since $\square$ is self-adjoint,

$$\int_M gR \square H = \int_M RH \square g.$$

Thus

$$\int_M (H \square H + HR \square g + gR \square H + gR^2 \square g) = \int_M (H \square H + 2RH \square g + R^2 g \square g).$$

Now, since $c=0$ and $R = \text{const.}$, the following expressions hold:

$$\square g = -n(n-1) R - \tfrac{1}{2} n^2 (n-1) RHg + 3S_3 g, \tag{4.3}$$

$$\begin{aligned} \square H = & -n \|\nabla H\|^2 + \frac{1}{n} \|\nabla B\|^2 + (n-1) R \|B\|^2 \\ & - \tfrac{1}{2} n^2 (n-1) H^2 R + 3HS_3. \end{aligned} \tag{4.4}$$

For (4.3), see [R, Eq. 13, p. 475], or compute it directly using the techniques of § 3. (4.4) follows from Lemma 3.7 for $c=0$. Therefore,

$$-\tfrac{1}{2}J''(0)(f)=\int_M H\left(\frac{1}{n}\|\nabla B\|^2-n\|\nabla H\|^2\right)$$
$$+\int_M \{(n-1)RH\|B\|^2-\tfrac{1}{2}n^2(n-1)H^3R+3H^2S_3\}$$
$$+\int_M \{-2n(n-1)HR^2-n^2(n-1)H^2R^2g+6HRgS_3\}$$
$$+\int_M \left\{-n(n-1)R^3g+3g^2R^2S_3-\frac{n^2}{2}(n-1)HR^3g^2\right\}$$
$$+\int_M \{\tfrac{1}{2}n^2(n-1)H^3R+n^2(n-1)gH^2R^2+\tfrac{1}{2}n^2(n-1)g^2HR^3\}$$
$$+\int_M \{-3H^2S_3-6gHRS_3-3g^2R^2S_3\}.$$

Fortunately, this expression simplifies into

$$-\tfrac{1}{2}J''(0)(f)=\int_M H\left(\frac{1}{n}\|\nabla B\|^2-n\|H\|^2\right)$$
$$+(n-1)R\int_M H\|B\|^2-n(n-1)R^2\int_m (H+gR)-n(n-1)R\int_M HR$$
$$=\int_M H\left(\frac{1}{n}\|\nabla B\|^2-n\|\nabla H\|^2\right)+(n-1)R\int_M H(\|B\|^2-nR)$$
$$=\int_M H\left(\frac{1}{n}\|\nabla B\|^2-n\|\nabla H\|^2\right)+(n-1)R\int_M Hn^2(H^2-R),$$

since $\|B\|^2-nR=n^2(H^2-R)$. Notice that, since M is compact, $R>0$ and

$$0<n(n-1)R=n^2H^2-\|B\|^2\leqq n^2H^2-nH^2=n(n-1)H^2;$$

we conclude that H is nowhere zero. Thus an orientation can be chosen so that $H>0$. Notice also that $H^2-R\geqq 0$ and the equality holds if and only if x is umbilic. It follows from Lemma 4.1 that

$$0\geqq -\tfrac{1}{2}J''(0)(f)=\int_M \left\{H\left(\frac{1}{n}\|\nabla B\|^2-n\|\nabla H\|^2\right)+n^2(n-1)RH(H^2-R)\right\}dM\geqq 0.$$

Thus if x is stable, $H^2-R=0$, hence x is umbilic and $x(M)$ is a geodesic sphere. The converse follows from Proposition 2.6. □

(4.5) *Proof of Theorem* (1.2) *for* $c>0$

Let $x\colon M^n\to S^{n+1}(c)\subset \mathbf{R}^{n+2}$. Fix a unit vector $v\in\mathbf{R}^{n+2}$ and define functions f and $\tilde{g}$ on M by

$$f(p)=\langle N(p),v\rangle,\qquad \tilde{g}(p)=\langle x(p),v\rangle,\qquad p\in M.$$

We will need the following expressions:

(4.6) $$\square f = cn(n-1)(R-c)\tilde{g} + 3S_3 f - \tfrac{1}{2}n^2(n-1)(R-c)Hf$$

(4.7) $$\square \tilde{g} = n(n-1)(R-c)f - n(n-1)cH\tilde{g}.$$

For (4.6) see [R, Proposition D, p. 473], and for (4.7) see [R, remark to Theorem E, p. 473]; they can both be computed directly with the techniques of § 3.

To find a convenient test-function, we set

$$h = (R-c)f - Hc\tilde{g}.$$

Notice that, by (4.7), $n(n-1)h = \square \tilde{g}$. Since $\square$ is self-adjoint, $\int_M h\,dM = 0$.

From now on, let us assume $c=1$. Choose v as an element of a canonical basis $a_0, a_1, \ldots, a_{n+1}$ of $\mathbf{R}^{n+2}$ and let f_A and $\tilde{g}_A$ be the above functions for $v = a_A$, $A = 0, 1, \ldots, n+1$. Set $h_A = (R-1)f_A - H\tilde{g}_A$.

Now, assume that x is stable. Then, for each A, $J''(0)(h_A) \geqq 0$, that is,

(4.8) $$2\int_M (h_A \square h_A + \{\tfrac{1}{2}n^2(n-1)H(R-1) + n(n-1)H - 3S_3\} h_A^2)\,dM \leqq 0.$$

We want to introduce $h_A = (R-1)f_A - H\tilde{g}_A$ in (4.8) and sum up in A. We divide the computation in two parts, and first compute the second term in the integrand:

$$\begin{aligned}
&\{\tfrac{1}{2}n^2(n-1)H(R-1) - 3S_3 + n(n-1)H\} h_A^2 \\
&\quad = \tfrac{1}{2}n^2(n-1)H(R-1)^3 f_A^2 - n^2(n-1)H^2(R-1)^2 f_A \tilde{g}_A \\
&\qquad + \tfrac{1}{2}n^2(n-1)H^3(R-1)\tilde{g}_A^2 - 3(R-1)^2 S_3 f_A^2 \\
&\qquad + 6(R-1)HS_3 f_A \tilde{g}_A - 3H^2 S_3 \tilde{g}_A^2 + n(n-1)(R-1)^2 H f_A^2 \\
&\qquad - 2n(n-1)(R-1)H^2 f_A \tilde{g}_A + n(n-1)H^3 \tilde{g}_A^2.
\end{aligned}$$

Now observe that, since $x(M)$ is contained in a unit sphere,

(4.9) $$\sum_A \tilde{g}_A^2 = \sum_A \langle x, a_A\rangle^2 = \langle x, x\rangle = 1,$$

(4.10) $$\sum_A f_A^2 = \sum_A \langle N, a_A\rangle^2 = \langle N, N\rangle = 1,$$

(4.11) $$\sum_A f_A \tilde{g}_A = \sum_A \langle N, a_A\rangle \langle x, a_A\rangle = \langle N, x\rangle = 0.$$

It follows that

(4.12) $$\begin{aligned}
&\sum_A \{\tfrac{1}{2}n^2(n-1)H(R-1) - 3S_3 + n(n-1)H\} h_A^2 \\
&\quad = \tfrac{1}{2}n^2(n-1)H(R-1)^3 + \tfrac{1}{2}n^2(n-1)H^3(R-1) \\
&\qquad - 3(R-1)^2 S_3 - 3H^2 S_3 + n(n-1)(R-1)^2 H \\
&\qquad + n(n-1)H^3.
\end{aligned}$$

We now compute the first term in the integral of (4.8):

$$\begin{aligned}\int_M h_A \square h_A &= (R-1)^2 \int_M f_A \square f_A - (R-1) \int_M f_A \square (H\tilde{g}_A)\\ &\quad -(R-1)\int_M H\tilde{g}_A \square f_A + \int_M H\tilde{g}_A \square (H\tilde{g}_A)\\ &= (R-1)^2 \int_M f_A \square f_A - 2(R-1)\int_M H\tilde{g}_A \square f_A + \int_M H\tilde{g}_A \square (H\tilde{g}_A)\\ &= (R-1)^2 \int_M f_A \{n(n-1)(R-1)\tilde{g}_A + 3S_3 f_A - \tfrac{1}{2}n^2(n-1)H(R-1)f_A\}\\ &\quad -2(R-1)\int_M H\tilde{g}_A\{n(n-1)(R-1)\tilde{g}_A\\ &\quad + 3S_3 f_A - \tfrac{1}{2}n^2(n-1)H(R-1)f_A\}\\ &\quad + \int_M H\tilde{g}_A\{H\square \tilde{g}_A + \tilde{g}_A \square H + 2n\sum_i (\nabla_{e_i}\tilde{g}_A)(\nabla_{e_i}H)\\ &\quad -2\sum_{i,j} h_{ij}(\nabla_{e_i}\tilde{g}_A)(\nabla_{e_i}H)\},\end{aligned}$$

where we have used the fact that (see § 3, Lemma 3.6) for any two functions $f, g\colon M^n \to \mathbf{R}$, we have

$$\square(fg) = g\square f + f\square g + 2nH\sum_{k=1}^{n} f_k g_k - 2\sum_{k,\ell=1}^{n} h_{k\ell} f_k g_\ell.$$

Therefore

$$\begin{aligned}&\int_M h_A \square h_A\\ &= n(n-1)(R-1)^3 \int_M f_A \tilde{g}_A + 3(R-1)^2 \int_M f_A^2 S_3\\ &\quad -\tfrac{1}{2}n^2(n-1)(R-1)^3 \int_M f_A^2 H - 2n(n-1)(R-1)^2 \int_M \tilde{g}_A^2 H\\ &\quad -6(R-1)\int_M f_A \tilde{g}_A H S_3 + n^2(n-1)(R-1)^2 \int_M f_A \tilde{g}_A H^2\\ &\quad + \int_M H\tilde{g}_A \Big\{ n(n-1)(R-1)Hf_A - n(n-1)H^2\tilde{g}_A - n\tilde{g}_A\|\nabla H\|^2\\ &\quad + \frac{1}{n}\tilde{g}_A\|\nabla B\|^2 + (n-1)(R-1)\tilde{g}_A\|B\|^2\\ &\quad -\frac{1}{2}n^2(n-1)H^2(R-1)\tilde{g}_A + 3H\tilde{g}_A S_3 + \|B\|^2\tilde{g}_A\\ &\quad -nH^2\tilde{g}_A + 2nH\sum_i (\nabla_{e_i}\tilde{g}_A)(\nabla_{e_i}H)\\ &\quad -2\sum_{i,j} h_{ij}(\nabla_{e_i}\tilde{g}_A)(\nabla_{e_j}H)\Big\}.\end{aligned}$$

We sum up the above expression in A, use (4.9), (4.10), (4.11) and the fact that

$$0=\nabla_{e_i}\big(\sum_A \tilde{g}_A^2\big)=2\sum_A \tilde{g}_A \nabla_{e_i} \tilde{g}_A$$

to obtain finally

$$\begin{aligned}(4.13)\qquad &\sum_A \int_M h_A \square h_A\\ &=3(R-1)^2 \int_M S_3-\tfrac{1}{2}n^2(n-1)(R-1)^3 \int_M H\\ &\quad -2n(n-1)(R-1)^2 \int_M H-n(n-1) \int_M H^3\\ &\quad -n \int_M H\|\nabla H\|^2+\frac{1}{n} \int_M H\|\nabla B\|^2\\ &\quad +(n-1)(R-1) \int_M H\|B\|^2-\tfrac{1}{2}n^2(n-1)(R-1) \int_M H^3\\ &\quad +3 \int_M H^2 S_3+\int_M H\|B\|^2-n \int_M H^3.\end{aligned}$$

Now, using (4.12) and (4.13) we obtain that the sum of (4.8) in A can be written as

$$\begin{aligned}&-\sum_A J''(0)(h_A)\\ &=\int_M \Big\{\tfrac{1}{2}n^2(n-1)H(R-1)^3+\tfrac{1}{2}n^2(n-1)H^3(R-1)-3(R-1)^2 S_3\\ &\quad -3H^2 S_3+n(n-1)(R-1)^2 H+n(n-1)H^3+3(R-1)^2 S_3\\ &\quad -\tfrac{1}{2}n^2(n-1)(R-1)^3 H-2n(n-1)(R-1)^2 H-n(n-1)H^3\\ &\quad -nH\|\nabla H\|^2+\frac{1}{n}\|\nabla B\|^2+(n-1)(R-1)H\|B\|^2\\ &\quad -\tfrac{1}{2}n^2(n-1)(R-1)H^3+3H^2 S_3+H\|B\|^2-nH^3\Big\}\leqq 0\end{aligned}$$

which simplifies into

$$\begin{aligned}(4.14)\qquad -\sum_A J''(0)(h_A)=\int_M &\Big\{H\Big[\frac{1}{n}\|\nabla B\|^2-n\|\nabla H\|^2\Big]\\ &+[-n(n-1)(R-1)^2 H+(n-1)(R-1)H\|B\|^2]\\ &+H(\|B\|^2-nH)\Big\}\leqq 0.\end{aligned}$$

We observe now that, since $\|B\|^2=n^2H^2-n(n-1)(R-1)$, we have $(n-1)(R-1)H\|B\|^2-n(n-1)(R-1)^2H=n^2(n-1)(R-1)H(H^2-(R-1))$. Furthermore,

$$n(n-1)(R-1)=n^2H^2-\|B\|^2\leqq n(n-1)H^2;$$

thus, since $R-1>0$ by hypothesis, H^2 never vanishes, and we can choose an orientation so that $H>0$.

It follows that if M is stable,

$$0 \geqq -\sum_A J''(0)(h_A) = \int_M \left\{ H\left[\frac{1}{n}\|\nabla B\|^2 - n\|\nabla H\|^2\right] \right.$$
$$\left. + n^2(n-1)H(R-1)[H^2-(R-1)] + H[\|B\|^2 - nH^2] \right\} \geqq 0.$$

Therefore, $\|B\|^2 = nH^2$ and M is umbilic as we wished. The converse follows from Proposition 2.6. □

References

[BdC] Barbosa, J.L., do Carmo, M.: Stability of hypersurfaces with constant mean curvature. Math. Z. **185**, 339–353 (1984)

[BdCE] Barbosa, J.L., do Carmo, M., Eschenburg, J.: Stability of hypersurfaces of constant mean curvature in Riemannian manifolds. Math. Z. **197**, 123–138 (1988)

[CY] Cheng, S.Y., Yau, S.Y.: Hypersurfaces with constant scalar curvature. Math. Ann. **225**, 195–204 (1977)

[H] Hsiung, C.C.: Some integral formulas for closed hypersurfaces. Math. Scand. **2**, 286–294 (1954)

[MR] Montiel, S., Ros, A.: Compact hypersurfaces: The Alexandrov theorem for higher order mean curvatures. In: Lawson, B., Tenenblat, K. (eds.) Differential Geometry, a symposium in honor of Manfredo do Carmo. (Pitman Monogr., vol. 52, pp. 279–296) Essex: Longman Scientific and Technical 1991

[R] Reilly, R.C.: Variational properties of functions of the mean curvatures for hypersurfaces in space forms. J. Differ. Geom. **8**, 465–477 (1973)