BURKARD POLSTER, ANDREAS E. SCHROTH AND HENDRIK VAN MALDEGHEM

Generalized Flatland

With illustrations by the author A Hexagon*
Based on the gospel of GENERALITY as proclaimed by the POLYGONS

Most of my readers will be familiar with the sad story of my grandfather, an honourable square and eminent mathematician of FLATLAND who was condemned to lifelong imprisonment for claiming to have been abducted to SPACELAND, a world somewhere "out there" that extends our two-dimensional FLATLAND by a third dimension. Of course nobody, not even I, his grandson (a hexagon), believed in his story until, on the eve of the new millennium, I myself was abducted to GENERALIZED FLATLAND. I discovered that this world extends our flat world and the worlds of graphs and projective planes in a completely natural manner. As our world is populated by polygons such as triangles, quadrangles/squares, pentagons, etc., this extension of our world contains generalized polygons, both us simple ones and much more complicated ones of breathtaking abstract beauty. I also found that GENERALIZED FLATLAND coincides with the land of mathematical buildings of rank 2 as conceived by one of our foremost mathematicians J. Tits. This means that all non-trivial mathematical buildings are made up of natives of this mysterious land.

Preface

I will tell you my story and, as evidence of my claims, show you drawings of my abductors, the four smallest natives of proper GENERALIZED FLATLAND. These drawings are extensions of beautiful renderings of closely related highly homogeneous graphs such as the complete graph on four vertices, the Petersen graph, and the Coxeter graph (Fig. 1). In fact, closer inspection discloses that my abductors share many of the remarkable properties of these graphs and are even more symmetric than the graphs they extend. I hope that the overwhelming evidence I have compiled will convince even the most sceptical among you that there is really life "out there" beyond FLATLAND, and that we are able, and have an obligation, to claim our rightful place in full GENERALITY.

A Painting in the Sand

It was the last day of our 2000th year. I spent this all-important day at the site of some recently discovered ruins in the desert of OZ. After unearthing some mysterious mathematical writings and drawings in the ruins they were excavating at the time, the archaeologists in charge had in-

*Dedicated to my dear grandfather Edwin E. Abbot (1838–1926), the author of the infamous *Flatland—a Romance in Many Dimensions* [1].

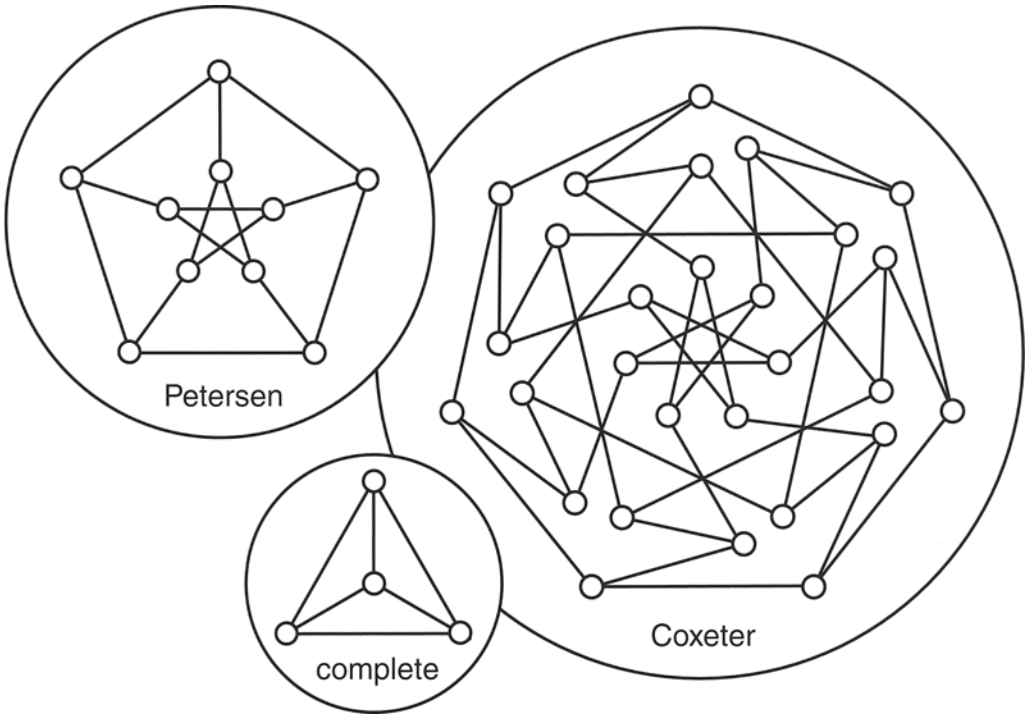

Figure 1. The complete graph on 4 vertices, the Petersen graph, and the Coxeter graph.

vited me to join their expedition as mathematical adviser. I had gladly accepted their offer and on that very day started deciphering the mathematical inscriptions that covered all the walls and floors. It soon became clear to me that what had been discovered here were some of the writings of the famous mathematical prophet J. Tits, in which he claims that there is a world he refers to as GENERALIZED FLATLAND that extends our world. Of course every child knows that these writings had been condemned as heresy and destroyed a long time ago. I was afraid to reveal my discovery to my colleagues in fear that they might destroy what turned out to be of true mathematical beauty, even though not referring to some real world as claimed by the prophet. My colleagues had already retired to their tents while I was still trying to unravel the mysteries of a pentagonal painting (Fig. 2) that occupied the interior of one of the rooms. After several hours of work, I summarized in mathematical language what I had learned so far from the inscriptions about GENERALIZED FLATLAND and its natives.

The geometry of GENERALIZED FLATLAND. Remember that a (point-line) *geometry* consists of a nonempty set of *points* and a nonempty set of subsets of the point set called *lines*, such that every point is contained in at least two lines and every line contains at least two points. Two geometries are *isomorphic* if and only if there is a bijection between the point sets of the two geometries that extends to a bijection between their line sets.

Every graph can be interpreted as a geometry. Here the vertices of the graph are the points, and associated with every edge is a line consisting of the two vertices contained in this edge. In particular, an *ordinary n-gon* is a geometry that is isomorphic to the geometry of vertices and edges of a regular n-gon in the plane, that is, one of the natives of FLATLAND.

Just as a graph can have multiple edges, that is, two or more edges that connect the same two vertices, a geometry can have multiple lines that cannot be distinguished by just looking at the points contained in them.

Let $\mathcal{G}$ be a geometry with point set P and line set L. A geometry $\mathcal{G}'$ with point set P' and line set L' is *contained in* $\mathcal{G}$, if the following three conditions are satisfied: (1) $P' \subseteq P$; (2) every line in L' is contained in a line in L; and (3) no two lines of L' are contained in one line of L.

Axioms for Generalized n-Gons of order (s, t)

(Q1) In a generalized n-gon $\mathcal{G}$ of order (s, t) every line contains $s + 1$ points and every point is contained in $t + 1$ lines.

(Q2) $\mathcal{G}$ does not contain any ordinary k-gons for $2 \leq k < n$.

(Q3) Given two points, two lines, or a point and a line, there is at least one ordinary n-gon in $\mathcal{G}$ that contains both objects.

The natives of GENERALIZED FLATLAND are the *generalized polygons*. Every generalized polygon is a *generalized n-gon* for some $n \geq 2$ and has an *order* (s, t), $1 \leq s, t$.

Generalized 2-gons, 3-gons, 4-gons, etc., are also called generalized digons, triangles, quadrangles, etc., respectively. Furthermore, an ordinary digon is just a graph consisting of two vertices that are connected by two edges. If a generalized polygon is of order (s, t), $s = t$, we also say that it is of order s. A generalized n-gon is *finite* if it contains only finitely many points and lines.

How I came to GENERALIZED FLATLAND and what I saw there

Although I did not fully understand these words, everything that I had learned in university seemed to suggest that we ordinary n-gons are the only generalized polygons and that therefore GENERALIZED FLATLAND = FLATLAND. At the same time I felt an irresistible urge to study the drawing (Fig. 2) further. At that point only a few grains of sand in the hourglass separated us from the new millennium.

Clearly, this drawing was supposed to be a picture of some geometry with 15 points and 15 lines (the 5 sides of the pentagon plus the 5 medians plus 5 circle segments), each point contained in 3 lines and each line containing 3 points. The overall shape seemed to suggest that this was a generalized pentagon, but I quickly discovered a number of quadrangles in the picture, no digons and no triangles,

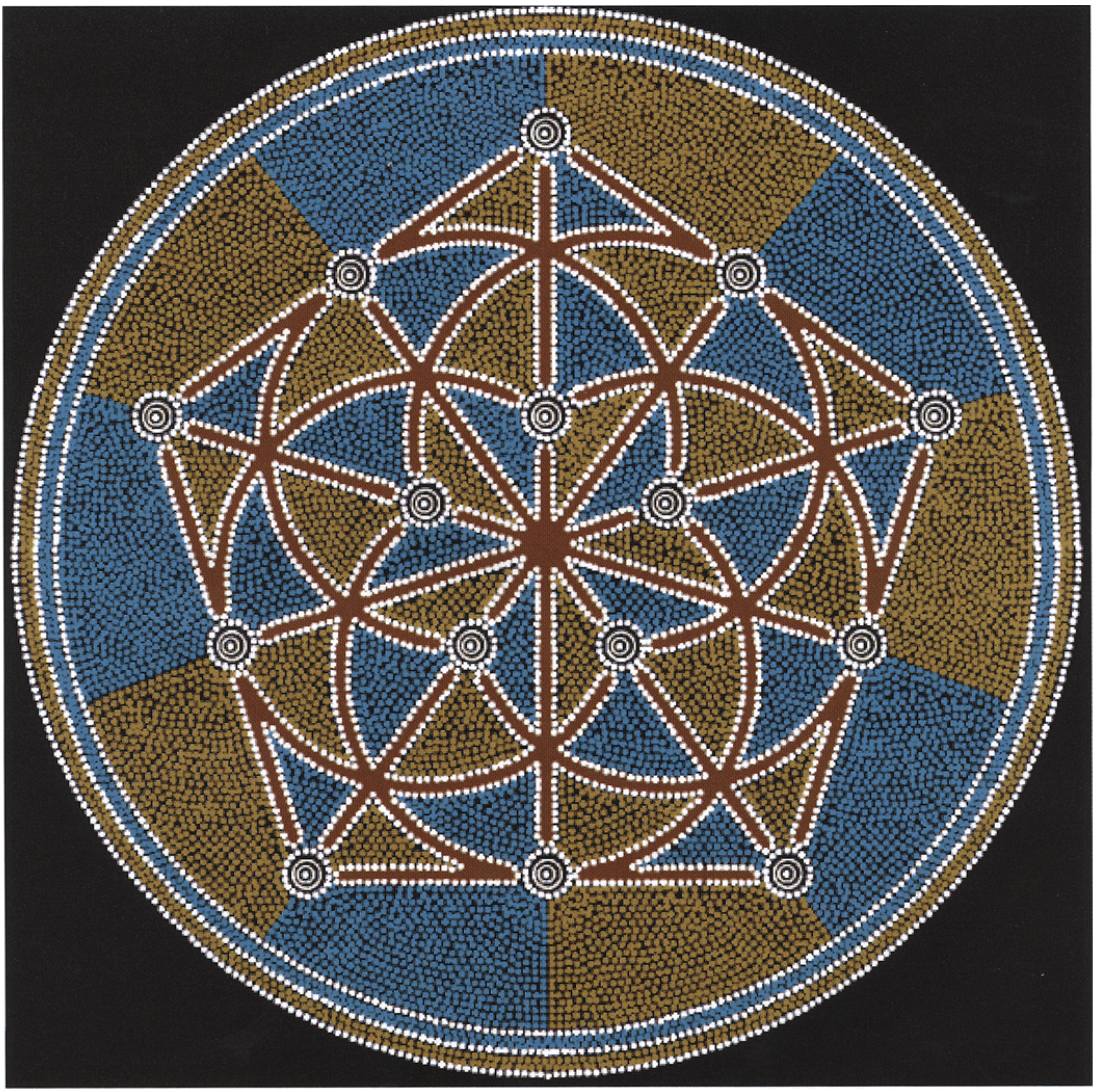

Figure 2. A sandpainting of the QUADRANGLE.

though. A true generalized quadrangle of order 2? Everything in me revolted against the mere idea and I exclaimed aloud: "GENERALIZED FLATLAND, what nonsense!"

Straightaway I became conscious of a presence and someone whispering, "Nonsense, is it indeed?" At the same time the painting on the floor seemed to come alive and right in front of my eyes first turned into a square, then into a pentagon, and then it took on ever-increasing "gonalities" until, finally, it consolidated into a 15-gon.

STRANGER: "What kind of mathematician are you not to believe in axioms and proofs that your own mind supplies to you? Do you really have to see to believe? Behold, then, as I am the generalized quadrangle whose shadow you have been staring at for such a long time."
I: "Pardon me, my Lord, but although you seem to have many gonalities they all seem to be distinct and unconnected, and at the moment I only see a 15-gon."
STRANGER: "Why, of course this is because I am *thick*—that is, a generalized polygon that has at least 3 points to every line and at least 3 lines through every point. One of us thick ones just does not fit into FLATLAND, and a *thin* generalized polygon like you can only see one of my ordinary n-gons at a time. Ah, I can see it in your eyes, you still don't believe me. Stay!" [I was inching my way towards the entrance of the room.] "I will prove to you that GENERALIZED FLATLAND $\neq$ FLATLAND. Let me first tell you about the generalized digons, the simplest generalized polygons. In a generalized digon there is a line that contains all the points of the geometry, and all lines are just copies of this line. Of course this also implies that every point is contained in all lines. Q.E.D."
I: "Dear Sir, with all due respect, but, being just a collection of identical copies of a line (which most certainly lives in FLATLAND), I cannot but think of these geometries as lines who pretend to be more than what they really are. If this is the only evidence you can muster, I have to say that I am not convinced."
STRANGER: "Of course you are right, but how can you be so hasty? Following good mathematical practice, I started by covering all those examples that, although not really having a life of their own at first sight, still fit the axioms. Now, listen further. The next step up from you ordinary polygons are the *slim* generalized polygons. Being neither thin nor thick, they are lost inanimate souls caught between your world and mine, present in both yet not belonging to either. For example, the simple square grids are examples of slim generalized quadrangles. Mistaking these pitiable beings as natural features, you have built your cities based on their underlying structure."

As he said this the stranger pointed to a drawing of a grid in one of the corners of the room (see Figure 3).

I: "Hmm, again these beings can still be considered as belonging to FLATLAND, and as you said yourself, they are really quite dead and amount to nothing special."
STRANGER: "Ah, you are really very hard to please. How shall I convince you? Well, all good things come in three, so let us move on to the generalized triangles. By Axiom Q2, a

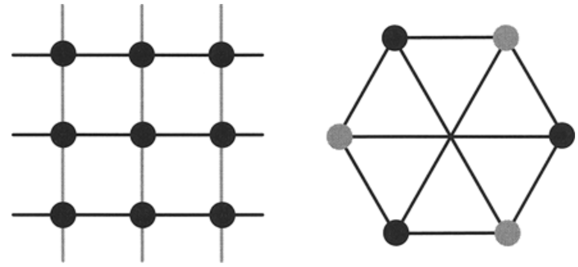

Figure 3. A grid and its dual, two slim generalized quadrangles. The black and gray lines in the grid correspond to the points of the corresponding colour in the graph. (The dual makes its appearance later in the story.)

generalized n-gon, $n > 2$, does not contain any digons. This just means that 2 of its points are connected by at most 1 line and that 2 of its lines intersect in at most 1 point. This observation and Axiom Q3 imply that in a generalized triangle 2 points are contained in exactly 1 line and 2 lines intersect in exactly 1 point."
I: "Wait, that sounds very familiar! Doesn't this mean that the thick generalized triangles are just the projective planes?"
STRANGER: "Finally, you are beginning to understand. Generalized triangles are something your mathematicians have known for a long time, only not by their rightful name."
I: "Oh yes, I see. But still, projective planes, or generalized triangles, are just abstract mathematical structures. They do not really exist."
STRANGER: "That does it! Deeds are called for and not words. I will introduce you to the members of my family: the POLYGONS consisting of the DIGON, the TRIANGLE (also known as the smallest projective plane or the Fano plane), the QUADRANGLE (myself), and the Siamese twins the HEXAGON and its dual. We are the only generalized polygons with exactly 3 points on every line and 3 lines through every point. As our names suggest, we are generalized digon, triangle, quadrangle, and hexagon, respectively. As a family we occupy as prominent a position in GENERALIZED FLATLAND as my brother the TRIANGLE does among the projective planes. Now, out of your plane you go!"

At this moment an unspeakable horror seized me and I was no longer "in" the room with the painting. Nevertheless, I was still able to see it in a strange way that reminded me very much of the account of SPACELAND that my grandfather had given. I noticed that the painting had changed, as now a small drawing of a hexagon was visible at the spot where I had stood just a moment ago. I turned "around" and for the first time I beheld the QUADRANGLE in all his glory and realized that just as the painting was his shadow, the addition to the painting was my own. Before I had been separated from FLATLAND I must have uttered a loud cry, because some of the archaeologists were running towards the very spot I had occupied just a moment earlier.

The TRIANGLE

QUADRANGLE: "Now listen, you need to learn as much as you can about us if, on your return, you don't want to share

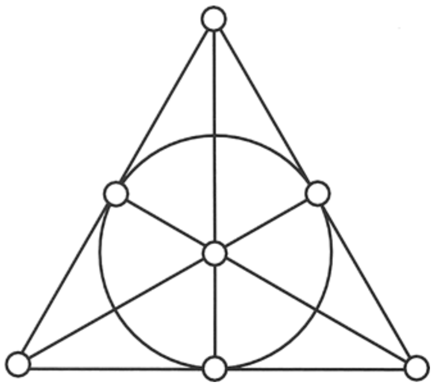

Figure 4. The most famous shadow of the TRIANGLE.

your grandfather's fate. Meet my brother the TRIANGLE, the smallest projective plane!"

The 7-Gonality of the TRIANGLE He vanished, and his place was taken by a stranger who introduced himself as the TRIANGLE. I noticed that my friends the archaeologists were frantically gesturing at the painting on the floor, which had also changed its shape. Clearly, this was the shadow of the TRIANGLE (see Figure 4 for a reconstruction of what my friends saw). As one person, my companions were seized by a great fear, and first fled the room, then the excavation site, and finally the desert itself, never to return.

TRIANGLE: "Although being very small and easily understood, I hold the key to a full understanding of my more complicated brothers the HEXAGONS and the QUADRANGLE! To illustrate what I mean by this, I need to show you a very special shadow of myself on a regular 7-gon and derive a neat labelling of my points and lines that will prove extremely useful in understanding the HEXAGONS. If you count carefully, you will find that I and my shadows have exactly 7 points and 7 lines."

While he was saying this he was turning inside out in a completely unexplainable manner and his shadow took on a 7-gonal appearance. Then he made a sudden movement that resulted in the mirror image of this 7-gonal shadow (Fig. 5).

In both cases the points of the shadow were the 7 points of the underlying 7-gon, and its lines were the images of one of the two triangles under successive rotations through 360/7 degrees around the center of the 7-gon.

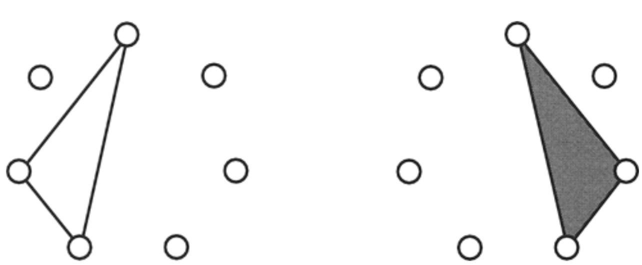

Figure 5. The TRIANGLE in terms of positive and negative Fano triangles.

I: "How marvellous! Who would have guessed by just looking at your initial shadow, which only exhibited symmetries of orders 2 and 3, that you also have a symmetry of order 7. By combining all these symmetries we arrive at a total of $2 \cdot 3 \cdot 7 = 42$ symmetries. Your brother mentioned that you are also an incarnation of the smallest projective plane. As such you should have even more symmetries, is that not so?"

TRIANGLE: "That is true indeed. In fact, just as your shadow does not capture your full being, our shadows only capture part of our complex structure. I trust that you are familiar with the concept of a symmetry group of a geometry? Well then, just remember that for me and my brothers our symmetry groups act sharply transitively on the ordered $(n + 1)$-gons contained in us, where n is our gonality. This means that the orders of these groups coincide with the number of ordered $(n + 1)$-gons contained in us."

I: "Wait, bear with me while I am trying to understand what you just said. You are a generalized triangle, therefore your gonality is 3. From what you just said it is clear that the ordered quadrangles contained in you are very important to you."

TRIANGLE: "So far your reasoning is flawless, but can you deduce how many such ordered quadrangles are contained in me?"

I: "I will try. You contain 7 points from which to choose the first vertex p of an ordered quadrangle. Any of the remaining 6 points can be chosen as the second vertex q. The connecting line of p and q contains one further point that cannot be chosen as the third vertex r. This means that there are only 4 points left to choose this vertex from. The lines connecting p and q, p and r, and q and r contain a total of 6 points. This means that the last vertex s in the quadrangle is the remaining 7th point. This implies that you contain a total of $7 \cdot 6 \cdot 4 = 168$ quadrangles! This means that your symmetry group has order 168. You are truly symmetric!"

TRIANGLE: "Very good. You really think that I am very symmetric? Wait until you encounter the QUADRANGLE and the HEXAGONS. Their symmetry groups have orders 720 and 6196, respectively!"

I: "Fantastic, but what about the DIGON?"

TRIANGLE: "Well spotted. Its symmetry group has order 36, and I am sure you will be able to verify this for yourself once you think about it for a moment. But enough of this. We do not have much time. Let us again consider my 7-gonal shadows. I call the lines in the shadow that correspond to the left and right diagrams *positive and negative Fano triangles*. One more model of me is hiding in this picture. Its points are the left Fano triangles and its lines are the right Fano triangles. Here a point is abstractly contained in a line if and only if the corresponding triangles have exactly one vertex of the underlying 7-gon in common. You will understand what I mean by this after you have been instructed in the mysteries of doubling."

The punishment of doubling

TRIANGLE: "The *double* (also incidence graph) of a point-line geometry is the graph whose vertices are the points and lines of the geometry. Two vertices are connected by an edge if and only if they correspond to a point and a line such that the point is contained in the line. Note that all the information about a geometry is contained in its double, which means that you don't really die when you are doubled. On the other hand, since doubles also get squashed into your FLATLAND, it is generally believed that we lose all awareness of ourselves after having been doubled. In fact, traditionally the worst punishment for a thick generalized polygon is to be doubled, and this is exactly what is going to happen to us POLYGONS if the other thick generalized polygons find out about us talking to you."

I: "If I understand you correctly, then the double of one of us ordinary n-gons should be an ordinary $2n$-gon and, since we live in FLATLAND to start with and our status is directly dependent on our gonality, doubling should be just about the best thing that can happen to one of us."

TRIANGLE: "You are very quick; but, unlike you thin ones, most thick generalized polygons cease to be generalized polygons after being doubled, and their gonality is irretrievably lost in the process of doubling."

I: "But why? After all, any ordinary k-gon in a geometry becomes an ordinary $2k$-gon in its double and any ordinary l-gon in the double comes from an ordinary $l/2$-gon in the original geometry. So, any doubled geometry contains only ordinary n-gons with even n. Also, if the original geometry contains no ordinary k-gons with $k < n$, then the double contains no ordinary k-gons with $k < 2n$. Doesn't this imply that the double of a generalized polygon is just another generalized polygon?"

TRIANGLE: "You forgot about Axiom Q1. Your arguments only take care of Axioms Q2 and Q3. In fact, it is exactly the generalized n-gons of order t that turn into slim generalized $2n$-gons after doubling. This means that, for example, we the POLYGONS are still generalized polygons after having been doubled. Using this revolutionary insight, my brothers and I discovered we could deliberately double ourselves, live in this state in FLATLAND for extended periods of time, and revert to our usual states whenever it pleased us. It was during those visits that we took a liking to your kind and decided to help you claim your rightful place in GENERALIZED FLATLAND."

He proceeded to demonstrate what he meant by doubling himself, a process too awful to describe in detail, at the end of which he (or his double) coincided with his shadow. Figure 6 shows the double of the TRIANGLE whose vertices have been labelled with the two different kinds of Fano triangles. It is clear that this double is a slim generalized hexagon.

The HEXAGONS

TRIANGLE: "So far it seems that we have found in you a fit apostle for the gospel of GENERALITY. But let us see how you fare in the presence of the HEXAGONS."

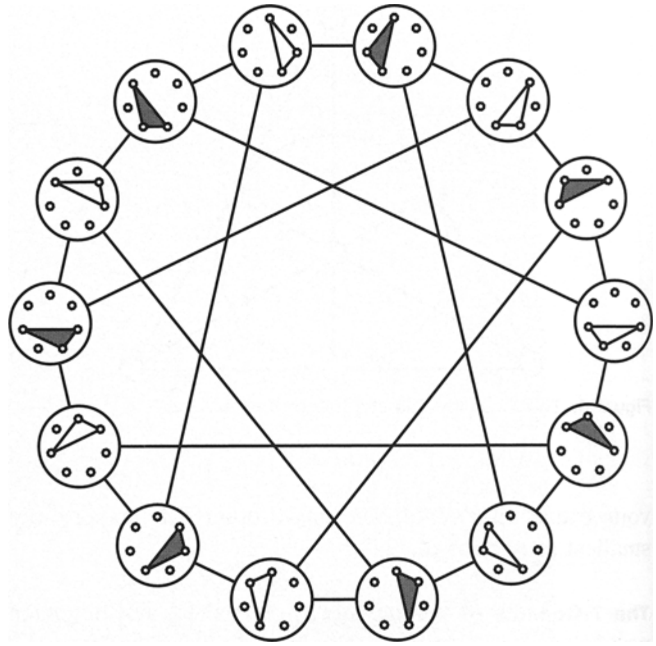

Figure 6. The double of the TRIANGLE. Vertices are connected by an edge if the triangles in their labels share exactly one point of the underlying 7-gon.

A narrow escape With a laugh he vanished and his place was taken by a being so glorious in appearance and complexity that at first I was too dazzled to pay any attention to what the being was saying. But even when I had recovered enough to pay attention to the noise emanating from it, I could not make out any words. Also, the being seemed to flicker between two completely different states. As I watched, it became more and more agitated and started making threatening moves towards me. Finally, something clicked into place in my mind, and flicking open a certain page in my notebook I reread a passage that I had translated earlier on and that had made no sense to me at that time.

> The *dual* of a geometry $\mathcal{G}$ is constructed by interchanging the roles of points and lines in $\mathcal{G}$. More precisely, its points are the lines of $\mathcal{G}$, and to every point of $\mathcal{G}$ corresponds a line of the dual consisting of all lines in $\mathcal{G}$ containing this point. A geometry is called *self-dual* if it is isomorphic to its dual, and an isomorphism between a geometry and its dual is called a *duality*.

Clearly, I thought to myself, if all this is true, then we ordinary n-gons are self-dual; the dual of a grid is a complete bipartite graph; and, come to think of it, there was such a graph drawn right next to the grid that the QUADRANGLE had pointed out to me (see again Figure 3). Also, since the TRIANGLE and the QUADRANGLE are the only smallest thick generalized triangle and quadrangle, they must both be self-dual, that is, coincide with their duals. In fact, the self-duality of the TRIANGLE follows immediately from its description in terms of Fano triangles, and a duality corresponds to a reflection of Figure 6 through its vertical symmetry axis. On the other hand, the HEXAGONS must be two separate geometries that are forever intertwined by a duality; what I was witnessing here was the two HEXAGONS speaking to me at the same time. Time was running out, and if I didn't want to be squashed under the weight of the HEXAGONS, I had to find a way to communicate with them. I started blinking my eyes and ears in unison with the flickering of the HEXAGONS and, lo and behold, I was able to see and hear only one of them.

HEXAGON: "Congratulations. One moment longer, and we would have doubled you (and your intelligence)."

Here he laughed a mischievous laugh, and it occurred to me in a flash that I had just missed a unique opportunity to raise my gonality from 6 to 12.

A simple numbers test

HEXAGON: "But then again, doubling is a fairly painful process and you might not have enjoyed it. Anyway, please refer to me as the (Cayley) HEXAGON. I will speak for both myself and my dual. In the future you may encounter other geometries who will pretend to be one of us POLYGONS and try to dissuade you from lighting the fire of GENERALITY in FLATLAND. Therefore, let us start by deducing some basic properties of us HEXAGONS, such as the number of our points and lines, and a simple test that will allow you to distinguish us from any impostor.

"You have seen that points and lines play similar roles. So let us refer to the points and lines of a geometry jointly as its *vertices*, and inductively define a distance between the vertices. The vertices at distance 1 from a point are the lines through this point, and the vertices at distance 1 from a line are the points on this line. Given a vertex e, the only vertex at distance 0 from e is e itself. A vertex is at distance $n > 1$ from e if it is not at distance $m < n$ and if it is at distance 1 from a vertex at distance $n - 1$ from e. The *diameter* of a geometry is the maximum distance between two of its vertices. As an immediate consequence of Axiom Q3 we see that a generalized n-gon has diameter n."

I: "Does this mean that if e is a point, then all vertices at odd and even distances from e are lines and points, respectively?"

HEXAGON: "That is correct. Given one of our vertices e, we can now count the number D_n^e of vertices at distance n from e using the axioms, the inductive definition above, and the fact that there are exactly 3 vertices at distance 1 from e. We conclude that $D_0^e = 1$, $D_1^e = 3$, $D_n^e = 2D_{n-1}^e$ for $2 \leq n \leq 5$. If e is a point, then a line is at distance 1, 3, or 5. This means that there is a total of $3 + 12 + 48 = 63$ lines. The dual argument yields that there are also 63 points. Consequently, $D_6^e = 63 - (1 + 6 + 24) = 32$.

"Simple counting arguments also show that a geometry with 63 points and 63 lines is one of us HEXAGONS if and only if, for all n with $1 \leq n \leq 5$ and vertices e, the numbers D_n^e coincide with the corresponding ones in us HEXAGONS. Using similar arguments, simple counting criteria can be deduced for any finite generalized polygon."

I: "Of course. In fact, I can see immediately that a geometry with 7 points and 7 lines is your brother the TRIANGLE if and only if $D_1^e = 3$ and $D_2^e = 6$ for all its vertices."

From TRIANGLE to HEXAGON

HEXAGON: "Very good. Now behold my shadow (Fig. 7). Pretty, isn't it, . . . but I am sure you would not be able to remember it, if I didn't tell you a little bit more about the way I am built. In the following I will describe my vertices in terms of the vertices of my brother the TRIANGLE. To avoid confusion, I will refer to vertices of the TRIANGLE as T-vertices and vertices of me the HEXAGON as H-vertices. A point-line pair $\{p, L\}$ of a geometry is called a *flag* or *anti-flag* if p is or is not contained in L, respectively.

"Look at me closely. Can you see that I have 4 different kinds of H-points? These are the T-points (7), T-lines (7), flags (7 T-points $\cdot$ 3 T-lines through a T-point = 21 flags) and anti-flags (7 T-points $\cdot$ 4 T-lines not through a T-point = 28 anti-flags) of the TRIANGLE. This gives a total of $7 + 7 + 21 + 28 = 63$ H-points. There are two different kinds of H-lines containing 3 H-points each. H-lines of the first kind are sets of the form $\{p, L, \{p, L\}\}$, where $\{p, L\}$ is a flag of the TRIANGLE. Clearly there are as many H-lines of this type as there are flags of the TRIANGLE; that is, there are 21 such H-lines. An H-line of the second kind is of the form $\{\{p, L\}, \{q, M\}, \{r, N\}\}$, where (1) $\{p, L\}$ is a flag; (2) p, q, and r are the 3 T-points contained in L; (3) L, M, and N are the 3 T-lines through p. This implies that both $\{q, M\}$ and $\{r, N\}$ are anti-flags. It is clear that there are two such H-lines associated with every flag of the TRIANGLE, that is, there are 42 such H-lines. This gives a total of $21 + 42 = 63$ H-lines."

I: "Wait, wait, let me check this using the numbers test. Hmm, from what the TRIANGLE has told me about itself I know that its symmetry group acts transitively on its sets of points, lines, flags, and anti-flags. This means that none of these objects is distinguished in any way, and it suffices to check that the numbers D_n^e pan out for the four essentially different kinds of H-points and the two essentially different kinds of H-lines. Now, if we take . . ."

A magic labelling of H-points and H-lines

HEXAGON: "Correct, correct, but the rest is just trivial bookkeeping. I am afraid we don't have the time for this right now. Instead, let us draw my shadow based on the above description. We start with the model of the TRIANGLE whose points and lines are positive and negative Fano triangles, as in Figure 6. With respect to this model there are 9 essentially different H-points of the HEXAGON as illustrated by the first row of labels in Figure 8. The remaining (labels of) H-points are the images of these 9 labels under successive rotations through 360/7 degrees around the center of the underlying 7-gon.

"We replace every label by a simpler label as indicated by the second row in Figure 8. Rotated labels get replaced by new labels that have been rotated in a corresponding

The POLYGONS

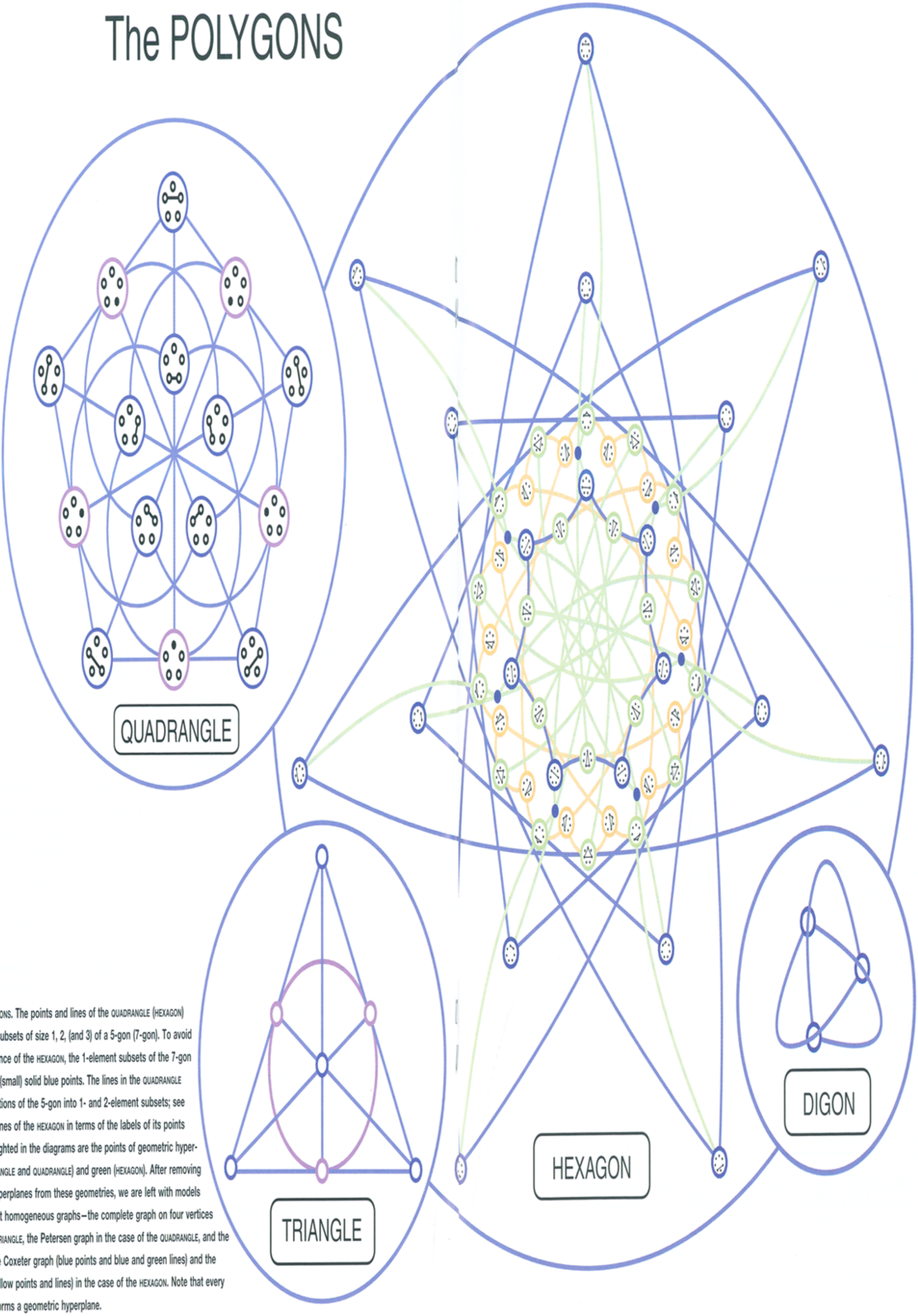

Figure 7. The POLYGONS. The points and lines of the QUADRANGLE (HEXAGON) correspond to the subsets of size 1, 2, (and 3) of a 5-gon (7-gon). To avoid a crowded appearance of the HEXAGON, the 1-element subsets of the 7-gon are represented by (small) solid blue points. The lines in the QUADRANGLE correspond to partitions of the 5-gon into 1- and 2-element subsets; see Figure 10. For the lines of the HEXAGON in terms of the labels of its points see Figure 9. Highlighted in the diagrams are the points of geometric hyperplanes—purple (TRIANGLE and QUADRANGLE) and green (HEXAGON). After removing these geometric hyperplanes from these geometries, we are left with models of some of the most homogeneous graphs—the complete graph on four vertices in the case of the TRIANGLE, the Petersen graph in the case of the QUADRANGLE, and the disjoint union of the Coxeter graph (blue points and blue and green lines) and the Heawood graph (yellow points and lines) in the case of the HEXAGON. Note that every point of the DIGON forms a geometric hyperplane.

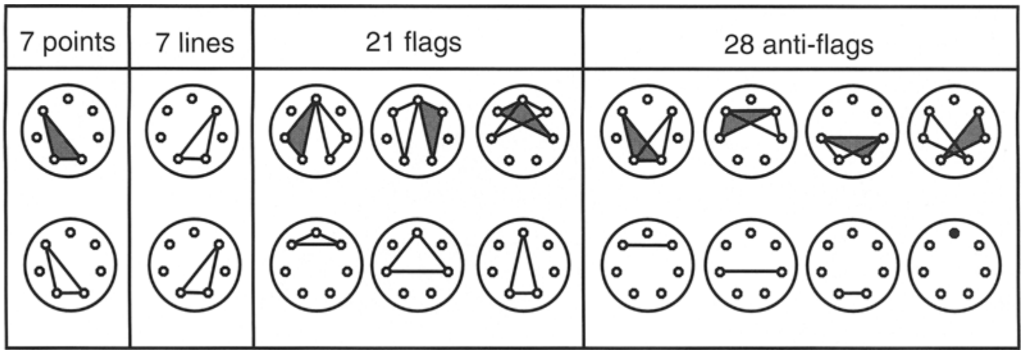

Figure 8. Labels for the points of the HEXAGON.

manner. Note that the new labels correspond in a natural way to all 1-, 2-, and 3-element subsets of the set consisting of the 7 vertices of the underlying 7-gon. In terms of the new labels there are 9 essentially different kinds of lines of the HEXAGON; see Figure 9.

"It is clear that every symmetry and duality of the TRIANGLE induces a symmetry of the HEXAGON. Encoded in the labels is an order 7 symmetry of the TRIANGLE and an order 2 symmetry that corresponds to a duality of the TRIANGLE. Using the labels, it is easy to reconstruct my shadow; see Figure 7."

I: "I understand all this. Except for the step where you replace the original labels by new labels. It seems that the new label associated with a label containing two Fano triangles is either the symmetric difference of the two triangles or the complement of this difference."

Strength in projective spaces

HEXAGON: "Ah, yes that is correct. In fact, the main source of our power can be explained using the mathematical operation that corresponds to this 'step.' Let S be a set with an odd number $|S| > 1$ of elements, and let $S_{1/2}$ be the set of all nonempty subsets of S with fewer than $|S|/2$ elements. If $A, B \in S_{1/2}$, $A \neq B$, let D be the symmetric difference of A and B and define $A \oplus B$ to be D if $D \in S_{1/2}$ or $S \setminus D$ otherwise. We define a geometry $\mathcal{G}(S)$ whose point set is S and whose lines are the sets $\{A, B, A \oplus B\}$ where A and B are distinct elements of S. Every line in this geometry contains 3 points. Furthermore, given two points P and Q on a line, the third point on the line is always $P \oplus Q$. This implies that any two points in the geometry are contained in exactly one line. Closer inspection reveals that the geometry is isomorphic to the projective space of dimension $|S| - 2$ over the field with two elements, for short PG($|S| - 2, 2$). See [10] and [8] for more details about this representation of PG($|S| - 2, 2$).

"As you have already observed, the rule that assigns a new label to one of the original labels can also be stated in terms of the operation $\oplus$. Here S consists of the vertices of the underlying 7-gon, and if a label consists of two Fano triangles A and B (sets of three vertices), then the new label is $A \oplus B$.

"With the above remarks it should be clear to you that my H-points coincide with the points of the 5-dimensional projective space PG(5, 2). Furthermore, . . . "

I: "I think I know what you are getting at. Your lines are also . . . wait, let me double-check this . . . Yes, any two H-points on any of your H-lines $\oplus$-add up to the third H-point on this H-line."

HEXAGON: "Exactly! This means that I am a subgeometry right at the center of this projective space, which is an important source of power for me."

I: "So there really are beings that live in spaces of a dimension greater than two, just as my grandfather claimed (although this dimension is quite different from the 'tangible' dimensions he had in mind!)."

Hyperplanes, Heawood graph, and Coxeter graph

I: "How miraculously all this fits together! But I am sure that there is much more beauty hiding in your shadow. For example, I just noticed that every one of the H-line labels in Figure 9 contains exactly one isosceles triangle. This seems to suggest that the H-points that correspond to these labels form a very special set of points."

HEXAGON: "We have indeed made the right choice in selecting you to be our messenger! Your remark reminds me of

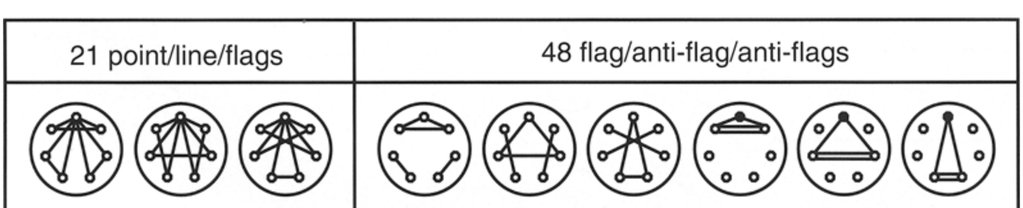

Figure 9. Labels for the lines of the HEXAGON.

something else we should talk about. By now you will probably have guessed that the kind of conversation we are having is extremely dangerous. It is only possible during the first hours of a new millennium, because at this time the BUILDINGS we are part of are too busy celebrating to broadcast every word that is said to the rest of (thick) GENERALIZED FLATLAND. To be able to communicate with us even after your return to FLATLAND, you have to know a little about the flat subgeometries that my different kinds of H-points and H-lines correspond to.

"*A geometric hyperplane H* of a geometry is a set of points such that every line either contains exactly one point of H or is completely contained in H. The set of all flag H-points (isosceles triangles) is a special geometric hyperplane that intersects every H-line in exactly one point (every one of the labels in Figure 9 contains exactly one such triangle). Imagine that we remove the points of this hyperplane from me and my H-lines. Then we are left with two famous graphs: the Coxeter graph, and the double of the TRIANGLE, which in FLATLAND is also known as the Heawood graph.

"The vertices of the Heawood graph are the H-points that correspond to points and lines of the TRIANGLE. The edges of this graph are induced by the H-lines of the point/line/flag type. The picture of the Heawood graph right in the middle of my shadow in Figure 7 corresponds to Figure 6.

"The vertices of the Coxeter graph are the H-points corresponding to the anti-flags of the TRIANGLE. The edges of this graph are induced by the H-lines of the flag/anti-flag/anti-flag type. This corresponds to a well-known representation of the Coxeter graph; see [6]. Also, the picture of this graph in the middle of Figure 7 corresponds, via some obvious rearrangements, to the most famous representation of this graph depicted in Figure 1 (three 7-gons joined together via 7 extra points).

"By the way, the presence of a special hyperplane as above distinguishes me from my dual. Also, after you are back in FLATLAND I will keep these two graphs immersed in FLATLAND so that you can communicate with me via either one of them."

Misfortune Strikes

At this moment the BUILDING we were hiding in started shaking violently.

HEXAGON: "We are discovered! Dear friend, always remember what we have told you today, and no matter what happens now you should be able to find me and my brothers again and finish what we have begun. Beware of the OCTAGON in the PENTAGON, because . . . "

THUNDERING VOICE: "HEXAGON, you and your brothers have committed the heinous crime of communicating with the thin ones. For this you will suffer the terrible fate of doubling."

At this moment the ceiling slammed down on my new friend and me, and we were both squashed back into FLATLAND. When I regained consciousness it was morning, and I found myself in the very room where all this had started. I automatically assumed that the night's adventure had been a dream induced by what I had read on the walls. But then I discovered that all the writings had vanished and that none of my companions was anywhere to be seen. I also found, to my utter amazement, that my gonality had been raised to 12—I had been doubled. Although still somewhat shaken, I immediately started looking for the doubles of the POLYGONS—to no avail. I realized that, using my doubled IQ and the unprocessed notes in my notebook, I first had to deduce as much as possible about the POLYGONS and their doubles; then, to convince you my fellow flatlanders of their existence, locate their whereabouts in FLATLAND, and with their help claim our rightful place in full GENERALITY.

The QUADRANGLE and the DIGON

It was a long journey back home. I spent most of the time organizing my notes and developing a mathematical theory of GENERALIZED FLATLAND.

Following the procedures the HEXAGON had introduced me to, it was easy to show that the QUADRANGLE has 15 points and 15 lines, that its diameter is 4, that $D_0^e = 1$, $D_1^e = 3$, $D_2^e = 6$, $D_3^e = 12$, $D_4^e = 8$ for all vertices of the QUADRANGLE, and that these numbers suffice to recognize the QUADRANGLE among geometries. I also found a geometric construction of the QUADRANGLE as a *derived geometry* at a point of the HEXAGON; see [3]. However, this construction is rather complicated, and executing it within the shadow of the HEXAGON yields a model of the QUADRANGLE with only very few symmetries. After two sleepless days and nights, I finally succeeded in reconstructing the shadow that I first saw in the ruins.

The Shadow of the QUADRANGLE revisited Let S be the set of vertices of a regular pentagon. The points of the shadow are all elements of $S_{1/2}$, that is, all 1- and 2-element subsets of S. The lines are the partitions of S into two 2-element subsets and one 1-element subset of S. Then there are essentially 3 different kinds of points and 3 different kinds of lines, as illustrated by the labels in Figure 10. Of course this representation parallels the representation of the HEXAGON as a subgeometry of the projective space PG(5, 2) and identifies the QUADRANGLE as a subgeometry right in the middle PG(3, 2). Using the labels, it is possible to reconstruct the shadow of the QUADRANGLE as in Figure 7.

Geometric hyperplane and Petersen graph Just like the HEXAGON, the QUADRANGLE also contains geometric hyperplanes that intersect every line in exactly one point. One is visible right in the centre of its shadow. It consists of the five 1-point subsets of S. If we remove the points of this hyperplane from the QUADRANGLE and its lines, we are left with the famous Petersen graph. Also, the picture of this graph in the diagram of the QUADRANGLE in Figure 7 corresponds to the most famous representation of this graph depicted in Figure 1 (two 5-gons joined together). I

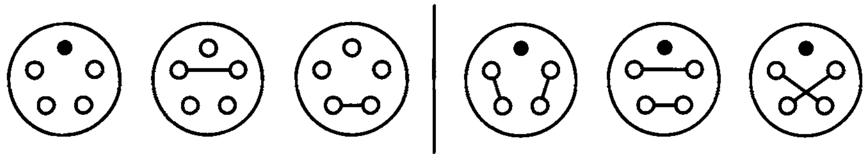

Figure 10. The points and lines of the QUADRANGLE.

assume that the QUADRANGLE planned to stay in touch with us in this form.

For completeness' sake I remark that the lines of the TRIANGLE are geometric hyperplanes. After deleting one of these hyperplanes from the TRIANGLE, we are left with the complete graph on four vertices. As you are probably aware, this graph, the Petersen graph, and the Coxeter graph are almost as homogeneous as the POLYGONS they are contained in; see [2].

The derived geometry and from DIGON to QUADRANGLE You are asking me where in all this the DIGON fits in? Although I never had the honor of meeting the DIGON, I found it very easy to reconstruct its shadow (see Fig. 7). Note that it contains 3 points and 3 lines, and that every line contains all the points. Your first reaction may be similar to mine when the QUADRANGLE first introduced me to generalized digons: "What's the big deal?" Well, it turns out that there is a labelling of the QUADRANGLE in terms of the DIGON that is the direct equivalent of the labelling of the HEXAGON in terms of the TRIANGLE: The points of the QUADRANGLE are the points, lines, and flags of the DIGON. There are two kinds of lines. The lines of the first kind are of the form $\{p, L, \{p, L\}\}$, where $\{p, L\}$ is a flag of the DIGON. The lines of the second kind are of the form $\{\{p, L\}, \{q, M\}, \{r, N\}\}$ such that $\{p, q, r\}$ and $\{L, M, N\}$ are the point and line sets of the DIGON.

The Doubles of the POLYGONS

It seems obvious to me that the POLYGONS intended to be present in FLATLAND in the form of some special graphs. According to their original plan they would be surveying proper GENERALIZED FLATLAND by using only the points of one of their special geometric hyperplanes, with the rest of their bodies immersed in FLATLAND (in this form they are almost invisible). If this is what they are doing, then to get in touch with them we have to locate the graphs in Figure 1 and Figure 6. Of course it is also possible that even surveying just using a geometric hyperplane is too risky at the moment and they are existing only as their doubles and are fully immersed in FLATLAND.

My investigations had confirmed my belief that the POLYGONS had revealed their most symmetric shadows and subgeometries to me. I therefore proceeded to reconstruct the most symmetric representations of their doubles.

I had already encountered an attractive picture of the double of the TRIANGLE in Figure 6. Also, it turned out that the double of the DIGON is the complete bipartite graph on 6 vertices in Figure 3. Of course this meant that, without my realising it at the time, the DIGON had been present in this form throughout my conversations with his brothers right next to their shadows.

To construct the best picture of the double of the QUADRANGLE, I considered the path in this geometry depicted in Figure 11. Since this is a path, two of its adjacent vertices correspond to a flag in the QUADRANGLE. Furthermore, this path contains the different kinds of points and lines in Figure 10 exactly once, except for its beginning and its end, which are two points of the same kind. If we fit together the 5 images of this path under rotations of the 5-gon underlying the labels, we arrive at a path that contains every point and line of the QUADRANGLE exactly once and is invariant under the rotations. This enables us to draw a picture of the double such that the vertices of the graph are the vertices of a 30-gon, two adjacent vertices of the 30-gon are connected by an edge, and rotations through 360/5 degrees around the center of the 30-gon leave the double invariant. Figure 12 is a picture of the double that has been constructed in this way. This also shows that the QUADRANGLE contains 15-gons like the one I saw in the ruins and that it is self-dual. Note that the reflection through the vertical symmetry axis of the diagram corresponds to a duality of the QUADRANGLE.

Figure 13 shows a similar path in the HEXAGON which can be used to model the double of this geometry on a regular 126-gon such that two adjacent vertices of this polygon are connected by an edge, and rotations through 360/7 degrees around the center of the polygon leave the double invariant. See [9, Section 13.5] for a picture of the double that has been constructed in this way.

Where to From Here?

When I finally arrived back in my hometown, I discovered that in my absence I had been accused of high treason and the police were looking for me everywhere. All this re-

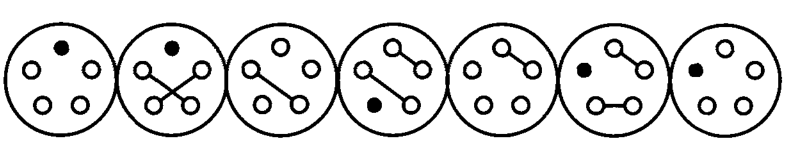

Figure 11. A special path in the QUADRANGLE.

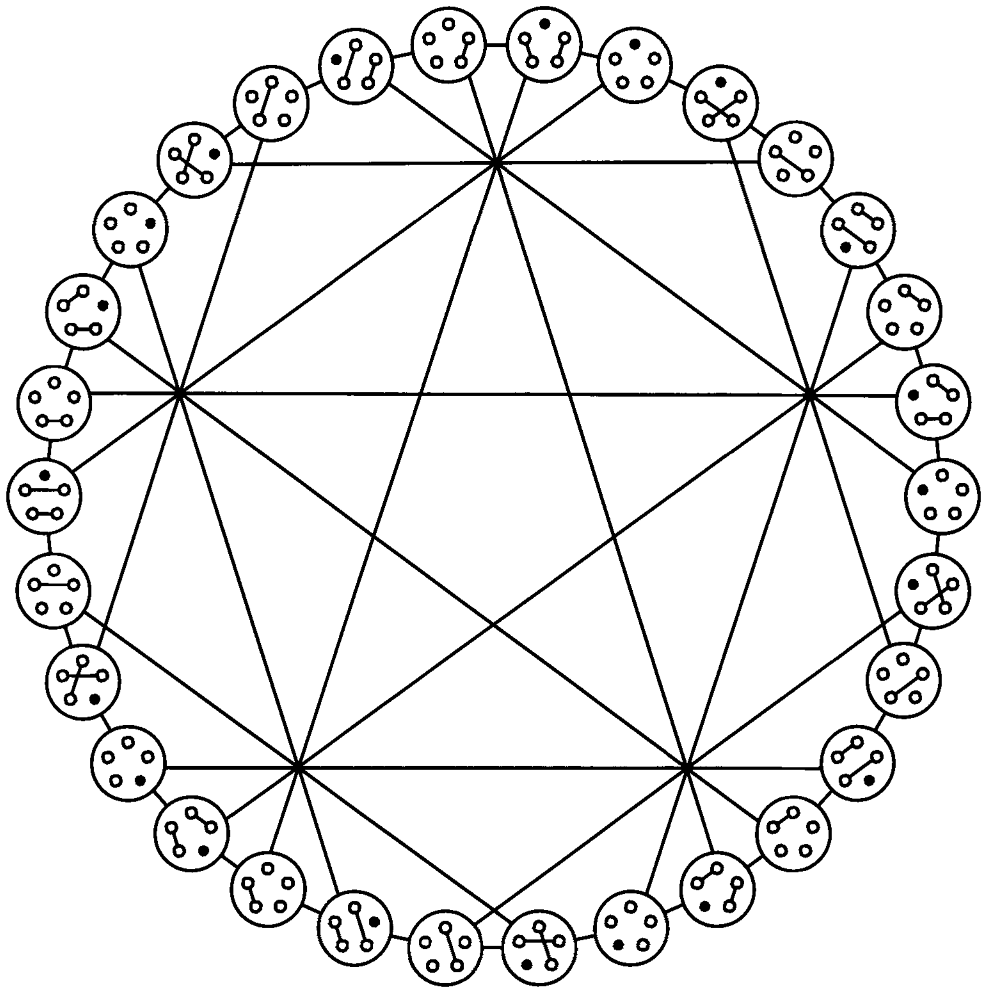

Figure 12. The double of the QUADRANGLE, a generalized octagon.

minded me so much of what had happened to my grandfather. Of course I was only a boy when he first told me about his abduction, and at that time his story sounded like the ramblings of a madman to me. But now that I had been abducted myself and reconsidered what he had told me with my doubled intellect, it all made perfect mathematical sense. So, why had he been locked away for something that our incredibly intelligent multigonal rulers should have recognized as the truth? And why were the authorities after me all of a sudden? I needed time to think. Since the police were looking for a hexagon I did not have to fear too much, of course.

But the HEXAGON had warned me to beware of the "OCTAGON in the PENTAGON." What had he meant by this? A (gen-

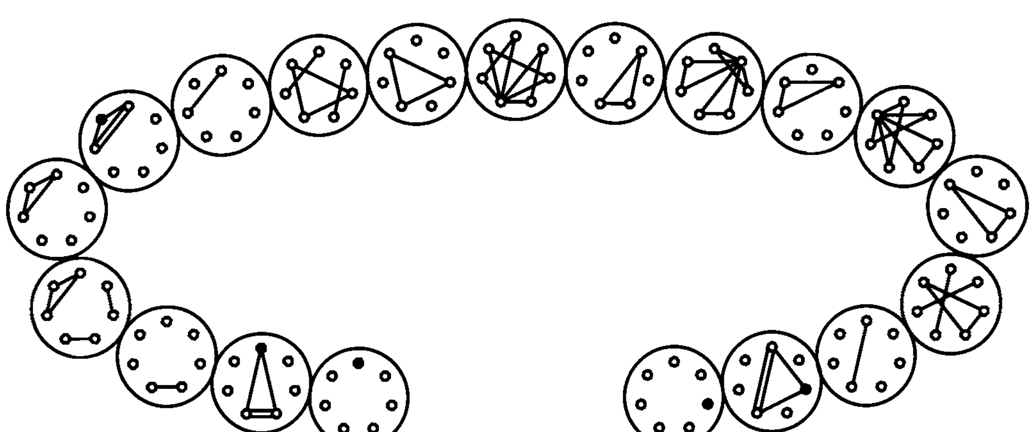

Figure 13. A special path in the HEXAGON.

eralized) octagon in a (generalized) pentagon? There must be infinitely many such combinations! On the other hand, the way he had pronounced PENTAGON and OCTAGON was very similar to the way he pronounced the names of his brothers. Did this suggest that I had to look for the smallest thick generalized pentagons and octagons and that these were perhaps somehow related to the POLYGONS? I returned to my studies, and after a couple of weeks of hard work I uncovered some more fundamental properties of generalized polygons that suggested an answer to my problem.

All generalized n-gons we have to worry about are *finite*, that is, both their point and line sets are finite sets. Remember that by Axiom Q1 a generalized n-gon $\mathscr{G}$ is of *order* (s, t), $s, t \geq 1$, if every line contains $s + 1$ points and every point is contained in $t + 1$ lines. If $s = t$, we also say that $\mathscr{G}$ is of order s. This means that the POLYGONS are the generalized polygons of order 2. Also, we ordinary n-gons are, up to isomorphism, the unique generalized n-gons of order 1. A generalized polygon is slim if either it or its dual is of order $(2, m)$ for some $m > 2$. If $\mathscr{G}$ is not an ordinary n-gon, then, by a celebrated result of Feit and Higman [7] (contemporaries of the prophet J. Tits), $n = 3$, 4, 6, 8, or 12, and, if $n = 12$, then $\mathscr{G}$ is slim.

The smallest slim generalized n-gons can be shown to be unique up to isomorphisms and duality. These geometries are the generalized 2-, 4-, 6-, 8-, and 12-gons of order $(1, 2)$ and their duals. The first (trivial) geometry is the graph consisting of 2 vertices that are connected by 3 edges (this is the DIGON minus one of its points, that is, minus one of its geometric hyperplanes). The remaining four geometries are the doubles of the POLYGONS. This means that all smallest non-trivial generalized polygons are related to the POLYGONS.

So, obviously, there are no non-ordinary generalized pentagons. Hence the PENTAGON must refer to something embedded in FLATLAND. Of course the shape of most of our buildings here in FLATLAND is that of a pentagon and the building that houses the best-kept secrets of our government is THE PENTAGON. Could that be it? Was the HEXAGON trying to warn me of my own government? All of a sudden everything seemed to make sense. Clearly, the OCTAGON was a thick generalized octagon that had immersed one of its multigons into FLATLAND and under the pretence of being a circle was ruling our land. Further study revealed that this OCTAGON is most probably a generalized octagon of order $(2, 4)$ having 1755 points and 2925 lines. So far I have been able to show the existence of only one such octagon. As I suspected it is a distant relative of the POLYGONS: Its derived geometry is the unique generalized quadrangle of order $(2, 4)$ which in turn contains the QUADRANGLE. I believe that this generalized octagon is unique but have not yet been able to prove it.

Following this discovery I joined the mathematical underground. Since governments are not interested in what mathematicians are writing and mathematicians are exactly the audience able to appreciate this report for what it is, I am submitting this account to a popular international mathematical journal, the perfect forum for subversive mathematical writings.

For a more detailed exposition of the mathematical theory of generalized polygons and the all-encompassing theory of mathematical buildings, see the recently discovered manuscripts [4], [12], [14], [16], and [17]. See [5], [9], [10], [11], and [13] for further information about the POLYGONS.

Enough said, my dear fellow flatlanders. Go forth and seek out the POLYGONS and then onwards to full GENERALITY!

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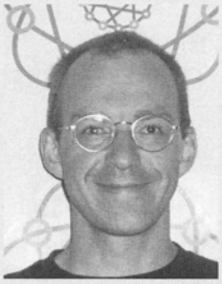

BURKARD POLSTER
Department of Mathematics and Statistics
P.O. Box 28M
Monash University, Victoria 3800
Australia
e-mail: Burkard.Polster@sci.monash.edu.au
http://www.maths.monash.edu.au/~bpolster

Burkard Polster joined the mathematical underground while studying arcane territories of finite and topological geometry. He has been on the run ever since, hastily completing his doctorate and working at eight universities on three continents over the last sixteen years. To maintain razor sharpness for this hectic existence, he practices daily: juggling, sculpting soap bubbles, and creating ambigrams. Some of these ambigrams have graced *The Mathematical Intelligencer*.

ANDREAS E. SCHROTH
Institut für Analysis
TU Braunschweig
D-38106 Braunschweig
Germany
e-mail: a.schroth@tu-bs.de
http://fb1.math.nat.tu-bs.de/~top/aschroth

Andreas E. Schroth was forced into the mathematical underground because his work on the connection between circle planes and generalized quadrangles verged dangerously close to circle-squaring. Cycling across the Indian subcontinent, he developed a persistent attachment to vegetarian Indian food (which he both cooks and eats) and to bollywood movies (which he only watches).

HENDRIK VAN MALDEGHEM
Department of Mathematics
University of Gent
9000 Gent
Belgium
e-mail: hvm@cage.rug.ac.be
http://cage.rug.ac.be/~hvm

Hendrik van Maldeghem was condemned to life in the mathematical underground by his addiction to numbers. Some of the numbers by which he lives:

6: his favorite number. His work on generalized hexagons earned him the 1999 Hall Medal of the Institute of Combinatorics and Applications.

40000: the number of kilometers he ran before the age of 38—and the approximate circumference of the earth.

4/4: the usual meter of the folk-rock band Lezzamie, in which he plays an electronic drum.