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Finding a Horseshoe on the Beaches of Rio[1]

What is Chaos?

A mathematician discussing chaos is featured in the movie *Jurassic Park*. James Gleick's book *Chaos* remains on the best-seller list for many months. The characters of Tom Stoppard's celebrated play *Arcadia* discourse on the meaning of chaos. What is the fuss about?

Chaos is a new science that establishes the omnipresence of unpredictability as a fundamental feature of common experience.

A belief in determinism, that the present state of the world determines the future precisely, dominated scientific thinking for two centuries. This credo was based on mechanics, where Newton's equations of motion describe the trajectories in time of states of nature. These equations have the mathematical property that the initial condition determines the solution for all time. This was taken as proving the validity of deterministic philosophy. Some went so far as to see in determinism a refutation of free will and hence even of human responsibility.

At the beginning of this century, with the advent of quantum mechanics, the untenability of determinism was exposed. At least on the level of electrons, protons, and atoms, it was discovered that uncertainty prevailed. The equations of motion of quantum mechanics produce solutions that are probabilities evolving in time.

In spite of quantum mechanics, Newton's equations govern the motion of a pendulum, the behavior of the solar system, the evolution of the weather, many macroscopic situations. Therefore the quantum revolution left intact many deterministic habits of thought. For example, well after the Second World War, scientists held the belief that long-range weather prediction would be successful when computer resources grew large enough.

In the 1970s the scientific community recognized another revolution, the theory of chaos, which seems to me to deal a death blow to the Newtonian picture of determinism. The world now knows that one must deal with unpredictability in understanding common experience. The coin-flipping syndrome is pervasive. "Sensitive dependence on initial conditions" has become a catchword of modern science.

Chaos contributes much more than extending the domain of indeterminacy, just as quantum mechanics did more than half a century earlier. The deeper understanding of dynamics underlying the theory of chaos has shed light on every branch of science. Its accomplishments range from analysis of electrocardiograms to aiding the construction of computational devices.

Chaos developed not from newly discovered physical laws, but by a deeper analysis of the equations underlying Newtonian physics. Chaos is a scientific revolution based on mathematics—deduction rather than induction. Chaos takes the equations of Newton, and uses mathematical analysis to establish the widespread unpredictability in the phenomena described by those equations. Via mathematics, one establishes the failure of Newtonian determinism by using Newton's own laws!

Taxpayers' Money

In 1960 in Rio de Janeiro I was receiving support from the National Science Foundation (NSF) of the United States as a postdoctoral fellow, while doing research in an area of mathematics which was to become the theory of chaos. Subsequently questions were raised about my having used U.S. taxpayers' money for this research done on the beaches of Rio. In fact none other than President Johnson's science adviser, Donald Hornig, wrote in 1968 in *Science:*

[1]This is an expanded version of a paper to appear in the proceedings of the International Congress of Science and Technology—45 years of the National Research Council of Brazil.

This blithe spirit leads mathematicians to seriously propose that the common man who pays the taxes ought to feel that mathematical creation should be supported with public funds on the beaches of Rio . . .

What happened during the eight years between the work on the beaches and this national condemnation?

The 1960s were turbulent in Berkeley where I was a professor; my students were arrested, tear gas frequently filled the campus air; dynamics conferences opened under curfew; Theodore Kaczynski, the suspected Unabomber, was a colleague of mine in the Math Department.

The Vietnam War was escalated by President Johnson in 1965, and I was moved to establish with Jerry Rubin a confrontational antiwar force. Our organization, the Vietnam Day Committee (VDC), with its teach-in, its troop train demonstrations and big marches, put me onto the front pages of the newspapers. These events led to a subpoena by the House Unamerican Activities Committee (HUAC), which was issued while I was en route to Moscow to receive the Fields Medal in 1966. The subsequent press conference I held in Moscow attacking U.S. policies in the Vietnam War (as well as Russian intervention in Hungary) created a long-lasting furor in Washington, D.C.

Let me hark back to what actually happened in that Spring of 1960 on those beaches of Rio de Janeiro.

Flying Down to Rio

In the 1950s there was an explosion of ideas in topology, which caught the imagination of many young research students such as myself. I finished a Ph.D. thesis in that domain at the University of Michigan in 1956. During that summer I, with my wife, Clara, attended in Mexico City a conference reflecting this great movement in mathematics, with the world's notables in topology present and giving lectures. There I met a Brazilian, Elon Lima, who was writing a thesis in topology at the University of Chicago—where I was about to take up the position of instructor—and we became good friends.

A couple of years later, Elon introduced me to Mauricio Peixoto, a young visiting professor from Brazil. Mauricio was from Rio, although he had come from a northern state of Brazil where his father had been governor. A good-humored pleasant fellow, Mauricio, in spite of his occasional bursts of excitement, was conservative in his manner and in his politics. As was typical for the rare mathematician working in Brazil at that time, he was employed as teacher in an engineering college. Mauricio also helped found a new institute of mathematics (IMPA), and his aspirations brought him to America to pursue research in 1957. Subsequently he was to become the President of the Brazilian Academy of Sciences.

Mauricio was working in the subject of differential equations or dynamics and showed me some beautiful results. Before long I myself had proved some theorems in dynamics.

In the summer of 1958, Clara and I with our newborn son, Nat, moved to the Institute for Advanced Study (IAS) in Princeton, New Jersey. I was supposed to spend two years there with an NSF postdoctoral fellowship. However, due to our common mathematical interests, Mauricio and Elon invited me to finish the second year in Rio de Janeiro. So Clara and I and our children, Nat and newly arrived Laura, left Princeton in December, 1959, to fly down to Rio.

The children were so young that most of our luggage consisted of diapers, but nevertheless we were able to realize an old ambition of seeing Latin America. After visiting the Panamanian jungle, the four of us left Quito, Ecuador, Christmas of 1959, on the famous Andean railroad down to the port of Guayaquil. Soon we were flying into Rio de Janeiro, recovering from sicknesses we had acquired in Lima. I still remember vividly, arriving at night, going out several times trying to get milk for our crying children, and returning with a substitute, cream or yogurt. We later learned that, in Rio, milk was sold only in the morning, on the street. At that time Brazil was truly part of the "third world."

However, our friends soon helped us settle into Brazilian life. We arrived just after a coup had been attempted by an air force colonel. He fled the country to take refuge in Argentina, and we were able to rent, from his wife, his luxurious 11-room apartment in the district of Rio called Leme. The U.S. dollar went a long way in those days, and we were even able to hire the colonel's two maids, all with our fellowship funds.

Sitting in our upper-story garden veranda, we could look across to the hill of the favela (called Babylonia) where *Black Orpheus* was filmed. In the hot humid evenings preceeding Carnaval, we would watch hundreds of the favela dwellers descend to samba in the streets. Sometimes I would join their wild dancing, which paraded for many miles.

In front of our apartment, away from the hill, lay the famous beach of Copacabana. I would spend my mornings on that wide, beautiful, sandy beach, swimming and body surfing. Also, I took a pen and paper and would work on mathematics.

Mathematics on the Beach

Very quickly after our arrival in Rio, I found myself working on mathematical research. My host institution, Instituto da Matematica, Pura e Aplicada (IMPA), funded by the Brazilian government, provided a pleasant office and working environment. Just two years earlier IMPA had set up its own quarters, a small colonial building in the old section of Rio called Botafogo. There were no undergraduates and only a handful of graduate mathematics students. There were also a very few research mathematicians, notably Peixoto, Lima, and an analyst named Leopoldo Nachbin. There was also a good math library. But no one could have guessed that in less than three decades IMPA would become a world center of dynamical systems, housed in a palatial building, as well as a focus for all Brazilian science.

In a typical afternoon I would take a bus to IMPA and soon be discussing topology with Elon or dynamics with Mauricio, or be browsing in the library. Mathematics research typically doesn't require much—a pad of paper and a ballpoint pen, library resources, and colleagues to query. I was satisfied.

Especially enjoyable were the times spent on the beach. My work was mostly scribbling down ideas and trying to see how arguments could be put together. I would sketch crude diagrams of geometric objects flowing through space, and try to link the pictures with formal deductions. Deep in this kind of thinking and writing on a pad of paper, I was not bothered by the distractions of the beach. It was good to be able to take time off from the research to swim.

The surf was an exciting challenge and even sometimes quite frightening. One time when Lima visited my "beach office," we entered the surf and were both caught in a current which took us out to sea. While Elon felt his life fading, bathers shouted the advice to swim parallel to the shore to a spot where we were able to return. (It was 34 years later, just before Carnaval, that once again those same beaches almost did me in. This time an oversized wave bounced me so hard on the sand it injured my wrist and tore my shoulder tendon; and then that same big wave carried me out to sea. I was lucky to get back using my good arm.)

Letter from America

At that time, as a topologist, I prided myself on a paper that I had just published in dynamics. I was delighted with a conjecture in that paper which had as a consequence that (in modern terminology) "chaos doesn't exist"!

This euphoria was soon shattered by a letter I received from Norman Levinson. I knew him as coauthor of the main graduate text in ordinary differential equations and as a scientist to be taken seriously.

Levinson wrote me of an earlier result of his which effectively contained a counterexample to my conjecture. His paper in turn was a clarification of extensive work of the British mathematicians Mary Cartwright and J. L. Littlewood done during World War II. Cartwright and Littlewood had been analysing some equations that arose in war-related studies involving radio waves. They had found unexpected and unusual behaviour of solutions of these equations. In fact Cartwright and Littlewood had found signs of chaos, even in equations that arose naturally in engineering. But the world wasn't ready to listen. I never met Littlewood, but in the mid-sixties, Dame Mary Cartwright, then head of a women's college (Girton) at Cambridge, invited me to high table.

I worked day and night to try to resolve the challenge to my beliefs that the letter posed. It was necessary to translate Levinson's analytic arguments into my own geometric way of thinking. At least in my own case, understanding mathematics doesn't come from reading or even listening. It comes from rethinking what I see or hear. I must redo the mathematics in the context of my particular background. And that background consists of many threads, some strong, some weak, some algebraic, some visual. My background is stronger in geometric analysis, but following a sequence of formulae gives me trouble. I tend to be slower than most mathematicians to understand an argument. The mathematical literature is useful in that it provides clues, and one can often use these clues to put together a cogent picture. When I have reorganized the mathematics in my own terms, then I feel an understanding, not before.

I eventually convinced myself that indeed Levinson was correct, and that my conjecture was wrong. Chaos was already implicit in the analyses of Cartwright and Littlewood. The paradox was resolved, I had guessed wrongly. But while learning that, I discovered the horseshoe!

The Horseshoe

The horseshoe is a natural consequence of a geometrical way of looking at the equations of Cartwright-Littlewood and Levinson. It helps understand the mechanism of chaos, and explain the widespread unpredictability in dynamics.

Chaos is a characteristic of dynamics, that is, of time evolution of a set of states of nature. Let me take time to be measured in discrete units. A state of nature will be idealized as a point in the two-dimensional plane.

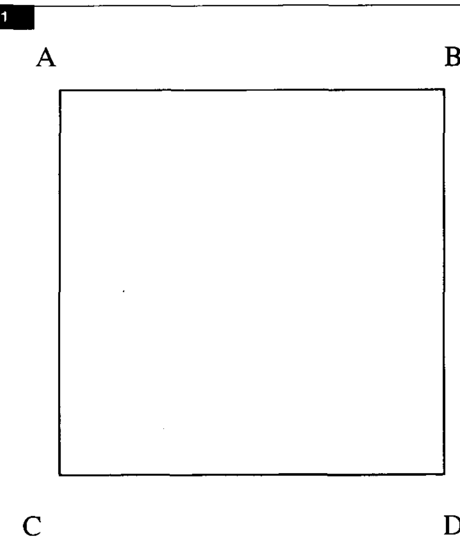
I will start by describing a non-chaotic linear example. The idea is to take a square, Figure 1, and to study what happens to a point on this square in one unit of time, under a transformation to be described.

The vertical dimension is shrunk uniformly towards the center of the square and the horizontal is expanded uniformly at the same time. Figure 2 shows the domain obtained by this process, A*B*D*C*, superimposed over the original square ABDC. I have also shaded in the set of points which don't move out of the square in this process.

The second of our three stages in understanding is the perturbed linear example. Now the square is moved into a bent version of the elongated rectangle of Figure 2. Thus Figure 3 describes the motion of our square obtained by a small modification of Figure 2.

The horseshoe is the fully non-linear version of what happens to points on the square, by an extension of the process expressed in Figures 2 and 3. This is the situation when motion makes a qualitative departure from the linear model. See Figure 4.

FIGURE 1



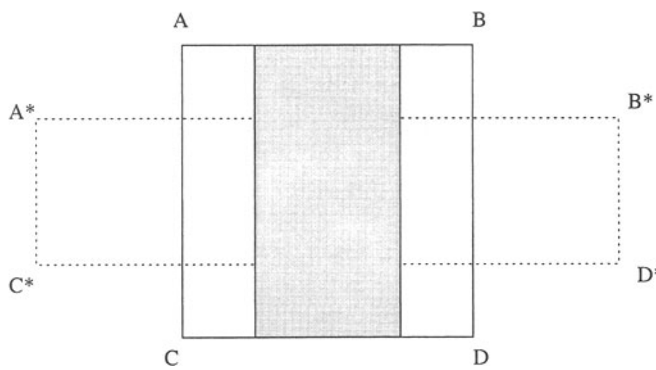


FIGURE 2

The horseshoe is the domain surrounded by the dotted line.

Instead of a state of nature evolving according to a mathematical formula, the evolution is given geometrically. The full advantage of the geometrical point of view is beginning to appear. The more traditional way of dealing with dynamics was with the use of algebraic expressions. But a description given by formulae would be cumbersome. That form of description wouldn't have led me to insights or to perceptive analysis. My background as a topologist, trained to bend objects like squares, helped to make it possible to see the horseshoe.

The dynamics of the horseshoe is described by moving a point in the square to a point in the horseshoe according to Figure 4. Thus the corner marked A moves to the point marked A* in one unit of time.

The motion of a general point x in the square is a sequence of points $x_0\ x_1\ x_2 \ldots$ Here $x_0 = x$ is the present state, x_1 is that state a unit of time later, x_2 that state two units of time later, etc.

Now imagine our visual field to be just the square itself. When a point is moved out of the square we will discard that motion. Figure 5 shades in the points which don't leave the square in one unit of time.

I will call a sequence which never leaves the square a "visual motion." The results in the next section concern visual motions.

In summary, the horseshoe is a fully non-linear motion. In the next section, I will show how chaos comes out of this picture.

The Horseshoe and Chaos: Coin Flipping

The laws of chance, with good reason, have traditionally been expressed in terms of flipping a coin. Guessing whether heads or tails is the outcome of a coin toss is the paradigm of pure chance. On the other hand it is a deterministic process that governs the whole motion of a real coin, and hence the result, heads or tails, depends only on very subtle factors of the initiation of the toss. This is "sensitive dependence on initial conditions."

A coin-flipping experiment is a sequence of coin tosses each of which has as outcome either heads (H) or tails (T). Thus it can be represented in the from HTTHHTTTTH. . . .[2] A general coin-flipping experiment is thus a sequence $s_0\ s_1\ s_2 \ldots$ where each of s_0, s_1, $s_2 \ldots$ is either H or T.

Here is the result of the horseshoe analysis that I found on that Copacabana beach. Consider all the points which, under the horseshoe mapping, stay in the square, i.e., don't drift out of our field of vision. These "visual motions" correspond precisely to the set of all coin-flipping experiments! This discovery demonstrates the occurrence of unpredictability in fully non-linear motion and gives a mechanism of how determinism produces uncertainty.

The correspondence is the following. To each visual motion there is an associated coin-flipping experiment. If $x_0\ x_1\ x_2 \ldots$ is the visual motion, at time $i = 0, 1, 2, 3, \ldots$ associate H or heads if x_i lies in the top half of the square and T or tails if it lies in the bottom half.

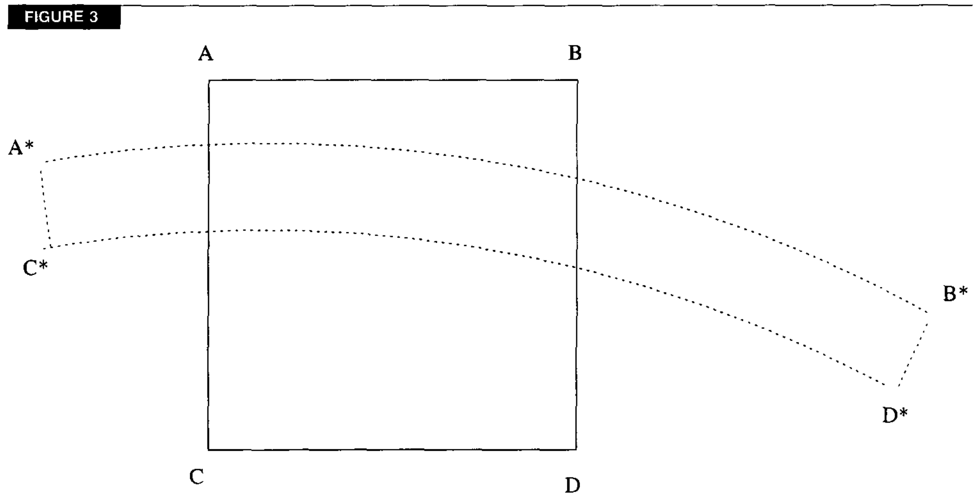
Moreover, and this is the crux of the matter, every possible sequence of coin flips is represented by a horseshoe motion. Therefore the dynamics is as unpredictable as coin-flipping. In the natural one–one correspondence

$$x_0\ x_1\ x_2 \ldots \rightarrow s_0\ s_1\ s_2 \ldots,$$

$x_0\ x_1\ x_2 \ldots$ is a motion lying in the square and $s_0\ s_1\ s_2 \ldots$ is a sequence of H's and T's. On the left is a deterministically generated motion and on the right a coin-flipping experiment.

We have seen complete unpredictability pop up within deterministic motion, the horseshoe. This is chaos.

FIGURE 3



The Hidden Origins of Chaos

As chaos is a mathematically based revolution, it is not surprising that a mathematician first saw evidence of chaos in dynamics.

[2]To give a complete picture in this section, one needs to reverse time and consider sequences of heads and tails which go back in time as well.

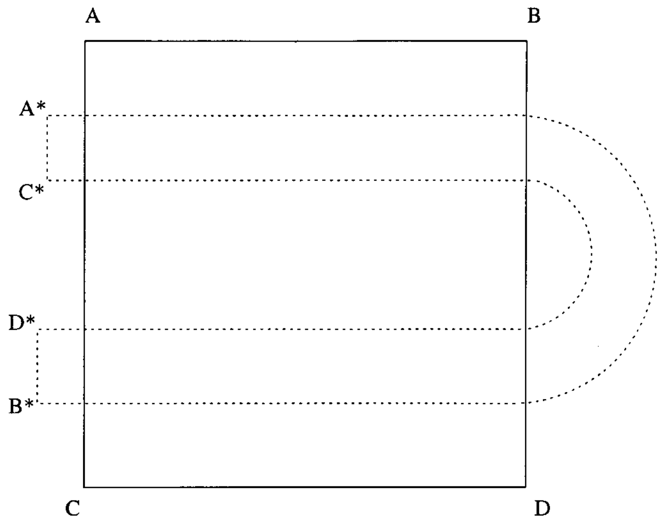


FIGURE 4

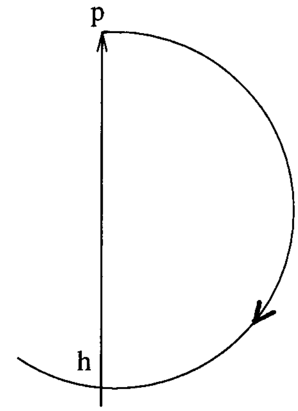


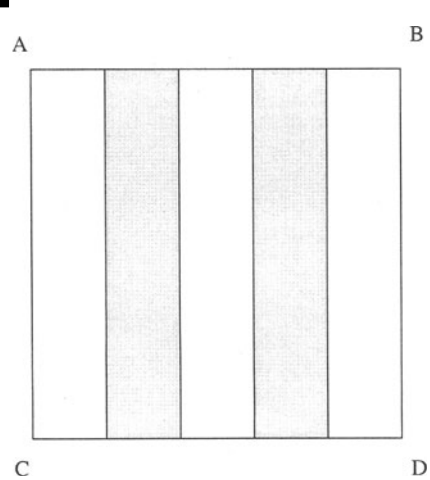
FIGURE 6

Henri Poincaré was (with David Hilbert) one of the two foremost mathematicians in the world active at the end of the last century. I heard of him first as an originator of topology, who had written an article claiming that a manifold with the same algebraic invariants as the n-dimensional sphere was topologically identical to the n-dimensional sphere. Then he found a mistake in his proof. Restricting himself now to 3 dimensions, he formulated the assertion as a problem, now called Poincaré's Conjecture, still one of the three or four great unsolved problems in mathematics today.

More germane to my present story is his contribution to the study of dynamics.

Poincaré made extensive studies in celestial mechanics, that is to say, the motions of the planets. At that time it was a celebrated problem to prove the solvability of those underlying equations, and in fact Poincaré at one time thought that he had proved it. Shortly thereafter, however, he became traumatized by a discovery which not only showed him wrong but showed the impossibility of ever solving the equations for even three bodies. This discovery he christened "homoclinic point."

FIGURE 5



A homoclinic point is a motion tending to an equilibrium as time increases and also to that same equilibrium as time recedes into the past. See Figure 6. Here p is an equilibrium and h marks the homoclinic point. The arrows represent the direction of time.

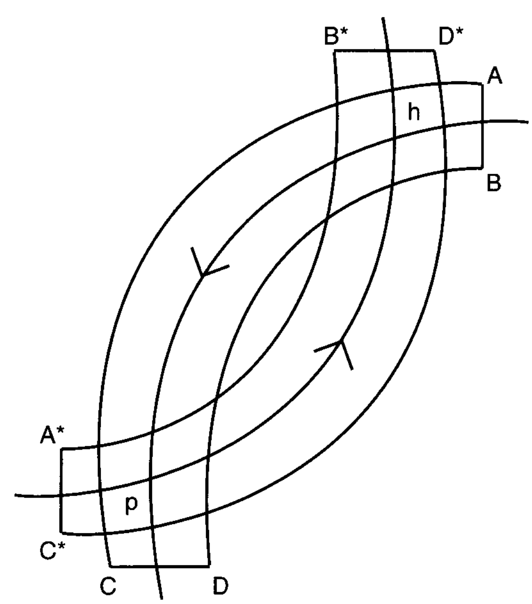
This definition sounds harmless enough but carries amazing consequences. Poincaré wrote concerning his discovery:[3]

One will be struck by the complexity of this figure, which I won't even try to draw. Nothing can more clearly give an idea of the complexity of the three-body problem and in general of all the problems of dynamics . . .

In addition to showing the impossibility of solving the equations of planetary motion, the homoclinic point has turned out to be the trademark of chaos; it is found in essentially every chaotic dynamical system.

It was in the first half of this century that American mathematics came into its own, and traditions stemming from Poincaré in topology and dynamics were central in this development. G.D. Birkhoff was the most well-known

FIGURE 7



[3]My own translation from the French.

American mathematician before World War II. He came from Michigan and did his graduate work at the University of Chicago, before settling down at Harvard. Birkhoff was heavily influenced by Poincaré's work in dynamics, and he developed these ideas and especially the properties of homoclinic points in his papers in the 20s and 30s.

Unfortunately, the scientific community soon lost track of the important ideas surrounding the homoclinic points of Poincaré. In the conferences in differential equations and dynamics that I attended in the late 50s, there was no awareness of this work. Even Levinson never showed in his book, papers, or correspondence with me that he was aware of homoclinic points.

It is astounding how important scientific ideas can get lost, even when they are aired by leading mathematicians of the preceding decades.

I learned about homoclinic points and Poincaré's work from browsing in Birkhoff's collected works, which I found in IMPA's library. It was because of the recently discovered horseshoe that the homoclinic landscape was to sink into my consciousness. In fact there was an important relation between horseshoes and homoclinic points.

If a dynamics possesses a homoclinic point then I proved that it also contains a horseshoe. This can be seen in Figure 7.

Thus the coin-flipping syndrome underlies the homoclinic phenomenon, and helps to comprehend it.

The Third Force

I was lucky to find myself in Rio at the confluence of three different historical traditions in dynamics. These three strains, while dealing with the same subject, were isolated from each other, and this isolation obstructed their development. I have already discussed two of these forces, Cartwright-Littlewood-Levinson and Poincaré-Birkhoff.

The third had its roots in Russia with the school of differential equations of A. Andronov in Gorki in the 1930s. Andronov had died before the first time I went to the Soviet Union, but in Kiev, in 1961, I did meet his wife, Andronova-Leontovich, who was still working in Gorki in differential equations.

In 1937 Andronov teamed up with the Soviet mathematician L. Pontryagin. Pontryagin had been blinded at the age of 14, yet went on to become a pioneering topologist. The pair described a geometric perspective of differential equations they called "rough," subsequently called structural stability. Chaos, in contrast to the two previously mentioned traditions, was absent in this development because of the restricted class of dynamics.

Fifteen years later the great American topologist Solomon Lefschetz became enthusiastic about Andronov and Pontryagin's work. Lefschetz had also suffered an accident, that of losing his arms, before turning to mathematics, and this perhaps generated some kind of bond between him and the blind Pontryagin. They first met at a topology conference in Moscow in 1938, and again after the war. It was through Lefschetz's influence, in particular from an article of his student De Baggis, that Mauricio Peixoto in Brazil learned of structural stability.

Peixoto came to Princeton to work with Lefschetz in 1957, and this is the route which led to our meeting each other through Elon. After this meeting, I studied Lefschetz's book on a geometric approach to differential equations, and eventually came to know Lefschetz in Princeton.

Thanks to Pontryagin and Lefschetz, there was the specter of topology in the concept of structural stability of ordinary differential equations. I believe that was why I listened to Mauricio.

Good Luck

Sometimes a horseshoe is considered an omen of good luck. The horseshoe I found on the beach of Rio certainly seemed to have such a property.

In that spring of 1960 I was primarily a topologist, mainly motivated by the problems of that subject, and most of all driven by the great unsolved problem posed by Poincaré. Since I had started doing research in mathematics, I had produced false proofs of the 3-dimensional Poincaré Conjecture, returning again and again to that problem.

Now on those beaches, within two months of finding the horseshoe, I found to my amazement an idea which seemed to succeed provided I returned to Poincaré's original assertion and then restricted the dimension to 5 or more. In fact the idea not only led to a solution of Poincaré's Conjecture in dimensions greater than 4, but it gave rise to a large number of other nice results in topology. It was for this work that I received the Fields Medal in 1966.

Thus "... the mathematics created on the beaches of Rio ..." (Hornig) was the horseshoe and the higher-dimensional Poincaré Conjecture.

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Our frequent contributor Steve Smale took early retirement from the University of California, Berkeley, after teaching there for more than 30 years. He arrived at his present job at a particularly fascinating stage: just as the territory prepared to pass from British to Chinese rule. (Photograph by G. Paul Bishop, Jr.)