SOME GEOMETRICAL APPLICATIONS OF FOURIER SERIES.

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I.

In « Festschrift zum 70. Geburtstag von H. WEBER », Leipzig, 1912, A. KNESER [1]) proved the beautiful theorem: An oval has at least four vertices, i. e. On an oval there are at least four points, where the radii of curvature are extremum. In « Kreis und Kugel », Leipzig, 1916, p. 160, W. BLASCHKE gave another clever proof of the same theorem, which depends upon the equations

$$\int_0^{2\pi} \rho \frac{\cos}{\sin} \varphi \, d\varphi = 0,$$

where ρ is the radius of curvature of the given curve, and φ is the inclination of the positive tangent referred to an x-direction.

The rectangular coordinates (x, y) of STEINER's curvature-centroid (Krümmungs-Schwerpunkt in German) are

$$x = \frac{1}{2\pi}\int \frac{1}{\rho} x \, ds, \qquad y = \frac{1}{2\pi}\int \frac{1}{\rho} y \, ds,$$

where ds is the curve-element. If p be the length of the perpendicular dropped from the curvature-centroid upon the tangent, then the coordinates are reduced to

$$x = \frac{1}{\pi}\int_0^{2\pi} p \sin\varphi \, d\varphi, \qquad y = -\frac{1}{\pi}\int_0^{2\pi} p \cos\varphi \, d\varphi \text{ [2])}.$$

[1]) A. KNESER, *Bemerkungen über die Anzahl der Extreme der Krümmung auf gechlossenen Kurven und über verwandte Fragen in einer nicht-euklidischen Geometrie* [Festschrift zum 70. Geburtstag von H. WEBER, pp. 170-180].

[2]) T. KUBOTA, *Über die Schwerpunkte der korwexen geschlossenen Kurven und Flächen* [Tôhoku

Hence if the origin of coordinates is the curvature-centroid itself

$$\int_0^{2\pi} p \begin{matrix}\sin\\ \cos\end{matrix} \varphi \, d\varphi = 0.$$

If we apply the method of proof due to BLASCHKE, using the unit-circle, on the circumference of which the mass proportional to p is laid at the extremity of the radius parallel to the positive tangent, we arrive at the theorem: *The perpendicular distance of the tangent to an oval from its curvature-centroid becomes maximum* $2+k$ *times and becomes minimum* $2+k$ *times,* k *being either* o *or a positive integer.* That is to say; *The radius vector of the pedal curve of an oval with respect to its curvature-centroid becomes maximum* $2+k$ *times and becomes minimum* $2+k$ *times.*

Let O be the curvature-centroid and F be the foot of the perpendicular p dropped from O on the tangent to the curve at the point P; and let OP be denoted by r. Then since

$$PF = \frac{dp}{d\varphi} = p',$$

we have

$$r^2 = p^2 + p'^2.$$

When p becomes extremum, $p' = 0$ and hence $r = p$.

When r becomes extremum,

$$\frac{dr^2}{dp} = 2(pp' + p'p'') = 2p'(p + p'') = 2p'\rho = 0,$$

where p'' is the second derivative of p with respect to φ.

Hence rejecting the case where ρ is always zero, $p' = 0$, and hence $r = p$, when r becomes extremum. Therefore when p becomes extremum, r becomes extremum, and vice versa; and OF coincides with OP. Therefore: *The radius vector of an oval drawn from its curvature-centroid becomes maximum* $2+k$ *times and becomes minimum* $2+k$ *times,* k *being either* o *or a positive integer, and the radius vector is a normal to the curve.*

Hence: *Normals to the oval at the points where the oval and its pedal curve with respect to its curvature-centroid touch each other, pass through the curvature-centroid.*

From the curvature-centroid of an oval, at least four normals can be drawn to the oval.

According to KNESER's theorem above mentioned, the evolute of an oval has

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Mathematical Journal, vol. 14 (1918), pp. 20-27], p. 23. Also see S. NAKAJIMA, *The circle and the straight line nearest to n given points, n given straight lines or a given curve* [Tôhoku Mathematical Journal, vol. 19 (1921), pp. 11-20], p. 18, and T. HAYASHI, *on* STEINER's *curvature-centroid* [Science Reports of the Tôhoku Imperial University, vol. 12 (1924), pp. 109-132].

$4+2k$ cusps and is closed, but not necessarily simple. From a point lying within the evolute, at least four normals can be drawn to the given oval. When the evolute has such a form that it cuts itself, it separates the plane into several parts, from a point within each of which 4, 6, 8, ..., $4+2k$ normals can be drawn to the oval, while if the point lie on the evolute some of the normals become coincident. The innermost region of the evolute is that region, from any point within which the most (just $4+2k$ in number) normals can be drawn to the given oval. The curvature-centroid lies within the evolute, and seems always to lie within the innermost region, from which the most normals (just $4+2k$ in number) to the oval can be drawn. But we have no proof for this.

It can be easily shown that if the origin of coordinates be the curvature-centroid, then

$$\int_0^{2\pi} p^{(n)} \frac{\cos}{\sin} \varphi \, d\varphi = 0,$$

where $p^{(n)}$ is the n-th derivative of p with respect to φ.

II.

In the BLASCHKE proof of KNESER's theorem above mentioned, we can take instead of ρ or p any one-valued and continuous function $f(\varphi)$, if it be periodic with period 2π. It need not be a positive function. Hence: *If a function $f(\varphi)$, be one-valued, continuous and periodic with period 2π, and satisfy the conditions*

$$\int_0^{2\pi} f(\varphi) \frac{\cos}{\sin} \varphi \, d\varphi = 0,$$

then the function has even extrema in the interval $0 \leqq \varphi < 2\pi$, one half maxima and the other half minima, and it has at least two maxima and two minima, the extrema taking place alternately excepting the case where $f(\varphi)$ is a constant.

If $f(\varphi)$ be expanded into FOURIER's series, then by the assumed conditions

$$f(\varphi) = \frac{a_0}{2} + \sum_{k=2}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi).$$

Since this function has at least two maxima and two minima, its derivative

$$f'(\varphi) = \sum_{k=2}^{\infty} (b'_k \cos k\varphi + a'_k \sin k\varphi),$$

where

$$a'_k = -k a_k, \quad b'_k = k b_k,$$

vanishes even times, and at least four times, in the interval $0 \leqq \varphi < 2\pi$.

Moreover, the function defined by

$$F(\varphi)=\sum_{k=2}^{\infty}(b'_k \cos k\varphi + a'_k \sin k\varphi)$$

always vanishes even times, and at least four times, in the interval $0 \leqq \varphi < 2\pi$.

For, by integration

$$\int F(\varphi)\,d\varphi = \frac{a_0}{2} + \sum_{k=2}^{\infty}(a_k \cos k\varphi + b_k \sin k\varphi),$$

where $a_0/2$ is a constant of integration, and

$$a_k = -\frac{1}{k}a'_k, \quad b_k = \frac{1}{k}b'_k;$$

and hence if we put

$$\int F(\varphi)\,d\varphi = f(\varphi),$$

then

$$\int_0^{2\pi} f(\varphi) \begin{matrix}\cos\\ \sin\end{matrix} \varphi\, d\varphi = 0.$$

Therefore by the preceding theorem $f(\varphi)$ has even extrema in the interval $0 \leqq \varphi < 2\pi$, one half maxima and the other half minima, and it has at least two maxima and two minima. Therefore $f'(\varphi)$, i. e. $F(\varphi)$ has at least four zeros in the same interval.

Therefore, by the virtue of BLASCHKE's proof, we can prove the theorem: *The trigonometric series (either finite or infinite)*

$$\sum_{k=2}^{\infty}(a_k \cos k\varphi + b_k \sin k\varphi)$$

has even zeros, and at least four zeros, in the interval $0 \leqq \varphi < 2\pi$.

This theorem is a particular case of HURWITZ's theorem on FOURIER's series lacking first terms [3]).

Though HURWITZ's proof for the general case is difficult to be applied, the proof here given for this particular case is not so difficult.

Conversely, if we apply HURWITZ's theorem to this particular case we can conclude that: *The trigonometric series (either finite or infinite)*

$$\frac{a_0}{2} + \sum_{k=2}^{\infty}(a_k \cos k\varphi + b_k \sin k\varphi)$$

has even extrema, and at least four extrema, in the said interval.

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[3]) A. HURWITZ, *Über die* FOURIER*schen konstanten integrierbarer Funktionen* [Mathematische Annalen, Bd. 57 (1903), S. 425-446], p. 444.

Therefore: *The trigonometric series*

$$\frac{a_0}{2} + \sum_{k=2}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi)$$

takes the value $a_0/2$ even times, and at least four times, in the said interval.

Now if we expand the radius of curvature ρ of an oval in FOURIER's series by means of the inclination φ of its positive tangent referred to a fixed x-direction, then

$$\rho = \frac{a_0}{2} + \sum_{k=2}^{\infty} (a_k \cos k\varphi + b_k \sin k\varphi),$$

where

$$a_0 = L/\pi,$$

L being the perimeter of the given oval. Hence ρ takes the value $L/2\pi$ even times, and at least four times, in the said interval. Therefore: *The even number, and at least four, of the osculating circles of an oval have the perimeters equal to the perimeter of the oval.*

The perimeter of an oval lies between the perimeters of the osculating circles whose radii are greatest and least respectively. This theorem is due to HURWITZ and BLASCHKE in BLASCHKE's « Kreis und Kugel », p. 116 [4]).

From my theorem above got, we can conclude that: *At least two of the maximum radii of curvature of an oval are not shorter than $L/2\pi$, and at least two of the minimum radii of curvature of an oval are not longer than $L/2\pi$, L being the perimeter of the oval.*

Similarly we get the following theorems on the perpendicular dropped from the curvature-centroid on the tangent.

The distance of the tangent to an oval from its curvature-centroid takes the value equal to $L/2\pi$ even times, and at least four times, L being the perimeter of the oval.

At least two of the maximum distances of the tangents from the curvature-centroid are not shorter than $L/2\pi$, and at least two of the minimum distances of the tangents from the curvature-centroid are not longer than $L/2\pi$, L being the perimeter of the oval.

III.

Now

$$\rho(\varphi) - \rho(\varphi + \pi) = \sum_{n=1}^{\infty} 2\{a_{2n+1} \cos(2n+1)\varphi + b_{2n+1} \sin(2n+1)\varphi\}.$$

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4) I have given another proof for this theorem. See T. HAYASHI, *The extremal chords of an oval* [Tôhoku Mathematical Journal, vol. 22 (1923), pp. 387-393], p. 393.

Hence for at least three values of φ in the interval $0 \leqq \varphi < 2\pi$,

$$\rho(\varphi) = \rho(\varphi + \pi).$$

Therefore: On an oval there are at least three pairs of points, such that the tangents at each pair are parallel and the radii of curvature are equal. This is also due to BLASCHKE and proved by SZEGÖ [5]).

Similarly: *On an oval there are at least three pairs of points, such that the tangents at each pair are parallel and the distances of the tangents from the curvature-centroid are equal.*

For a central oval, i. e. an oval having a point, all the chords through which are bisected at that point, the radius vector r drawn from the point, called the centre of the oval, to a point on the oval, expanded in FOURIER's series, has the form

$$r = \sum_{n=0}^{\infty} (a_{2n} \cos 2n\varphi + b_{2n} \sin 2n\varphi),$$

and therefore the perpendicular p from the centre on the tangent to the oval, and therefore the radius of curvature ρ, expanded in FOURIER's series have similar forms. Hence for all values of φ,

$$p(\varphi) = p(\varphi + \pi),$$

$$\rho(\varphi) = \rho(\varphi + \pi),$$

and the centre coincides with the curvature-centroid.

Conversely, if for all values of φ

$$\rho(\varphi) = \rho(\varphi + \pi),$$

i. e.

$$p(\varphi) + p''(\varphi) = p(\varphi + \pi) + p''(\varphi + \pi),$$

we find

$$p(\varphi) - p(\varphi + \pi) = a \cos\varphi + b \sin\varphi,$$

where a and b are some constants of integration. Solving this functional equation, we find

$$p(\varphi) = \frac{1}{2}(a \cos\varphi + b \sin\varphi) + \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_{2n} \cos 2n\varphi + b_{2n} \sin 2n\varphi).$$

Transferring the origin of coordinates from which the perpendicular is dropped on the tangent to the oval, to the curvature-centroid of the oval, a and b vanishes,

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5) BLASCHKE: Archiv d. Math. und Physik, vol. 26 (1917) Aufgabe 540, p. 65; and SZEGÖ: Archiv, vol. 28 (1920), Lösung, p. 183. W. SÜSS gave another proof which is to be published in a near future issue of Tôhoku Math. Jour.

and the series for $p(\varphi)$ takes the form

$$p(\varphi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_{2n} \cos 2n\varphi + b_{2n} \sin 2n\varphi),$$

so that

$$p(\varphi) = p(\varphi + \pi).$$

Hence

$$p'(\varphi) = p'(\varphi + \pi).$$

Therefore

$$r(\varphi) = r(\varphi + \pi)$$

and the radii vectores $r(\varphi)$ and $r(\varphi + \pi)$ are on the same straight line.

Therefore: *The necessary and sufficient condition that an oval is central is, that for all pairs of points on it, where the tangents are parallel, the radii of curvature are equal.*

Since for a curve of constant breadth

$$p(\varphi) + p(\varphi + \pi) = \text{const.},$$

the FOURIER series for $p(\varphi)$ takes the form

$$p(\varphi) = \frac{L}{2\pi} + \sum_{n=0}^{\infty} \{a'_{2n+1} \cos(2n+1)\varphi + b'_{2n+1} \sin(2n+1)\varphi\}.$$

Therefore: *The distance of the tangent to a curve of constant breadth from any point within it becomes equal even times to $L/2\pi$.*

Let the point from which the distance is measured be the curvature-centroid. Then: *The distance of the tangent to a curve of constant breadth from its curvature-centroid becomes extremum even times, and at least six times, one half maximum and the other half minimum.*

The distance of the tangent to a curve of constant breadth from its curvature-centroid becomes equal even times, and at least six times, to $L/2\pi$.

At least three of the maximum distances of the tangents from the curvature-centroid are not shorter than $L/2\pi$ and at least three of the minimum distances of the tangents from the curvature-centroid are not longer than $L/2\pi$.

Finally, for any oval

$$p(\varphi) + p(\varphi + \pi) = \frac{L}{\pi} + \sum_{n=1}^{\infty} (a'_{2n} \cos 2n\varphi + b'_{2n} \sin 2n\varphi).$$

Therefore: *The breadth of an oval becomes equal even times, and at least twice, to L/π.*

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