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k-OUT-OF-n-SYSTEM WITH REPAIR : T–POLICY

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ABSTRACT. We consider a k-out-of-n system with repair under T-policy. Life time of each component is exponentially distributed with parameter λ. Server is called to the system after the elapse of T time units since his departure after completion of repair of all failed units in the previous cycle or until accumulation of $n-k$ failed units, whichever occurs first. Service time is assumed to be exponential with rate μ. T is also exponentially distributed with parameter α. System state probabilities in finite time and long run are derived for (i) cold (ii) warm (iii) hot systems. Several characteristics of these systems are obtained. A control problem is also investigated and numerical illustrations are provided. It is proved that the expected profit to the system is concave in α and hence global maximum exists.

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1. Introduction

In this paper we consider a k-out-of-n system with repair. The repair is according to T-policy. Server is called to the system after the elapse of T time units where T is exponentially distributed with parameter α since his departure after completion of repair of all failed units in the previous cycle or the moment $n-k$ failed units accumulate whichever occurs first. Thus server is brought to the system at the moment which is min $\{T$, epoch of faliure of $n-k$ units$\}$ after his previous departure. He continues to remain in the system until all the failed units are repaired, once he arrives. The process continues in this fashion. The repaired units are assumed to be as good as new. Life time of each unit is assumed to have independent exponential distributions with parameter λ_i, when i units are functioning. Repair time is also assumed to be exponentially distributed with rate μ. We consider three

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different situations (a) cold system (b) warm system and (c) hot system which are defined in Section 2. In all these cases we derive the time dependent and steady state system probabilities. Control problems are investigated in all cases.

We aim at finding out optimal T to maximize the profit that is to minimize the running cost and maximize the system reliability. N-policy for repair of the k-out-of-n system has been studied extensively in Krishnamoorthy, Ushakumari and Lakshmi [2] and Ushakumari and Krishnamoorthy [4]. In these, the optimal number of components to fail before repairman is called in order to minimize the running cost and maximize the system reliability is investigated. Under suitable conditions they established convexity of the cost functions.

Waiting until a large number of units (very close to $n-k$) fail in order for the server to be called may lead to the system being down for longer duration thereby decreasing its uptime and hence the reliability. Calling the server frequently results in high fixed cost. Hence we go for T-policy and determine its optimal parameter value that maximizes the profit and system reliability.

T-policy in the queueing set up has been extensively studied (See Artalejo [1] for some references). However this has not been brought to investigation of the reliability of k-out-of-n system with repair in order to minimize the system running cost (maximize profit) simultaneously increasing system reliability.

This paper is presented as follows. Section 2 gives some preliminaries, notations, modelling and analysis of the problem under investigation. We outline the system state distribution in the finite time and in the long run for all the three models. Section 3 is devoted to some measures of performance and section 4 discusses the control problem. It also provides some numerical illustrations.

2. Mathematical Preliminaries

Definition 1. The k-out-of-n system is called a cold system if once the system is down (that is exactly $k-1$ functional units) there is no further failure of units that are not in failed state, until system starts functioning.

Definition 2. The system is called a warm system if functional units continue to deteriorate and so fail even when the system is down, but now at a lesser rate.

Definition 3. A hot system is one whose components deteriorate at the same rate during the system down state as they deteriorate when the system is up.

We discuss these three situations separately. First we introduce some notations.
$X(t)$: number of functional components at time t.
$Y(t)$: server state at time t.

$$\text{Write } Y(t) = \begin{cases} 1 & \text{if the server is available at time } t \\ 0 & \text{otherwise} \end{cases}$$

under assumptions made on the distribution of repair time, lifetime of components and on T, we see that $\{(X(t), Y(t)),\ t \in R_+\}$ is a Markov chain on $E_1 = \{k+1, \ldots, n\} \times \{0,1\} \cup \{(k-1,1)\} \cup \{(k,1)\}$ for model a (Definition 1) and $E_2 = \{k+1, \ldots, n\} \times \{0,1\} \cup \{(0,1), (1,1), \ldots, (k-1,1)\}$ for models b and c (Definitions 2 and 3, respectively). Denote by $P_{ij}(t)$ the system state probability at time t given $X(0) = n$, $Y(0) = 0$ that is $P_{ij}(t) = P((X(t), Y(t)) = (i,j)/(X(0), Y(0)) = (n,0))$ for $(i,j) \in E_1(E_2)$.

Model a

Transient Solution.

Here the functioning units do not deteriorate while the system is down. The Kolmogorov forward differential difference equations satisfied by $P_{ij}(t)$ are

$$\begin{aligned}
P'_{k1}(t) &= -(k\lambda_k + \mu)P_{k1}(t) + (k+1)\lambda_{k+1}P_{k+1,0}(t) \\
&\quad + (k+1)\lambda_{k+1}P_{k+1,1}(t) + \mu P_{k-1,1}(t) \\
P'_{m0}(t) &= -(m\lambda_m + \alpha)P_{m0}(t) + (m+1)\lambda_{m+1}P_{m+1,0}(t), k+1 \le m < n \\
P'_{n0}(t) &= -(n\lambda_n + \alpha)P_{n0}(t) + \mu P_{n-1,1}(t) \\
P'_{m1}(t) &= -(m\lambda_m + \mu)P_{m1}(t) + (m+1)\lambda_{m+1}P_{m+1,1}(t) \\
&\qquad + \mu P_{m-1,1}(t) + \alpha P_{m0}(t), \qquad k+1 \le m < n \\
P'_{n1}(t) &= -n\lambda_n P_{n1}(t) + \alpha P_{n0}(t) \\
P'_{k-1,1}(t) &= k\lambda_k P_{k1}(t) - \mu P_{k-1,1}(t)
\end{aligned}$$

whose solution is given by $\mathbf{P}(t) = e^{tA}\mathbf{P}(0)$, where $\mathbf{P}(0)$ is the initial probability vector which has 1 corresponding to state $(n,0)$ and rest zeros. A is the matrix of coefficients on the right side of the systems of equations.

Steady State Probabilities.

From the above equations by setting $q_{ij} = \lim_{t\to\infty} P_{ij}(t)$, $(i,j) \in E_1$ we get steady state probabilities

$$q_{n1} = \frac{\alpha}{n\lambda_n} q_{n0}, \qquad q_{n-1,1} = \frac{(n\lambda_n + \alpha)}{\mu} q_{n0},$$

$$q_{r0} = \prod_{l=r}^{n-1} \frac{(l+1)\lambda_{l+1}}{l\lambda_l}\, q_{n0}, \quad k+1 \le r \le n-1,$$

$$\begin{aligned} q_{n-r,1} &= \frac{(n-r+1)\lambda_{n-r+1} + \mu}{\mu}\, q_{n-r+1,1} - \frac{\alpha}{\mu} q_{n-r+1,0} \\ &- \frac{(n-r+2)\lambda_{n-r+2}}{\mu}\, q_{n-r+2,1}, \quad 2 \le r \le n-k, \end{aligned}$$

where $q_{n-l,1}$, $l = 1, 2, \ldots, n-k$ and $q_{n-l,0}$, $l = 1, 2, \ldots, n-k-1$ can be expressed in terms of q_{n0}, $q_{k-1,1} = \frac{k\lambda_k}{\mu}\, q_{k1}$. q_{n0} can be determined from the relation $\sum_{i=k+1}^{n} q_{i0} + \sum_{i=k-1}^{n} q_{i1} = 1$. However the expressions for $q_{n-i,1}$ for different i values is unwieldy and so we consider the particular case of $\lambda_i = \frac{\lambda}{i}$ in further developments.

Model b

In this model, when the number of functioning components reduce to $k-1$, the units that have not failed start deteriorating at a rate $\delta_i < \lambda_i$. Then life times of functioning components are exponential with parameter δ_i. The Kolmogorov forward differential equations are

$$\begin{aligned} P'_{k1}(t) &= -(k\lambda_k + \mu)P_{k1}(t) + (k+1)\lambda_{k+1}P_{k+1,0}(t) + (k+1)\lambda_{k+1}P_{k+1,1}(t) \\ &\quad + \mu P_{k-1,1}(t) \\ P'_{m0}(t) &= -(m\lambda_m + \alpha)P_{m0}(t) + (m+1)\lambda_{m+1}P_{m+1,0}(t), \quad k+1 \le m < n \\ P'_{n0}(t) &= -(n\lambda_n + \alpha)P_{n0}(t) + \mu P_{n-1,1}(t) \\ P'_{m1}(t) &= -(m\lambda_m + \mu)P_{m1}(t) + \alpha P_{m0}(t) \\ &\quad + (m+1)\lambda_{m+1}P_{m+1,1}(t) + \mu P_{m-1,1}(t), \quad k+1 \le m < n \\ P'_{n1}(t) &= -n\lambda_n P_{n1}(t) + \alpha P_{n0}(t) \\ P'_{k-1,1}(t) &= -((k-1)\delta_{k-1} + \mu)P_{k-1,1}(t) + k\lambda_k P_{k1}(t) + \mu P_{k-2,1}(t) \\ P'_{m1}(t) &= -((k-1)\delta_{k-1} + \mu)P_{m1}(t) + (m+1)\delta_{m+1}P_{m+1,1}(t) \\ &\quad + \mu P_{m-1,1}(t), \quad 0 < m < k-1 \\ P'_{01}(t) &= -\mu P_{01}(t) + \delta P_{11}(t) \end{aligned}$$

These lead to the system state probabilities in steady state

$$q_{k-l,1} = \frac{(k-l+1)\delta_{k-l+1}+\mu}{\mu} q_{k-l+1,1} - \frac{(k-l+2)\delta_{k-l+2}}{\mu} q_{k-l+2,1},\ 2 \le l \le k.$$

The rest of the system state probabilities are as in model a. $q_{k-l+1,1}$ and $q_{k-l+2,1}$, $l = 2, 3, \ldots, k$ are available in terms of q_{n0} and q_{n0} can be obtained from the relation $\sum_{i=k+1}^{n} q_{i0} + \sum_{i=0}^{n} q_{i1} = 1$.

Model c

Here the functional components, when the system is down start to deteriorate at the same rate as that when the system is up. The time dependent system state distribution can be obtained as in model a. The long run system state probabilities are given by

$$q_{k-l,1} = \frac{(k-l+1)\lambda_{k-l+1}+\mu}{\mu} q_{k-l+1,1} - \frac{(k-l+2)\lambda_{k-l+2}}{\mu} q_{k-l+2,1},\ 2 \le l \le k$$

and the rest of the system state probabilities are as in model a with the normalizing condition $\sum_{(ij)\in E_2} q_{ij} = 1$.

3. Some performance measures

We compute the optimal α for the three models. To do this we need to compute the distribution of time during which the server is continuosly available. We assume $\lambda_i = \frac{\lambda}{i}$ for $i = k, \ldots, n$ for model a, $\lambda_i = \frac{\lambda}{i}$, $i = k, \ldots, n$ and $\delta_i = \frac{\delta}{i}$ for $i = 1, 2, \ldots, k-1$ for model b and $\lambda_i = \frac{\lambda}{i}$, $i = 1, 2, \ldots, n$ for model c. This assumption states that failure rate decreases with increasing number of functioning units, which is quite reasonable.

Model a. The system state probabilities are given by

$$q_{n1} = \frac{\alpha}{\lambda} q_{n0}, \qquad q_{n-1,1} = \frac{\lambda+\alpha}{\mu} q_{n0}.$$

$$q_{n-r,1} = \Big[\lambda^{r-1}(\lambda+\alpha)^{n-r} + \lambda^r \mu((\lambda+\alpha)^{r-2} + \mu^{r-3}(\lambda+\alpha+\mu) + \ldots + \mu(\lambda+\alpha)^{r-3} + \ldots + \mu^{r-4}(\lambda+\alpha)^2)\Big] q_{n0} \Big/ \mu^r(\lambda+\alpha)^{r-1},$$
$$2 \le r \le n-k.$$

$$q_{k-1,1} = \frac{\lambda}{\mu} q_{k1}, \qquad q_{r0} = \Big(\frac{\lambda}{\lambda+\alpha}\Big)^{n-r} q_{n0},\ k+1 \le r \le n-1.$$

The system availability at any epoch is given by $1 - q_{k-1,1}$. Hence the fraction of time the system is not available is $q_{k-1,1}$. Under the normalizing condition, we get q_{n0}.

Distribution of the server availability.

Consider the Markov chain on the state space $\{(k-1,1), \ldots (n,0)\}$ with state $(n,0)$ absorbing. We have to compute the distribution of time until reaching $(n,0)$ starting from one of the transient states (corresponding to server arrival). The infinitesimal generator of this chain is

$$\begin{array}{c} \\ (k-1,1) \\ (k,1) \\ \vdots \\ \\ (n-1,1) \\ (n,1) \\ (n,0) \end{array} \begin{array}{c} \begin{array}{ccccccc} (k-1,1) & (k,1) & \ldots & \ldots & (n-1,1) & (n,1) & (n,0) \end{array} \\ \left(\begin{array}{ccccccc} -\mu & \mu & \ldots & & 0 & 0 & 0 \\ \lambda & -(\lambda+\mu) & \mu & \ldots & & & \\ & & & & & & \\ \vdots & & & & & & \\ 0 & \ldots & & \lambda & -(\lambda+\mu) & 0 & \mu \\ 0 & \ldots & & 0 & \lambda & -\lambda & 0 \\ 0 & \ldots & & 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

$$= \begin{bmatrix} M_1 & \underline{e_\mu} \\ \underline{0} & 0 \end{bmatrix}$$

where M_1 is the matrix obtained by deleting the last row and last column of the generator and $\underline{e_\mu}$ is the column vector with last entry μ and all others zero. $\underline{0}$ is a row vector of zeros. The distribution of time till absorption is of Phase type given by $F_1(x) = 1 - \underline{\alpha_1}\ \exp(M_1 x)\underline{e_1}$ for $x \geq 0$, where $\underline{\alpha_1}$ is the row vector of initial probability with entries $\alpha_{k-1}, \alpha_k, \ldots, \alpha_n$ where $\alpha_{k-1} = 0, \alpha_k = 1 - (\alpha_{k-1} + \ldots + \alpha_n); \alpha_i = P(S_{n-i} < T < S_{n-i+1})$ where the random variable S_i is the time till i failures take place starting from the instant at which all units function write $S_0 = 0$, then we have $S_0 < S_1 < \ldots < S_{n-k}$ and $\underline{e_1} = (1, 1, \ldots 1)'$.

Expected duration of time the server is busy in a cycle.

Expected time the server is busy is the time to reach $(n,0)$ starting from $(i,1)\,, i = k \ldots n-1$ Let T_i denote the time to reach $(i+1,1)$ starting from $(i,1), i \geq k-1$ we can recursively compute $E(T_i), i \geq k-1$ from the relation $E(T_i) = \frac{1}{\mu} + \frac{\lambda}{\mu}E(T_{i-1})$ starting from $E(T_{k-1}) = \frac{1}{\mu}$ then

$$E(T_j) = \frac{1 - (\lambda/\mu)^{j-k+2}}{(\mu - \lambda)}$$

The expected time to reach $(n,0)$ conditional on server arrival between $(n-i)th$ and $(n-i+1)$th component failures is $\sum_{j=i}^{n-1} E(T_j/S_i < T < S_{i+1})P(S_i < T < S_{i+1})$, where the random variable S_i is the time till i failures take place starting from the instant at which all units function. Write $S_0 = 0$ then we have $S_0 < S_1 < \ldots < S_{n-k}$. With this the expected time to reach $(n, 0)$ is

$$\sum_{i=k}^{n-2} \frac{1}{(\mu-\lambda)}\Big((n-i) - (\frac{\lambda}{\mu})^{i-k+2}\mu\frac{1-(\lambda/\mu)^{n-i}}{\mu-\lambda}\Big)$$
$$P(S_{n-i-1} < T < S_{n-i}) + \frac{1}{(\mu-\lambda)}(1-(\frac{\lambda}{\mu})^{n-k+1})\frac{\alpha}{\alpha+\lambda}$$

where $P(S_{n-i-1} < T < S_{n-i}) = \alpha\frac{\lambda^{n-i-1}}{(\lambda+\alpha)^{n-i}}$, $k \le i \le n-1$

Expected time the server is not in the system in a cycle.

From the state $(n, 0)$ the system can move either to $(n, 1)$ or $(n-1, 0)$. If it goes to $(n-1, 0)$ then from this the system further moves to $(n-2, 0)$ or $(n-1, 1)$. Thus process goes on for all the states till $(k+1, 0)$. From $(k+1, 0)$ it can either go to $(k+1, 1)$ or $(k, 1)$. At $(k+1, 1)$ on failure of one unit the system goes to $(k, 1)$. Thus expected amount of time the server is not in the system in a cycle is

$$\begin{aligned}\frac{1}{\alpha}P(T < S_1) &+ (\frac{1}{\alpha}+\frac{1}{\lambda})P(S_1 < T < S_2) + \ldots\\ &+ (\frac{1}{\alpha}+\frac{n-k-1}{\lambda})P(S_{n-k-1} < T < S_{n-k}) + P(T > S_{n-k})\frac{n-k}{\lambda}\\ &= \frac{2}{\alpha}\big(1-(\frac{\lambda}{\lambda+\alpha})^{n-k}\big)\end{aligned}$$

Expected duration of time the system is down in a cycle.

It is well known that $\frac{q_{k-1,1}}{q_{n0}}$ gives the expected number of visits to $(k-1, 1)$ before first return to $(n, 0)$ (starting from $(n, 0)$) (see Tijms (1994) [3]). Further $\frac{1}{\mu}$ is the expected amount of time system remains in $(k-1, 1)$ during each visit to that state .

Hence expected duration of time system is down is $q_{k-1,1}/(\mu q_{n0})$ which is equal to

$$\Big((\frac{\lambda}{\mu})^{n-k}\frac{1}{\mu^2}(\lambda+\alpha) + (\frac{\lambda}{\mu})^{n-k+1}\mu^{n-k-3}\frac{(\lambda+\alpha+\mu)}{(\lambda+\alpha)^{n-k-1}}$$
$$+ (\frac{\lambda}{\mu})^{n-k+1}\frac{1-(\frac{\mu}{\lambda+\alpha})^{n-k-3}}{\lambda+\alpha-\mu}\Big)$$

Model b

System state probabilities in the long run are the same as in model a for states $((k-1, 1), \ldots, (n-1, 1), (k+1, 0), \ldots, (n, 0))$. Further since the functional units deteriorate even when the system is down, we have for $l = 2, \ldots, k$, $q_{k-l,1} =$

$(\delta/\mu)^{l-1}q_{k-1,1}, 2 \leq l \leq k$. The system is down for the fraction of time $\sum_{i=0}^{k-1} q_{i1}$. So the system reliability is

$$1-\sum_{i=0}^{k-1} q_{i1} = 1-\frac{(1-(\delta/\mu)^k)}{(1-\delta/\mu)}q_{k-1,1}$$

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Distribution of server availability.

Consider the Markov chain on the state space $\{(0,1),(1,1),\ldots,(k,1),\ldots,(n-1,1),(n,0)\}$ with state $(n,0)$ absorbing. This distribution is Phase type; $F_2(x) = 1-\underline{\alpha_2}\exp(M_2(x))\underline{e_2}$, where M_2 is the matrix

$$\begin{array}{c} \\ (0,1) \\ (1,1) \\ \vdots \\ (n-1,1) \end{array} \begin{array}{c} \begin{array}{cccccc} (0,1) & (1,1) & \ldots & \ldots & & (n-1,1) \end{array} \\ \begin{pmatrix} -\mu & \mu & \ldots & 0 & 0 & 0 \\ 0 & -(\mu+\delta) & \mu & \ldots & & \\ \vdots & & & & & \\ 0 & \ldots & & 0 & \lambda & -(\lambda+\mu) \end{pmatrix} \end{array}$$

$\underline{\alpha_2}$ is the row vector of initial probabilities with first k entries zero, the rest of the entries are $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n, \alpha_{n+1}$ where $\alpha_{k+1} = 1-(\alpha_{k+2}+\ldots+\alpha_{n+1})$ and $\alpha_i = P(S_{n-i+1} < T < S_{n-i+2})$, $i = k+2,\ldots,n$, $\alpha_{n+1} = P(T < S_1)$, $\underline{e_2} = (1,1,\ldots,1)'$

Expected duration of time the server is continuously busy.

As in the earlier model, T_i denote the time to enter state $(i+1,1)$ starting from $(i,1)$. Here $E(T_0) = \frac{1}{\mu}$, $E(T_1) = \frac{1}{\mu}(1+\frac{\lambda}{\mu})$,

$$E(T_{k-1}) = \frac{1}{\mu}\frac{1-(\delta/\mu)^k}{1-\delta/\mu}, \quad E(T_k) = \frac{1}{\mu}+\frac{\lambda}{\mu}\frac{1}{\mu}\frac{1-(\delta/\mu)^k}{1-\delta/\mu}.$$

We can recursively compute $E(T_i)$, $i \geq 0$ from the relation $E(T_i) = \frac{1}{\mu}+\frac{\lambda}{\mu}E(T_{i-1})$ starting from $E(T_{k-1}) = \frac{1}{\mu}\frac{1-(\delta/\mu)^k}{1-\delta/\mu}$.
Thus

$$E(T_j) = \frac{1}{\mu}[\frac{1-(\lambda/\mu)^{j+1-k}}{1-(\lambda/\mu)}+(\frac{\lambda}{\mu})^{j+1-k}\frac{1-(\delta/\mu)^k}{1-(\delta/\mu)}].$$

The expected time to reach $(n,0)$ conditional on server reaches between $(n-i)^{th}$ and $(n-i+1)^{th}$ component failures is $\sum_{j=i}^{n-1} E(T_j/S_i < T < S_{i+1})P(S_i < T < S_{i+1})$ which is equal to

$$\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\frac{1}{\mu}[\frac{1-(\lambda/\mu)^{j+1-k}}{1-(\lambda/\mu)}+(\frac{\lambda}{\mu})^{j+1-k}\frac{1-(\delta/\mu)^k}{1-(\delta/\mu)}]P(S_{n-i-1} < T < S_{n-i}),$$

where $P(S_{n-i-1} < T < S_{n-i}) = \frac{\alpha\lambda^{n-i-1}}{(\lambda+\alpha)^{n-i}}$, $k \leq i \leq n-1$.

$$P(T < S_1) = \frac{\alpha}{\lambda + \alpha}.$$

Expected time the server is not in the system during a cycle is same as in model a.

Expected duration of time the system is down in a cycle.

$\frac{q_{k-1,1}}{q_{n0}}$ gives the expected number of visits to $(k-1, 1)$ before first return to $(n, 0)$.

Consider the class $\{(0,1), (1,1), \ldots, (k-1,1)\}$. The process spends on the average $\frac{1}{\mu}\frac{1-(\delta/\mu)^k}{1-(\delta/\mu)}$ amount of time in this class during each visit before returning to state $(k, 1)$. Hence expected duration of time the system is down in a cycle is

$$\frac{1}{\mu}\frac{1-(\delta/\mu)^k}{1-(\delta/\mu)}\frac{q_{k-1,1}}{q_{n0}} = \frac{1}{\mu}\frac{1-(\delta/\mu)^k}{1-\delta/\mu}\Big((\frac{\lambda}{\mu})^{n-k}\frac{1}{\mu^2}(\lambda+\alpha) + (\frac{\lambda}{\mu})^{n-k+1} \mu^{n-k-3}\frac{\lambda+\alpha+\mu}{(\lambda+\alpha)^{n-k-1}} + (\frac{\lambda}{\mu})^{n-k+1}\frac{1-(\frac{\mu}{\lambda+\alpha})^{n-k-3}}{\lambda+\alpha-\mu}\Big)$$

Model c

System state probabilities in the long run are the same as in model a for states $(k-1, 1) \ldots, (n-1, 1), (k+1, 0), \ldots, (n, 0)$. Further since the functional units deteriorate even when the system is down at the same rate, we have for $l = 2, \ldots, k$ $q_{k-l,1} = (\frac{\lambda}{\mu})^{l-1} q_{k-1,1}$ can be expressed in terms of $q(n, 0)$. System reliability is computed as earlier. $q(n, 0)$ can be obtained using the normalizing condition, $\sum_{(i,j)\in E_2} q_{i,j} = 1$.

The distribution of the duration of time the server continuously remains in the system is given by $F_3(x) = 1 - \underline{\alpha_3}\exp(M_3 x)\underline{e_3}$ where $\underline{\alpha_3}$ is a $(n+1)$ component row vector with first k entries zero the rest of the entries are $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n$, α_{n+1} where $\alpha_{k+1} = 1 - (\alpha_{k+2} + \ldots + \alpha_n)$ and $\alpha_i = P(S_{n-i+1} < T < S_{n-i+2}), i = k+2, \ldots n$, $\alpha_{n+1} = P(T < S_1)$. $\underline{e_3}$ is also of the same dimension with all entries,1. M_3 is a non singular matrix of order n given by first n rows and n columns of the matrix $I - \mathbf{P}$ where I is of order $(n+1)$ and $\mathbf{P}$ is the transition probability matrix of the chain on the set $\{(0,1), (1,1), \ldots, (n-1,1)(n,0)\}$.

Expected amount of time the server is continuously busy.

In this case $E(T_j) = \frac{1}{\mu}\frac{(1-(\lambda/\mu)^{j+1})}{(1-\lambda/\mu)}$, $j \geq 0$, starting with $E(T_0) = \frac{1}{\mu}$. As in model b, we get

$$E(T_i) = \sum_{j=i}^{n-1} E(T_j/S_i < T < S_{i+1})P(S_i < T < S_{i+1})$$

$$= \sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\frac{1}{\mu}\Big[\frac{1-(\lambda/\mu)^{j+1}}{1-(\lambda/\mu)}\Big]P(S_{n-i-1} < T < S_{n-i}),$$

where $P(S_{n-i-1} < T < S_{n-i}) = \alpha\frac{\lambda^{n-i-1}}{(\lambda+\alpha)^{n-i}}$, $k \leq i \leq n-1$. Thus

$$E(T_i) = \sum_{i=0}^{n-2}(n-i) - (\frac{\lambda}{\mu})^{i+1}\frac{(1-(\lambda/\mu)^{n-i})}{(1-\lambda/\mu)}\alpha\frac{\lambda^{n-i-1}}{(\lambda+\alpha)^{n-i}} + (1-(\lambda/\mu)^n)\frac{\alpha}{\lambda+\alpha}.$$

Here also the expected time the server is not in the system is the same as in the above two models.

Expected amount of time the system is non functional.

The process spends on the average $\frac{1}{\mu}\frac{(1-(\lambda/\mu)^k)}{(1-\lambda/\mu)}$ amount of time in the class $\{(0,1)(1,1),\ldots,(k-1,1)\}$. Expected amount of time the system is non functional in a cycle is $\frac{1}{\mu}\frac{1-(\lambda/\mu)^k}{(1-\lambda/\mu)}\frac{q_{k-1,1}}{q_{n0}}$.

4. Control Problem

Here we attempt to find the optimal value of α by maximizing the profit and the system reliability. The following costs are considered :

(1) Cost per unit time due to the machine remaining non functional.
(2) Profit per unit time when the server is not in the system.

Let C denote the cost per unit time due to the machine remaining non functional and w denote the wages given to the server.

Model a

Profit per unit time when the server is not in the system $= w\Big(\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})\Big)$
Expected cost per unit time due to idleness is

$$C(\frac{1}{\mu})\frac{q_{k-1,1}}{q_{n0}} = C(\frac{\lambda}{\mu})^{n-k+1}\Big((\frac{\mu}{\lambda})\frac{1}{\mu^2}(\lambda+\alpha) + \mu^{n-k-3}\frac{(\lambda+\alpha+\mu)}{(\lambda+\alpha)^{n-k-1}} + \frac{(1-(\frac{\mu}{\lambda+\alpha})^{n-k-3})}{(\lambda+\alpha-\mu)}\Big)$$

Therefore the total expected profit per unit time $(TEP)_a$ is

$$w\Big(\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})\Big) - C(\frac{\lambda}{\mu})^{n-k+1}$$
$$\Big((\frac{\mu}{\lambda})\frac{1}{\mu^2}(\lambda+\alpha) + \mu^{n-k-3}\frac{\lambda+\alpha-\mu}{(\lambda+\alpha)^{n-k-1}} + \frac{1-(\mu/(\lambda+\alpha))^{n-k-3}}{\lambda+\alpha-\mu}\Big)$$

The above function is concave in α as can be seen by differentiating the profit function twice with respect to α. However it is difficult to find optimal α value from the first derivative equated to zero.

Model b

In model b the total expected profit per unit time $(TEP)_b$ is

$$\begin{aligned}(TEP)_b = w\Big(\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})\Big)\\ - C\Big((\frac{\lambda}{\mu})^{n-k}\frac{1}{\mu^2}(\lambda+\alpha)+(\frac{\lambda}{\mu})^{n-k+1}\mu^{n-k-3}\frac{\lambda+\alpha+\mu}{(\lambda+\alpha)^{n-k-1}}\\ +(\frac{\lambda}{\mu})^{n-k+1}\frac{1-(\mu/(\lambda+\alpha))^{n-k-3}}{\lambda+\alpha-\mu}\Big)\frac{1-(\delta/\mu)^k}{\mu-\delta}\end{aligned}$$

$(TEP)_b$ is a concave function in α. It can also be seen by differentiating the profit function twice with respect to α.

Model c

In this case the total expected profit per unit time $(TEP)_c$ is

$$\begin{aligned}(TEP)_c = w\Big(\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})\Big)\\ - C\Big((\frac{\lambda}{\mu})^{n-k}\frac{1}{\mu^2}(\lambda+\alpha)+(\frac{\lambda}{\mu})^{n-k+1}\mu^{n-k-3}\frac{\lambda+\alpha+\mu}{(\lambda+\alpha)^{n-k-1}}\\ +(\frac{\lambda}{\mu})^{n-k+1}\frac{1-(\mu/(\lambda+\alpha))^{n-k-3}}{\lambda+\alpha-\mu}\Big)\frac{1-(\lambda/\mu)^k}{\mu-\lambda}\end{aligned}$$

Numerical illustration.

For illustration we calculate the total expected profit per unit time for given parameters for the three models and for various values of α. On comparing the three models for different set of parameters, we can see that total expected profit is maximum for model b.

Comparison of three models

$n = 12$, $\lambda = 7.5$, $\mu = 13$, $k = 6$, $w = 70$, $C = 80$, $\delta = 5$

Total expected profit/unit time			
α	$(TEP)_a$	$(TEP)_b$	$(TEP)_c$
3	38.986	40.284	40.209
3.1	38.052	39.315	39.242
3.2	37.156	38.386	38.315
3.3	36.296	37.495	37.426
3.4	35.471	36.64	36.573
3.5	34.679	35.819	35.753
3.6	33.917	35.03	34.966

Total expected profit/unit time			
α	$(TEP)_a$	$(TEP)_b$	$(TEP)_c$
3.7	33.184	34.271	34.209
3.8	32.479	33.541	33.48
3.9	31.8	32.839	32.779
4	31.146	32.162	32.104
4.1	30.516	31.51	31.453
4.2	29.908	30.882	30.826
4.3	29.322	30.276	30.221
4.4	28.756	29.691	29.637
4.5	28.21	29.126	29.073
4.6	27.682	28.581	28.529
4.7	27.171	28.054	28.003
4.8	26.678	27.544	27.494
4.9	26.2	27.051	27.002
5	25.738	26.575	26.526

$n = 18$, $k = 7$, $\lambda = 9.5$, $\mu = 14$, $w = 80$, $C = 110$, $\delta = 4$

Total expected profit/unit time			
α	$(TEP)_a$	$(TEP)_b$	$(TEP)_c$
2	49.157	50.57	50.402
2.1	47.805	49.153	48.992
2.2	46.518	47.805	47.652
2.3	45.292	46.523	46.376
2.4	44.124	45.302	45.161
2.5	43.01	44.138	44.003
2.6	41.946	43.028	42.898
2.7	40.929	41.968	41.844
2.8	39.957	40.955	40.835
2.9	39.027	39.986	39.871
3	38.136	39.059	38.949
3.1	37.282	38.172	38.065
3.2	36.463	37.321	37.218
3.3	35.677	36.505	36.406
3.4	34.922	35.722	35.627
3.5	34.796	35.03	35.002
3.6	33.499	34.248	34.158

Total expected profit/unit time			
α	$(TEP)_a$	$(TEP)_b$	$(TEP)_c$
3.7	32.828	33.553	33.466
3.8	32.182	32.885	32.806
3.9	31.56	32.241	32.216
4	30.96	31.621	31.542

$n = 10$, $K = 5$, $\lambda = 5.5$, $\mu = 10$, $w = 50$, $C = 100$, $\delta = 3$

Total expected profit/unit time			
α	$(TEP)_a$	$(TEP)_b$	$(TEP)_c$
3	27.435	29.251	29.106
3.1	26.737	28.512	28.37
3.2	26.069	27.806	27.667
3.3	25.43	27.131	26.995
3.4	24.819	26.484	26.351
3.5	24.232	24.865	25.734
3.6	23.67	25.271	25.143
3.7	23.13	24.702	24.576
3.8	22.612	24.156	24.032
3.9	22.114	23.631	23.509
4	21.635	23.126	23.007
4.1	21.174	22.641	22.524
4.2	20.73	22.174	22.059
4.3	20.303	21.725	21.611
4.4	19.891	21.292	21.18
4.5	20.047	20.953	20.88
4.6	19.11	20.472	20.363
4.7	18.739	20.083	19.976
4.8	18.382	19.708	19.602
4.9	18.036	19.346	19.241
5	17.701	18.996	18.893

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