INJECTIVITY OF THE POMPEIU TRANSFORM IN THE HEISENBERG GROUP

By

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1. Introduction

The Pompeiu problem can be formulated as follows [BZ]. Let X be a locally compact topological space, dm a non-negative measure, G a Lie group acting transitively on X, leaving dm invariant and K a compact subset of X (or $\mathcal{K}$ a family of compact subsets of X) then the Pompeiu transform $P = P_K$ (or $P = P_{\mathcal{K}}$) is the map

$$P : C(X) \to C(G),$$
$$P(f)(\sigma) := \int_{\sigma(K)} f\, dm.$$

The Pompeiu problem is to decide whether P is injective for a given K (or family $\mathcal{K}$). Clearly, we can pose this problem for other spaces of functions in X, *e.g.* L^p, $1 \le p \le \infty$. A typical example is $X = \mathbb{R}^n$, $G = \mathbb{R}^n$ (acting as a group of translations), dm = Lebesgue measure. Then

$$Pf(\sigma) = (\check{\chi}_K * f)(-\sigma),$$

* Partially supported by grants from the National Science Foundation, DMS-9000619 and CDR-8803012.

† Partially supported by a grant from the National Science Foundation DMS-9000968.

where χ_K is the characteristic function of K, $\check{\chi}_K(x) = \chi_K(-x)$. If $f \in L^p$, $1 \le p \le 2$, and $Pf = 0$ almost everywhere, then

$$\mathcal{F}(f) \cdot \mathcal{F}(\check{\chi}_K) = 0,$$

where $\mathcal{F}$ denotes the Fourier transform. If $m(K) > 0$, then $\mathcal{F}(\check{\chi}_K)$ is an entire function which does not vanish identically. It follows that $\mathcal{F}(f) = 0$ almost everywhere, so $f = 0$ almost everywhere. Hence P is injective, when acting in $L^p(\mathbb{R}^n)$, $1 \le p \le 2$, for any compact set of positive measure. This reasoning fails for $L^p(\mathbb{R}^n)$, $p > 2$, since we do not have the Fourier transform at our disposal. It is easy to see that there are always exponential solutions $f(x) = \exp\{i(\alpha_1 x_1 + \cdots + \alpha_n x_n)\} = e^{i\alpha \cdot x}$ so that convolution equation $\check{\chi}_K * f = 0$. Namely, $\alpha \in \mathbb{C}^n$ such that $\mathcal{F}(\check{\chi}_K)(\alpha) = 0$. This shows that P_K cannot be injective in $C(\mathbb{R}^n)$ when $G = \mathbb{R}^n$. Moreover, if one can find $\alpha \in \mathbb{R}^n$ as above then P_K is not injective in $L^\infty(\mathbb{R}^n)$. An instance of this occurs when K a ball of radius r, then every solution α of $\alpha \cdot \alpha = \frac{x_\nu}{r}$, x_ν a non-zero root of the Bessel function $J_{\frac{n-2}{2}}$ provides a counterexample to the injectivity. These examples explain why it is customary to consider the larger group of Euclidean motions $G = M_n$ in the case of $\mathbb{R}^n$ for the Pompeiu problem. There is presently a substantial lore of knowledge on the Pompeiu problem for symmetric spaces of rank 1, $X = G/K$, and we refer the reader to the delightful survey [Z2] for references and applications. More recently, some interesting variations of the Pompeiu problem have been posed, for instance a local version [BG1], [BG2], [BGY1], [BGY2], [Z1], *i.e.*, only an open subset of a Lie group acts on X. The need of considering an invariant measure is challenged in the recent result [BP], Badertscher [B], Quinto [Q] and in many lectures by the second author. In Carey–Kaniuth–Moran [CKM] a different version of the Pompeiu problem is considered (Actually it can be reduced to the one above for special choice of X and G.) The present article is a natural continuation of our work about the Heisenberg group $\mathbb{H}^n$. This group can be represented as the boundary of Siegel domain in $\mathbb{C}^{n+1}$, and as such, it makes sense to try to find out which functions in $\mathbb{H}^n$ are boundary values of holomorphic functions (CR functions). In [ABCP] we characterized which elements of $L^2(\mathbb{H}^n) \cap C^1(\mathbb{H}^n)$ are CR functions in terms of vanishing integrals over spheres. We found out that a single radius suffices for the Morera theorem in $L^2(\mathbb{H}^n)$. To explain what this means in a simple way let us restrict ourselves to $n = 1$, then $\mathbb{H}^1$ can be identified as a set with $\mathbb{C} \times \mathbb{R}$. Let $r > 0$, $S = S(r) = \{(z, 0) : |z| = r\}$, $f \in L^2(\mathbb{H}^1) \cap C^1(\mathbb{H}^1)$, then if

$$\int_S f(a \cdot (z, 0))dz = 0 \tag{0.1}$$

for all $a \in \mathbb{H}^1$, f is a CR function. This condition is also necessary. The reader can easily supply the n-dimensional version of this result. This theorem holds for

$L^p(\mathbb{H}^n)$, $1 \le p \le 2$, instead of $L^2(\mathbb{H}^n)$, and it is definitely false for $L^\infty(\mathbb{H}^n)$. The proof depended very strongly on having a good Fourier decomposition of functions in $L^2(\mathbb{H}^n)$, due to Gindikin, and that the equation (0.1) is $\mathbf{U}(n)$ invariant.

In this paper we consider Pompeiu problems in $L^\infty(\mathbb{H}^n)$ which are either $\mathbf{U}(n)$-invariant or $\mathbb{T}^n$-invariant, that is for collections $\mathcal{K}$ of spheres or polydisks, and characterize exactly the conditions on the radii that are necessary and sufficient for the injectivity of $P_{\mathcal{K}}$ in $L^\infty(\mathbb{H}^n)$ and, a fortiori, in $L^p(\mathbb{H}^n)$, $1 \le p \le \infty$. The Wiener Tauberian theorem of Hulanicki and Ricci [HR] plays an essential part in our proof. The lack of a supple version of the Paley–Weiner theorem for distributions of compact support in $\mathbb{H}^n$ is what prevents us for the moment from extending our investigation to the space $C(\mathbb{H}^n)$. We conclude this paper showing that the theorems obtained can be thought of as a quantized version of the corresponding results in $\mathbb{R}^n$. We would like to thank Fulvio Ricci, Yitzhak Weit, and Lawrence Zalcman for stimulating comments. We would also like to thank the referee for his careful reading of the manuscript, which detected several errata, and for pointing out to us the references in the Remark after Corollary 4.5. Agranovsky and Pascuas would like to thank the National Science Foundation, which through the grant DMS-9000619, made our collaboration possible.

2. Preliminaries

In this paper, we shall use the usual notations $\mathbb{Z}_+ = \{0, 1, \ldots\}$, $\mathbb{R}_+ = (0, \infty)$, and $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Consider the Heisenberg group $\mathbb{H}^n$, $n \ge 1$, as the set $\{(z, t) : z = (z_1, \ldots, z_n) \in \mathbb{C}^n, t \in \mathbb{R}\}$ with the group operation

$$(z, t) \cdot (w, s) = (z + w,\ t + s + 2\mathrm{Im} z \cdot \overline{w})$$

where $z \cdot \overline{w} = \sum_{j=1}^n z_j \overline{w}_j$.

It is well-known (see Folland–Stein [FS]) that a basis of the left-invariant vector fields (the Lie algebra $\mathcal{H}^n$) on $\mathbb{H}^n$ is formed by the vector fields Z_j, $\overline{Z}_j$, where

$$Z_j = \frac{\partial}{\partial z_j} + i\,\overline{z}_j \frac{\partial}{\partial t}, \qquad \text{for } j = 1, \ldots, n,$$
$$\overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - i z_j \frac{\partial}{\partial t}, \qquad \text{for } j = 1, \ldots, n,$$

and

$$T = \frac{\partial}{\partial t}.$$

Let $a \in \mathbb{H}^n$ be given, we denote by L_a the operator of left translation:

$$(L_a f)(b) = f(a^{-1} \cdot b).$$

For nice functions f, g on $\mathbb{H}^n$, the convolution $f * g$ is defined by

$$(f * g)(a) = \int_{\mathbb{H}^n} f(a \cdot b^{-1}) g(b) dm(b)$$

where $dm(b)$ is the Haar measure on $\mathbb{H}^n$ with $dm(b) = dV(w)ds$, $b = (w, s)$ and dV is the Lebesgue measure in $\mathbb{C}^n$. It is clear that $f * g \in L^1(\mathbb{H}^n)$ when $f, g \in L^1(\mathbb{H}^n)$.

For $m \in \mathbb{Z}$ we define the mapping $\chi_m : \mathbb{C} \to \mathbb{C}$ by

$$\chi_m(z) := \begin{cases} z^m, & \text{if } m \geq 0 \\ \bar{z}^{-m}, & \text{if } m \leq 0. \end{cases}$$

(Note that $\chi_0 \equiv 1$.)

Now let $\mathbf{m} \in \mathbb{Z}^n$, $\mathbf{m} = (m_1, \ldots, m_n)$. Define $\chi_{\mathbf{m}} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\chi_{\mathbf{m}}(z) := \prod_{j=1}^{n} \chi_{m_j}(z_j).$$

We now introduce the rotation operator by

$$I_\sigma f(z, t) = (f \circ \sigma)(z, t) = f(\sigma_1 z_1, \ldots, \sigma_n z_n, t), \qquad \sigma \in \mathbb{T}^n,$$

$\mathbb{T}^n$ is the n-dimensional torus. Then the following relation between operators of rotations and translations is true

$$I_\sigma L_a = L_{\sigma^{-1}a} I_\sigma \qquad \text{for every } a \in \mathbb{H}^n, \qquad \text{every } \sigma \in \mathbb{T}^n.$$

A function f on $\mathbb{H}^n$ is said to be $\mathbf{m}$-homogeneous if it satisfies

$$f \circ \sigma = \chi_{\mathbf{m}}(\sigma) \cdot f, \qquad \text{for every } \sigma \in \mathbb{T}^n.$$

Similarly, a distribution T on $\mathbb{H}^n$ is said to be $\mathbf{m}$-homogeneous if it satisfies

$$\langle T, \varphi \circ \sigma^{-1} \rangle = \chi_{\mathbf{m}}(\sigma) \cdot \langle T, \varphi \rangle,$$

for every $\varphi \in C_c^\infty(\mathbb{H}^n)$, the space of smooth functions with compact support and $\sigma \in \mathbb{T}^n$. Denote by $\mathcal{P}_{\mathbf{m}}$ the class of all $\mathbf{m}$-homogeneous functions in $\mathbb{H}^n$. Then the formula

$$\begin{aligned} \mathbf{P}_{\mathbf{m}} f &= \int_{\mathbb{T}^n} \chi_{-\mathbf{m}}(\sigma)(f \circ \sigma) d\sigma \\ &= \int_0^1 \cdots \int_0^1 f(\sigma_1 z_1, \ldots, \sigma_n z_n, t) e^{-2\pi i (m_1 \varphi_1 + \cdots + m_n \varphi_n)} d\varphi_1 \cdots d\varphi_n \end{aligned}$$

gives the projection onto $\mathcal{P}_{\mathbf{m}}$. Here $d\sigma$ is the Haar measure on $\mathbb{T}^n$.

Lemma 2.1 *Let $f \in \mathcal{P}_{\mathbf{m}}$, $g \in \mathcal{P}_{\mathbf{m}'}$, and the convolution $f * g$ is well defined. Then $f * g \in \mathcal{P}_{\mathbf{m}+\mathbf{m}'}$.*

Proof We need to compute $I_\sigma(f * g)$ for $\sigma \in \mathbb{T}^n$:

$$\begin{aligned}
I_\sigma(f * g)(a) &= (f * g)(\sigma a) \\
&= \int_{\mathbb{H}^n} (I_\sigma R_{b^{-1}} f)(a) g(b) dm(b) \\
&= \int_{\mathbb{H}^n} (R_{\sigma^{-1}b^{-1}} I_\sigma f)(a) g(b) dm(b) \\
&= \chi_{\mathbf{m}}(\sigma) \int_{\mathbb{H}^n} (R_{\sigma^{-1}b^{-1}} f)(a) g(b) dm(b) \\
&= \chi_{\mathbf{m}}(\sigma) \int_{\mathbb{H}^n} (R_{b^{-1}} f)(a) g(\sigma b) dm(b) \\
&= \chi_{\mathbf{m}}(\sigma)\chi_{\mathbf{m}'}(\sigma) \int_{\mathbb{H}^n} (R_{b^{-1}} f)(a) g(b) dm(b) = \chi_{\mathbf{m}+\mathbf{m}'}(\sigma)(f * g)(a).
\end{aligned}$$

This completes the proof. □

Lemma 2.2 *Let f be $\mathbf{m}$-homogeneous and $f * g$ is well-defined. Then*

$$\mathbf{P}_{\mathbf{m}'}(f * g) = f * \mathbf{P}_{\mathbf{m}'-\mathbf{m}}(g).$$

Proof According to the definition of the projection operator $\mathbf{P}_{m'}$, we know that

$$\begin{aligned}
\mathbf{P}_{\mathbf{m}'}(f * g)(\xi) &= \int_{\mathbb{T}^n} \chi_{-\mathbf{m}'}(\sigma)(f * g)(\sigma\xi) d\sigma \\
&= \int_{\mathbb{T}^n} \chi_{-\mathbf{m}'}(\sigma) \int_{\mathbb{H}^n} (I_\sigma R_{\zeta^{-1}} f)(\xi) g(\zeta) d\zeta d\sigma \\
&= \int_{\mathbb{T}^n} \chi_{-\mathbf{m}'}(\sigma) \int_{\mathbb{H}^n} (R_{\sigma^{-1}\zeta^{-1}} I_\sigma f)(\xi) g(\zeta) d\zeta d\sigma \\
&= \int_{\mathbb{T}^n} \chi_{-\mathbf{m}'}(\sigma) \int_{\mathbb{H}^n} \chi_{\mathbf{m}}(\sigma)(R_{\sigma^{-1}\zeta^{-1}} f)(\xi) g(\zeta) d\zeta d\sigma \\
&= \int_{\mathbb{T}^n} \chi_{-\mathbf{m}'}(\sigma) \int_{\mathbb{H}^n} \chi_{\mathbf{m}}(\sigma)(R_{\zeta^{-1}} f)(\xi) g(\sigma\zeta) d\zeta d\sigma \\
&= \int_{\mathbb{H}^n} (R_{\zeta^{-1}} f)(\xi) \int_{\mathbb{T}^n} \chi_{-\mathbf{m}'+\mathbf{m}}(\sigma) g(\sigma\zeta) d\sigma d\zeta \\
&= f * \mathbf{P}_{\mathbf{m}'-\mathbf{m}}(g)(\xi).
\end{aligned}$$

This completes the proof. □

Definition 2.3 *We define $\mathcal{P}_0$ as the class of all locally integrable functions in $\mathbb{H}^n$, invariant under the unitary group $\mathbf{U}(n)$:*

$$\mathcal{P}_0 = \{f(U(a)) = f(a),\ a \in \mathbb{H}^n,\ U \in \mathbf{U}(n)\}.$$

Similarly we may define the operator $\mathbf{P}_0$ the projection operator onto $\mathcal{P}_0$ by

$$\mathbf{P}_0 f = \int_{\mathbf{U}(n)} (f \circ U) dU.$$

3. The group algebras $L^1_*(\mathbb{H}^n)$ and $L^1_{\mathbf{0}}(\mathbb{H}^n)$

3.1 Let us denote by $L^1_{\mathbf{0}}(\mathbb{H}^n)$ (respectively $L^1_*(\mathbb{H}^n)$) the set of all **0**-homogeneous integrable functions on $\mathbb{H}^n$, i.e.

$$L^1_{\mathbf{0}}(\mathbb{H}^n) = L^1(\mathbb{H}^n) \cap \mathcal{P}_{\mathbf{0}} \qquad \text{(respectively } L^1_*(\mathbb{H}^n) = L^1(\mathbb{H}^n) \cap \mathcal{P}_0).$$

Obviously, $L^1_{\mathbf{0}}(\mathbb{H}^n)$ (respectively $L^1_*(\mathbb{H}^n)$) is a closed subspace of $L^1(\mathbb{H}^n)$ and, moreover, it is a closed subalgebra by Lemma 2.1. Furthermore, we have the following lemma:

Lemma 3.1 *The algebras $L^1_{\mathbf{0}}(\mathbb{H}^n)$ and $L^1_*(\mathbb{H}^n)$ are commutative.*

Proof We only need to prove the lemma for $L^1_{\mathbf{0}}(\mathbb{H}^n)$. Let us introduce the mapping of complex conjugation of the variable z:

$$\omega : (z,t) \to (\bar{z}, t).$$

Let f be a function defined on $\mathbb{H}^n$, we set $f^\omega = f \circ \omega$.

For two elements $a = (z,t)$, $b = (w,s)$ in $\mathbb{H}^n$, we have

$$\begin{aligned}\omega(a)\omega(b) &= (\bar{z} + \overline{w}, t + s + 2\mathrm{Im}\bar{z} \cdot w) \\ &= (\overline{w} + \bar{z}, s + t + 2\mathrm{Im}\bar{z} \cdot w) = \omega(b \cdot a).\end{aligned}$$

Since ω preserves the Haar measure of $\mathbb{H}^n$, for any two functions f, $g \in L^1(\mathbb{H}^n)$, we can write

$$\begin{aligned}(f^\omega * g)(a) &= \int_{\mathbb{H}^n} f^\omega(a \cdot b^{-1}) g(b) dm(b) \\ &= \int_{\mathbb{H}^n} f(\omega(b^{-1}) \cdot \omega(a)) g(b) dm(b) \\ &= \int_{\mathbb{H}^n} f(b^{-1} \cdot \omega(a)) g(\omega(b)) dm(b) \\ &= \int_{\mathbb{H}^n} f(b^{-1}) g(\omega(\omega(a)b)) dm(b) = (g^\omega * f)^\omega(a).\end{aligned}$$

Thus we have

$$f^\omega * g = (g^\omega * f)^\omega.$$

Now if $f, g \in \mathcal{P}_0$, then $f^\omega = f$, $g^\omega = g$, and $(g^\omega * f)^\omega = g * f$. Hence we obtain

$$f * g = g * f$$

and therefore complete the proof of the lemma. □

3.2 We now deal with the commutative Banach algebra $L^1_0(\mathbb{H}^n)$ (respectively $L^1_*(\mathbb{H}^n)$). The first goal of this section is to describe the maximal ideal space of this algebra. We start by observing that $L^1_0(\mathbb{H}^n)$ (respectively $L^1_*(\mathbb{H}^n)$) is actually the L^1 space of $\mathbb{H}^n/\mathbb{T}^n$ (respectively $\mathbb{H}^n/\mathbf{U}(n)$), which is not a group. Otherwise everything will be a consequence of general theory of $L^1(G)$ for G locally compact Abelian (see Loomis [Lo]).

Recall that the maximal ideal space $\mathcal{M}(\mathcal{A})$ of a commutative Banach algebra $\mathcal{A}$ is the set

$$\mathcal{M}(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$$

of nonzero continuous homomorphisms (characters) of the algebra $\mathcal{A}$ into the complex number field $\mathbb{C}$. In other words, $\mathcal{M}(\mathcal{A})$ consists of all bounded linear multiplicative functionals on $\mathcal{A}$.

The Gelfand transformation Γ is the mapping

$$\begin{aligned} &\Gamma : \mathcal{A} \to C(\mathcal{M}(\mathcal{A}), \mathbb{C}), \\ &\Gamma : a \to \tilde{a}, \qquad \text{where } \tilde{a}(m) = m(a) \end{aligned}$$

for all $m \in \mathcal{M}(\mathcal{A})$. The topology on $\mathcal{M}(\mathcal{A})$ is the weak topology of the dual space $\mathcal{A}^*$, i.e. the weakest topology with respect which all functions $\tilde{a}$, $a \in \mathcal{A}$, are continuous. The topological space $\mathcal{M}(\mathcal{A})$ is locally compact Hausdorff space and is compact if $\mathcal{A}$ is an algebra with the unit. In the case $\mathcal{A}$ does not have a unit, the functionals $\tilde{a}$ lie in $C_0(\mathcal{M}(\mathcal{A}))$, the space of continuous functions vanishing at ∞.

Following the general theory of algebras on homogeneous spaces of groups (see e.g., Helgason [H, Chapter 4]) one can obtain the description of the space $\mathcal{M}(L^1_0(\mathbb{H}^n))$ in terms of spherical functions. A nonzero function ψ on $\mathbb{H}^n$ is said to be $\mathbb{T}^n$-spherical, if $\psi(0) = 1$ and the following identity holds

$$\int_{\mathbb{T}^n} \psi(a \cdot \sigma b) d\sigma = \psi(a)\psi(b), \qquad a, b \in \mathbb{H}^n.$$

One can prove that $\psi \in C^\infty(\mathbb{H}^n)$ (see [H]). Similarly, a nonzero function ϕ on $\mathbb{H}^n$ is said to be $\mathbf{U}(n)$-spherical, if $\phi(0) = 1$ and the following identity holds:

$$\int_{\mathbf{U}(n)} \phi(a \cdot U(b)) d\sigma = \phi(a)\phi(b), \qquad a, b \in \mathbb{H}^n.$$

The spherical functions can be also described as eigenfunctions of all left-invariant differential operators, acting on $C^\infty(\mathbb{H}^n/\mathbb{T}^n)$ or $C^\infty(\mathbb{H}^n/\mathbf{U}(n))$. Thus, if we denote by $\mathcal{D}$ or $\mathcal{D}^*$ the set of all such operators, then $\psi \in C^\infty(\mathbb{H}^n)$ is $\mathbb{T}^n$-spherical function (respectively $\phi \in C^\infty(\mathbb{H}^n)$ is $\mathbf{U}(n)$-spherical function), if ψ is $\mathbf{0}$-homogeneous, $\psi(0) = 1$ and

$$D\psi = \lambda_D \psi, \qquad D \in \mathcal{D};$$

or respectively ϕ is 0-homogeneous, $\phi(0) = 1$ and

$$D\phi = \lambda_D \phi, \qquad D \in \mathcal{D}^*.$$

Theorem 3.2 *All characters of the algebra $L^1_{\mathbf{0}}(\mathbb{H}^n)$ are functionals of the form:*

$$m(f) = \int_{\mathbb{H}^n} f(a)\psi(a)dm(a),$$

where ψ is a bounded $\mathbb{T}^n$-spherical function on $\mathbb{H}^n$.

Theorem 3.3 *All characters of the algebra $L^1_*(\mathbb{H}^n)$ are functionals of the form:*

$$m(f) = \int_{\mathbb{H}^n} f(a)\phi(a)dm(a),$$

where ϕ is a bounded $\mathbf{U}(n)$-spherical function on $\mathbb{H}^n$.

The proofs of these theorems are given in a more general form in [H, Theorem 3.3, Chapter 4]; we omit the details here.

4. Description of $\mathbf{U}(n)$ and $\mathbb{T}^n$-spherical functions and the spaces $\mathcal{M}^*$ and $\mathcal{M}$

4.1 In order to compute the $\mathbb{T}^n$-spherical functions on $\mathbb{H}^n$, we have to recall the basic properties of the Fourier transform in the Heisenberg group.

For $f \in L^1(\mathbb{H}^n)$ denote by

$$\widehat{f}^\lambda(z) = \widehat{f}(z, \lambda)$$

the Fourier transform of f in the real variable t:

$$\widehat{f}^\lambda(z) = \widehat{f}(z, \lambda) = \int_{-\infty}^{+\infty} e^{-2\pi i \lambda t} f(z, t)dt.$$

The function $\widehat{f}^\lambda$ is well-defined for almost all $z \in \mathbb{C}^n$ and belongs to $L^1(\mathbb{C}^n)$.

We denote by $\mathcal{F}_{2n+1}(f)(\eta_1, \ldots, \eta_n, \lambda)$ the full Fourier transform of the function f as a function defined on $\mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n} \times \mathbb{R}$;

$$\mathcal{F}_{2n+1}(f)(\eta_1, \ldots, \eta_n, \lambda) = \int_{\mathbb{R}^{2n+1}} f(z,t) e^{-2\pi i \mathrm{Re} z \cdot \overline{\eta}} e^{-2\pi i \lambda t} dV(z) dt.$$

If $f \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$, then the Plancherel formula shows that

$$\|f\|_{L^2(\mathbb{H}^n)} = \int_{-\infty}^{+\infty} \|\widehat{f}^\lambda\|_{L^2(\mathbb{C}^n)} d\lambda,$$

and then, as usual the Fourier transform can be extended to a Hilbert space isomorphism from $L^2(\mathbb{H}^n)$ onto itself.

For $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, we define the λ-convolution (twisted convolution) by

$$(f \overset{\lambda}{*} g)(z) = (f \star g)(z) \equiv \int_{\mathbb{C}^n} e^{-4\pi i \lambda \mathrm{Im} z \cdot \overline{w}} f(z-w) g(w) dV(w),$$

where we use only the symbol $\star$ when the index of λ is clear from the context. Then we have

Lemma 4.1 *For all $f, g \in L^1(\mathbb{H}^n)$,*

$$\widehat{(f * g)}^\lambda = \widehat{f}^\lambda \overset{\lambda}{*} \widehat{g}^\lambda = \widehat{f}^\lambda \star \widehat{g}^\lambda.$$

Proof For $f, g \in L^1(\mathbb{H}^n)$ we have

$$\begin{aligned}
&\widehat{(f * g)}^\lambda(z) \\
&= \int_{-\infty}^{+\infty} e^{-2\pi i \lambda t} (f * g)(z,t) dt \\
&= \int_{-\infty}^{+\infty} \left(\int_{\mathbb{H}^n} f((z,t) \cdot (w,s)^{-1}) g(w,s) dV(w) ds \right) e^{-2\pi i \lambda t} dt \\
&= \int_{\mathbb{C}^n} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(z-w, t-s-2\mathrm{Im} z \cdot \overline{w}) g(w,s) e^{-2\pi i \lambda t} dt \right) ds dV(w) \\
&= \int_{\mathbb{C}^n} \left(\int_{-\infty}^{+\infty} f(z-w,t) e^{-2\pi i \lambda t} dt \right) \left(\int_{-\infty}^{+\infty} g(w,s) e^{-2\pi i \lambda s} ds \right) e^{-4\pi i \lambda \mathrm{Im} z \cdot \overline{w}} dV(w) \\
&= \widehat{f}^\lambda \star \widehat{g}^\lambda(z).
\end{aligned}$$

□

4.2 In this sub-section we describe the construction of the Fock representation, which is our basic tool. Let us introduce the operator $\tau^\lambda(b)$, $b = (w, s) \in \mathbb{H}^n$, of λ-translations as follows:

$$(\tau^\lambda(b) f)(z) = e^{-2\pi i\lambda s - 4\pi i \lambda \operatorname{Im} z\cdot\overline{w}} f(z - w).$$

These operators are unitary operators in $L^2(\mathbb{H}^n)$ and the mapping

$$\tau^\lambda : b \in \mathbb{H}^n \to \tau^\lambda(b)$$

gives a unitary representation of the Heisenberg group $\mathbb{H}^n$.

Now we are going to define a basis of $L^2(\mathbb{H}^n)$ adapted to the action of the group $\mathbb{T}^n$ on $\mathbb{H}^n$.

For $\alpha > -1$ and $\nu \in \mathbb{Z}_+$, let $L_\nu^{(\alpha)}$ be the generalized Laguerre polynomial defined by

$$L_\nu^{(\alpha)}(x) = e^x \frac{x^{-\alpha}}{\nu!} \frac{d^\nu}{dx^\nu}(e^{-x} x^{\nu+\alpha}).$$

For $\mu, \nu \in \mathbb{Z}_+$, we define the function $w_{\mu,\nu}^\lambda$ on $\mathbb{C}$ by

$$\lambda > 0 \qquad w_{\mu,\nu}^\lambda(z) = \begin{cases} z^{\mu-\nu} e^{-2\pi\lambda|z|^2} L_\nu^{(\mu-\nu)}(4\pi\lambda|z|^2) & \text{if } \mu \geq \nu, \\ \overline{z}^{\nu-\mu} e^{-2\pi\lambda|z|^2} L_\mu^{(\nu-\mu)}(4\pi\lambda|z|^2) & \text{if } \mu \leq \nu; \end{cases}$$

$$\lambda < 0 \qquad w_{\mu,\nu}^\lambda(z) = w_{\nu,\mu}^{-\lambda}(z).$$

For μ, $\nu \in (\mathbb{Z}_+)^n$, we define the function $w_{\mu,\nu}^\lambda$ on $\mathbb{C}^n$ by

$$w_{\mu,\nu}^\lambda(z) = \prod_{j=1}^n w_{\mu_j,\nu_j}^\lambda(z_j).$$

The function $w_{\mu,\nu}^\lambda$ is $\operatorname{sgn}(\lambda)(\mu - \nu)$-homogeneous. We also introduce the normalized function:

$$\mathcal{W}_{\mu,\nu}^\lambda(z) = c_{\mu,\nu}^\lambda w_{\mu,\nu}^\lambda(z)$$

where the constants $c_{\mu,\nu}^\lambda$ are positive and chosen so that $\|\mathcal{W}_{\mu,\nu}^\lambda\|_{L^2(\mathbb{C}^n)} = 1$. In fact

$$c_{\mu,\nu}^\lambda = \prod_{j=1}^n \left\{ \frac{\pi}{(4\pi|\lambda|)^{|\mu_j-\nu_j|+1}} \frac{(\max\{\mu_j, \nu_j\})!}{(\min\{\mu_j, \nu_j\})!} \right\}^{-\frac{1}{2}}.$$

Moreover, we have the following theorem (see Ogden–Vági [OV]):

Theorem *For any $\lambda \in \mathbb{R}^*$, the family $\{\mathcal{W}^\lambda_{\mu,\nu}\}_{\mu,\nu\in(\mathbb{Z}_+)^n}$ is an orthonormal basis of $L^2(\mathbb{C}^n)$, which is well related to λ-convolution*:

$$\mathcal{W}^\lambda_{\mu,\nu} \overset{\lambda}{*} \mathcal{W}^\lambda_{\mu',\nu'} = \pm\left(\frac{1}{2\sqrt{|\lambda|}}\right)^n \delta_{\nu,\mu'}\mathcal{W}^\lambda_{\mu,\nu'}. \tag{4.1}$$

The Fock representation is realized in the space

$$\Phi^\lambda_\nu = \bigoplus_{\mu\in(\mathbb{Z}_+)^n} \mathbb{C}\cdot\mathcal{W}^\lambda_{\mu,\nu}$$

by means of the operators

$$a \in \mathbb{H}^n \mapsto \tau^\lambda(a)$$

and any representation $(\Phi^\lambda_\nu, \tau^\lambda)$ is irreducible, *i.e.*, the space Φ^λ_ν is the minimal closed invariant subspace for the operators $\tau^\lambda(a)$.

It is worth mentioning that all representations $(\Phi^\lambda_\nu, \tau^\lambda)$ are equivalent. The intertwinning operators are the operators of λ-convolution with elements of the λ-basis:

$$f \in \Phi^\lambda_\nu \mapsto f \star w^\lambda_{\nu,\nu'} \in \Phi^\lambda_{\nu'}.$$

4.3 Let $f \in L^1(\mathbb{H}^n)$ and $\lambda \in \mathbb{R}^*$. We define the following transform:

$$\tilde{f}(\lambda;\mu,\nu) = \int_{\mathbb{C}^n} \hat{f}^\lambda(z)\overline{w^\lambda_{\mu,\nu}(z)}\,dV(z), \qquad \text{for } \mu,\nu \in (\mathbb{Z}_+)^n.$$

We can express those values in terms of integration over the whole group $\mathbb{H}^n$:

$$\tilde{f}(\lambda;\mu,\nu) = \int_{\mathbb{H}^n} f(a)\overline{\psi^\lambda_{\mu,\nu}(a)}\,dm(a), \qquad \text{for } \mu,\nu \in (\mathbb{Z}_+)^n,$$

where

$$\psi^\lambda_{\mu,\nu}(z,t) = e^{2\pi i\lambda t} w^\lambda_{\mu,\nu}(z), \qquad \text{for } \mu,\nu \in (\mathbb{Z}_+)^n.$$

The relation $\tilde{f}(-\lambda;\mu,\nu) = \tilde{f}(\lambda;\nu,\mu)$ follows from the definition of $w^\lambda_{\mu,\nu}$.

Lemma 4.2 *Let $f \in L^1(\mathbb{H}^n)$ of homogeneity degree $\mathbf{m} \in (\mathbb{Z}_+)^n$, i.e., $f \in \mathcal{P}_{\mathbf{m}}$. Then*

$$\tilde{f}(\lambda;\mu,\nu) = 0, \qquad \text{if} \qquad \mu-\nu \neq \mathbf{m},\ \ \lambda > 0;$$

and

$$\tilde{f}(\lambda;\mu,\nu) = 0, \qquad \text{if} \qquad \mu-\nu \neq -\mathbf{m},\ \ \lambda < 0.$$

In particular, if f is a $\mathbf{0}$-homogeneous integrable function, i.e., $f \in L^1_{\mathbf{0}}(\mathbb{H}^n)$, then $\tilde{f}(\lambda;\mu,\nu) = 0$ unless $\mu = \nu$.

Proposition 4.3 *The following relation for elements of λ-basis is true:*

$$\mathbf{P_0}(\tau^\lambda(z,0)w^\lambda_{\nu,\nu})(w) = w^\lambda_{\nu,\nu}(z)w^\lambda_{\nu,\nu}(w). \tag{4.2}$$

Proof Since the space Φ^λ_ν is invariant under the operator $\tau^\lambda(a)$, the function $\tau^\lambda(z,0)w^\lambda_{\nu,\nu}$ belongs to the space Φ^λ_ν and therefore it can be decomposed into a series:

$$\tau^\lambda(z,0)w^\lambda_{\nu,\nu} = \sum_{\mu\in(\mathbb{Z}_+)^n} \gamma_\mu(z)^\lambda_{\mu,\nu}$$

where

$$\gamma_\mu(z) = \langle \tau^\lambda(z,0)w^\lambda_{\nu,\nu}, \mathcal{W}^\lambda_{\mu,\nu}\rangle_{L^2(\mathbb{C}^n)}.$$

On the other hand,

$$\langle \mathbf{P}_0(\tau^\lambda(z,0)w^\lambda_{\mu,\nu}), \mathcal{W}^\lambda_{\mu,\nu}\rangle = \langle \tau^\lambda(z,0)w^\lambda_{\mu,\nu}, \mathbf{P}_0\mathcal{W}^\lambda_{\mu,\nu}\rangle = \begin{cases} 0 & \text{if } \mu \neq \nu, \\ \gamma_\nu(z) & \text{if } \mu = \nu. \end{cases}$$

Therefore,

$$\mathbf{P}_0(\tau^\lambda(z,0)w^\lambda_{\mu,\nu}) = \gamma_\nu(z)\mathcal{W}^\lambda_{\nu,\nu}. \tag{4.3}$$

In order to finish the proof we have to compute the coefficient $\gamma_\nu(z)$. By definition and the 0-homogeneity of $\mathcal{W}^\lambda_{\nu,\nu}$, we have

$$\begin{aligned}
\gamma_\nu(z) &= \frac{1}{c^\lambda_{\nu,\nu}} \int_{\mathbb{C}^n} (\tau^\lambda(z,0)\mathcal{W}^\lambda_{\nu,\nu})(w)\overline{\mathcal{W}^\lambda_{\nu,\nu}}(w)dV(w) \\
&= \frac{1}{c^\lambda_{\nu,\nu}} \int_{\mathbb{C}^n} \mathcal{W}^\lambda_{\nu,\nu}(w-z)e^{-4\pi i\lambda \mathrm{Im} w\cdot\bar{z}}\overline{\mathcal{W}^\lambda_{\nu,\nu}}(w)dV(w) \\
&= \frac{1}{c^\lambda_{\nu,\nu}} \int_{\mathbb{C}^n} \mathcal{W}^\lambda_{\nu,\nu}(w-z)e^{4\pi i\lambda \mathrm{Im}\bar{w}\cdot z}\mathcal{W}^\lambda_{\nu,\nu}(w)dV(w) \\
&= \frac{1}{c^\lambda_{\nu,\nu}} \int_{\mathbb{C}^n} \mathcal{W}^{-\lambda}_{\nu,\nu}(w-z)e^{4\pi i\lambda \mathrm{Im}\bar{w}\cdot z}\mathcal{W}^{-\lambda}_{\nu,\nu}(w)dV(w) \\
&= \frac{1}{c^\lambda_{\nu,\nu}} \left(\mathcal{W}^{-\lambda}_{\nu,\nu} \overset{-\lambda}{*} \mathcal{W}^{-\lambda}_{\nu,\nu}\right)(z).
\end{aligned}$$

Thus, by formula (4.1), we have

$$\gamma_\nu(z) = \frac{1}{(c^\lambda_{\nu,\nu})^2}\mathcal{W}^{-\lambda}_{\nu,\nu}(z) = \frac{1}{c^\lambda_{\nu,\nu}} w^\lambda_{\nu,\nu}(z).$$

Using this relation in the formula (4.3) we obtained the desired result. □

Now we are able to come to the description of the bounded $\mathbf{U}(n)$-spherical functions on $\mathbb{H}^n$, and, as a consequence, we obtain that of the bounded $\mathbb{T}^n$-spherical functions on $\mathbb{H}^n$.

Theorem 4.4 *The bounded* $\mathbf{U}(n)$*-spherical functions on* $\mathbb{H}^n$ *are the following functions:*

$$\Psi^{\lambda}_{\nu,\nu}(z,t) = \binom{\nu+n-1}{\nu}^{-1} e^{2\pi i(\lambda t + i|\lambda||z|^2)} L_{\nu}^{(n-1)}(4\pi|\lambda||z|^2), \qquad (\lambda,\nu) \in \mathbb{R}^* \times \mathbb{N},$$

$$\mathcal{J}^{\rho}_{n-1}(z,t) = (n-1)!2^{n-1}\frac{J_{n-1}(\rho|z|)}{(\rho|z|)^{n-1}}, \qquad \rho \in \mathbb{R}_+,$$

J_{n-1} being the Bessel function of the first kind of order $n-1$. (Note that $\mathcal{J}^{0}_{n-1}(z,t) = 1$.)

Proof Part I. First we show that, if ψ is a bounded $\mathbf{U}(n)$-spherical function on $\mathbb{H}^n$, it coincides with either some function $\Psi^{\lambda}_{\nu,\nu}$ or some function $\mathcal{J}^{\rho}_{n-1}$.

Recall that a $\mathbf{U}(n)$-spherical function ψ on $\mathbb{H}^n$ is an eigenfunction of every left-invariant differential operator $D \in \mathcal{D}(\mathbb{H}^n/\mathbf{U}(n))$:

$$D\psi = \lambda_D \psi.$$

Taking $D = \frac{\partial}{\partial t}$, we obtain that ψ is of the form

$$\psi(z,t) = e^{\lambda_D t}\psi_0(z).$$

Since ψ is bounded, we conclude that $\lambda_D = 2\pi i\lambda$ for some $\lambda \in \mathbb{R}$.

Let us now consider D as the Kohn Laplacian:

$$\mathcal{L}_0 = \sum_{j=1}^{n} \Box_j$$

where $\Box_j$ is the j-th subLaplacian in $\mathbb{H}^n$, i.e.

$$\Box_j = Z_j\overline{Z}_j + \overline{Z}_jZ_j.$$

Then

$$\mathcal{L}_0\psi = \lambda_0 e^{2\pi i\lambda t}\psi_0, \qquad \text{with } \lambda_0 = \lambda_{\mathcal{L}_0}.$$

But

$$\Box_j = 2\left\{\frac{\partial^2}{\partial z_j\partial\overline{z}_j} + |z_j|^2\frac{\partial^2}{\partial t^2} - i\frac{\partial}{\partial t}\left(z_j\frac{\partial}{\partial z_j} - \overline{z}_j\frac{\partial}{\partial\overline{z}_j}\right)\right\},$$

and, since ψ is 0-homogeneous, it is easy to see that

$$\left(z_j\frac{\partial}{\partial z_j}-\bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right)\psi=0.$$

Thus

$$\begin{aligned}\mathcal{L}_0\psi &= 2\sum_{j=1}^{n}\left(\frac{\partial^2}{\partial z_j\partial \bar{z}_j}+|z_j|^2\frac{\partial^2}{\partial t^2}\right)e^{2\pi i\lambda t}\psi_0\\ &= e^{2\pi i\lambda t}\left(\frac{1}{2}\,\Delta\,\psi_0-2(2\pi\lambda|z|)^2\psi_0\right).\end{aligned}$$

Therefore the function ψ_0 satisfies the equation

$$\Delta\psi_0-(4\pi\lambda)^2|z|^2\psi_0=2\lambda_0\psi_0. \tag{4.4}$$

We distinguish two cases for the parameter λ:

Case 1: $\lambda\neq 0$. Then we can make the following change of variable in the equation (4.4):

$$z'=(4\pi|\lambda|)^{\frac{1}{2}}z,$$

and we obtain the equation

$$\Delta\tilde{\psi}_0-|z'|^2\tilde{\psi}_0=\mu\tilde{\psi}_0, \tag{4.5}$$

where

$$\tilde{\psi}_0(z')=\psi_0((4\pi|\lambda|)^{-\frac{1}{2}}z)\qquad\text{and}\qquad \mu=\frac{\lambda_0}{2\pi|\lambda|}.$$

If we use polar coordinates, $z'=r\zeta'$, $|\zeta'|=1$, (4.5) becomes

$$\frac{\partial^2\tilde{\psi}_0}{\partial r^2}+\frac{2n-1}{r}\frac{\partial\tilde{\psi}_0}{\partial r}-(r^2+\mu)\tilde{\psi}_0=0$$

and, if we let $u(r^2)=\tilde{\psi}_0(r)$, then we get

$$t\,u''(t)+n\,u'(t)-\frac{\mu+t}{4}\,u(t)=0\qquad (t>0).$$

By considering the change $v(t)=e^{t/2}u(t)$, it turns out that this equation is equivalent to the following confluent hypergeometric equation:

$$t\,v''(t)+(n-t)\,v'(t)-\frac{\mu+2n}{4}\,v(t)=0. \tag{4.6}$$

When $a = \frac{\mu+2n}{4}$ does not coincide with any nonpositive integer, the equation (4.6) has two linearly independent solutions, v_1 and v_2, with the following asymptotic behaviour:

$$v_1(t) \sim \frac{(n-1)!}{\Gamma(a)} e^t t^{a-n}, \quad v_2(t) \sim t^{-a}, \qquad \text{as } t \to +\infty;$$

$$v_1(t) \sim 1, \qquad v_2(t) \sim \begin{Bmatrix} -\frac{\log t}{\Gamma(a)}, & \text{if } n = 1 \\ \frac{c}{t^{n-1}}, & \text{if } n \geq 2 \end{Bmatrix} \qquad \text{as } t \to 0^+,$$

where $c = c(a,n)$ is a nonzero constant. (See Olver [O, pp. 254–259]; in his notations, $v_1(t) = M(a,n,t)$ and $v_2(t) = U(a,n,t)$.) Therefore $u(r^2) = e^{-\frac{r^2}{2}} v(r^2)$ is unbounded for every non-identically zero solution v of equation (4.6). Hence a must be equal to some nonpositive integer $-\nu$, that is, $\mu = -(4\nu + 2)$, for some $\nu \in \mathbb{N}$. Then there are two linearly independent solutions, v_1 and v_2, of (4.6) such that

$$v_1(t) = L_\nu^{(n-1)}(t) \qquad \text{and} \qquad v_2(t) \sim e^t(-t)^{-\nu-n} \quad \text{as } t \to +\infty.$$

(See [O, pp.256–259]; in Olver's notations, $v_1(t) = U(-\nu, n, t)$ and $v_2(t) = V(-\nu, n, t)$.) It follows that the only solutions v of (4.6) which make $u(r^2) = e^{\frac{-r^2}{2}} v(r^2)$ bounded are $v = c \cdot v_1$, where c is an arbitrary constant.

Returning to the old function ψ_0, it turns out that

$$\psi_0(z) = c \cdot e^{-2\pi|\lambda|\cdot|z|^2} L_\nu^{(n-1)}(4\pi|\lambda| \cdot |z|^2),$$

i.e.

$$\psi(z,t) = c \cdot e^{2\pi i(\lambda t + i|\lambda|\cdot|z|^2)} L_\nu^{(n-1)}(4\pi|\lambda| \cdot |z|^2).$$

Finally, the condition $1 = \psi(0) = c \cdot L_\nu^{(n-1)}(0)$ implies that

$$c = \binom{\nu + n - 1}{\nu}^{-1},$$

and therefore $\psi = \Psi_{\nu,\nu}^\lambda$.

Case 2: $\lambda = 0$. In this case the equation (4.4) has the form

$$\Delta_j \psi_0 = 2\lambda_0 \psi_0.$$

It is well known that the radial solutions of this equation are

$$\psi_0(z) = c \cdot \frac{J_{n-1}(\rho|z|)}{(\rho|z|)^{n-1}},$$

where $\eta \in \mathbb{C}$ satisfies $\eta^2 = -\lambda_0$, $|\arg \eta| \leq \frac{\pi}{2}$, and c is an arbitrary constant.

Since

$$1 = \psi(z) = \psi_0(z) = c\eta^{n-1} \lim_{z\to 0} \frac{J_{n-1}(\eta|z|)}{(\eta|z|)^{n-1}} = c\frac{\eta^{n-1}}{(n-1)!2^{n-1}},$$

it is clear that

$$c = \frac{(n-1)!2^{n-1}}{\eta^{n-1}}.$$

On the other hand the boundedness of

$$\psi_0(z) = (n-1)!2^{n-1}\frac{J_{n-1}(\eta|z|)}{(\eta|z|)^{n-1}}$$

implies that η coincides with a nonnegative real number ρ.

In fact, taking into account that

(4.7)

$$J_{n-1}(z) \sim \left(\frac{2}{\pi|z|}\right)^{\frac{1}{2}} \left\{\cos\left(z - \frac{2n-3}{4}\pi\right) - \frac{4(n-1)^2-1}{8z}\sin\left(z - \frac{2n-3}{4}\pi\right)\right\}$$

as $|z| \to +\infty$, $|\arg z| \le \frac{\pi}{2}$ (see [O, p.133 (9.09)]), a straighforward (and tedious) calculation shows that

$$\frac{|J_{n-1}(re^{\pm i\frac{\pi}{2}})|}{r^{n-1}} \sim \frac{1}{\sqrt{2\pi}}\frac{e^r}{r^{n-\frac{1}{2}}}, \qquad \text{as } r \to +\infty,$$

and, for every $\theta \in \mathbb{R}$, $0 < |\theta| < \frac{\pi}{2}$,

$$\frac{|J_{n-1}(r_k e^{i\theta})|}{r_k^{n-1}} \sim \frac{1}{\sqrt{2\pi}}\left(\frac{\cos\theta}{2k\pi}\right)^{n-\frac{1}{2}} e^{2k\pi|\tan\theta|}, \qquad \text{as } k \to +\infty,$$

where $r_k = \frac{2k\pi}{\cos\theta}$, $k \ge 1$. Thus $\frac{J_{n-1}(\eta|z|)}{(\eta|z|)^{n-1}}$ is unbounded if $0 < |\arg\eta| \le \frac{\pi}{2}$, so η must be equal to some $\rho \in \mathbb{R}_+$. Hence we conclude that $\psi = \psi_0 = \mathfrak{J}^{\rho}_{n-1}$.

Part II. To complete the proof of the theorem we are going to show that the functions $\Psi^{\lambda}_{\nu,\nu}$ and $\mathcal{J}^{\rho}_{n-1}$ are bounded $\mathbf{U}(n)$-spherical functions on $\mathbb{H}^n$.

It is clear that $\Psi^{\lambda}_{\nu,\nu}$ is bounded, and, by (4.7), it is also clear that $\mathcal{J}^{\rho}_{n-1}$ is bounded. Moreover, $\Psi^{\lambda}_{\nu,\nu}(0) = \mathcal{J}^{\rho}_{n-1}(0) = 1$.

Thus we only have to show that $\psi = \Psi^{\lambda}_{\nu,\nu}$, $\mathcal{J}^{\rho}_{n-1}$ satisfy the functional equation

$$\int_{\mathbf{U}(n)} \psi(a \cdot U(b))\, dU = \psi(a)\,\psi(b) \qquad \text{for every } a, b \in \mathbb{H}^n.$$

Let $a = (z,t)$ and $b = (w,s)$. We distinguish two cases depending on the form of ψ:

Case 1: $\psi = \Psi^\lambda_{\nu,\nu}$.

Since $\Psi^{-\lambda}_{\nu,\nu} = \overline{\Psi^\lambda_{\nu,\nu}}$ we may assume that $\lambda > 0$. Then observe that, for every $U \in \mathbf{U}(n)$, we have

$$\Psi^{-\lambda}_{\nu,\nu}(a \cdot U(b)) = e^{2\pi i(t+s)} e^{-4\pi\lambda(U(w)\cdot\bar{z})} L^{(n-1)}_\nu(4\pi\lambda|z + U(w)|^2).$$

Thus, after changing z and w by $\frac{z}{\sqrt{4\pi\lambda}}$ and $\frac{w}{\sqrt{4\pi\lambda}}$, respectively, it turns out that we have to prove

(4.8)

$$\int_{\mathbf{U}(n)} e^{-U(w)\cdot\bar{z}} L^{(n-1)}_\nu(|z + U(w)|^2)\, dU = \binom{\nu+n-1}{\nu}^{-1} L^{(n-1)}_\nu(|z|^2) L^{(n-1)}_\nu(|w|^2),$$

for every $z, w \in \mathbb{C}^n$.

By the bi-invariance of the Haar measure dU on $\mathbf{U}(n)$, the value of the integral $I_{n,\nu}$ in (4.8) only depends on $|z|$ and $|w|$, so

$$I_{n,\nu} = \int_S \left(\int_{\mathbf{U}(n)} e^{-|z| U_1(|w|b)} L^{(n-1)}_\nu(||z|e_1 + U(|w|b)|^2)\, dU \right) d\sigma(b),$$

where σ is the normalized surface measure on the unit sphere S of $\mathbb{C}^n$. By Lemma 1.4.2 and 1.4.5(2) of Rudin [R] we obtain

$$\begin{aligned} I_{n,\nu} &= \int_S e^{-|z|\cdot|w|\cdot w_1} L^{(n-1)}_\nu(||z|e_1 + |w|b|^2)\, d\sigma(b) \\ &= \int_S e^{-|z|\cdot|w|\cdot w_1} L^{(n-1)}_\nu(|z|^2 + 2|z||w|\mathrm{Re} w_1 + |w|^2)\, d\sigma(w) \\ &= \begin{cases} K_{1,\nu}(1), & \text{if } n = 1, \\ (n-1)\int_0^1 (1-r^2)^{n-2} J_{n,\nu}(r) 2r\, dr, & \text{if } n \geq 2, \end{cases} \end{aligned}$$

where

$$K_{n,\nu}(r) = \int_0^{2\pi} e^{-|z|\cdot|w|e^{i\theta}} L^{(n-1)}_\nu(|z|^2 + 2|z|\cdot|w|\cdot r\cos\theta + |w|^2)\, \frac{d\theta}{2\pi}.$$

By using the formula

$$\sum_{\nu=0}^\infty L^{(n-1)}_\nu(x) y^\nu = (1-y)^{-n} e^{\frac{xy}{y-1}}, \qquad |y| < 1$$

(see Erdélyi–Magnus–Oberhettinger–Tricomi [E1, p.189, 10.12(17)]) we have

$$\sum_{\nu=0}^\infty K_{n,\nu}(r) y^\nu = (1-y)^{-n} e^{\frac{y}{y-1}(|z|^2+|w|^2)} \int_0^{2\pi} e^{|z|\cdot|w|\cdot r\left(\frac{2y\cos\theta}{y-1} - e^{i\theta}\right)} \frac{d\theta}{2\pi}.$$

Integrating term by term the exponential series we get

$$\int_0^{2\pi} e^{|z|\cdot|w|\cdot r\left(\frac{2y\cos\theta}{y-1}-e^{i\theta}\right)}\frac{d\theta}{2\pi}=\sum_{\nu=0}^{\infty}\frac{r^{2\nu}}{\nu!\nu!}\frac{|z|^{2\nu}|w|^{2\nu}y^{\nu}}{(1-y)^{2\nu}}.$$

Thus

$$\sum_{\nu=0}^{\infty}K_{n,\nu}(r)y^{\nu}=e^{\frac{y}{y-1}(|z|^2+|w|^2)}\sum_{\nu=0}^{\infty}\frac{r^{2\nu}}{\nu!\nu!}\frac{|z|^{2\nu}|w|^{2\nu}y^{\nu}}{(1-y)^{2\nu+n}}. \tag{4.9}$$

For $n=1$ and $r=1$ we obtain that

$$\begin{aligned}\sum_{\nu=0}^{\infty}K_{1,\nu}(1)y^{\nu}&=e^{\frac{y}{y-1}(|z|^2+|w|^2)}\sum_{\nu=0}^{\infty}\frac{|z|^{2\nu}|w|^{2\nu}}{\nu!\nu!}\frac{y^{\nu}}{(1-y)^{2\nu+1}}\\&=\sum_{\nu=0}^{\infty}L_{\nu}^{(0)}(|z|^2)L_{\nu}^{(0)}(|w|^2)y^{\nu},\end{aligned}$$

where the last identity follows from the formula

(4.10)

$$\sum_{\nu=0}^{\infty}\frac{\nu!}{(\nu+n-1)!}L_{\nu}^{(n-1)}(s)L_{\nu}^{(n-1)}(t)y^{\nu}=e^{y\frac{s+t}{y-1}}\sum_{\nu=0}^{\infty}\frac{s^{\nu}t^{\nu}}{\nu!(\nu+n-1)!}\frac{y^{\nu}}{(1-y)^{2\nu}+n},$$

that holds for $s,t\in\mathbb{R}$ and $y\in\mathbb{C}$, $|y|<1$ (see [E1, p.189, 10.12(20)]). Therefore

$$I_{1,\nu}=K_{1,\nu}(1)=L_{\nu}^{(0)}(|z|^2)L_{\nu}^{(0)}(|w|^2),$$

which proves (4.8) for $n=1$.

Now let $n\geq 2$. Then integrating term by term the series in (4.9) we have that

$$\sum_{\nu=0}^{\infty}I_{n,\nu}y^{\nu}=(n-1)e^{\frac{y}{y-1}(|z|^2+|w|^2)}\sum_{\nu=0}^{\infty}a_{n,\nu}\frac{|z|^{2\nu}|w|^{2\nu}y^{\nu}}{\nu!\nu!(1-y)^{2\nu+n}}$$

where

$$a_{n,\nu}=\int_0^1(1-r^2)^{n-2}r^{2\nu}2r\,dr=\int_0^1(1-r)^{n-2}r^{\nu}\,dr=\frac{(n-2)!\nu!}{(\nu+n-1)!}.$$

(See Gradshteyn–Ryzhik [GR, p.284, 3.191.1].) Therefore, by (4.10),

$$\begin{aligned}\sum_{\nu=0}^{\infty}I_{n,\nu}y^{\nu}&=(n-1)!e^{\frac{y}{y-1}(|z|^2+|w|^2)}\sum_{\nu=0}^{\infty}\frac{|z|^{2\nu}|w|^{2\nu}}{\nu!(\nu+n-1)!}\frac{y^{\nu}}{(1-y)^{2\nu+n}}\\&=\sum_{\nu=0}^{\infty}\binom{\nu+n-1}{\nu}^{-1}L_{\nu}^{(n-1)}(|z|^2)L_{\nu}^{(n-1)}(|w|^2)y^{\nu}.\end{aligned}$$

Hence

$$I_{n,\nu} = \binom{\nu+n-1}{\nu}^{-1} L_\nu^{(n-1)}(|z|^2) L_\nu^{(n-1)}(|w|^2),$$

and we have proved (4.8) for $n \geq 2$.

Case 2: $\psi = \mathcal{J}^\rho_{n-1}$.

For $n = 1$ the functional equation easily follows from a simple addition theorem for the Bessel functions of the first kind of order 0 (see Lebedev [L, p. 124, 5.12.2]):

$$\begin{aligned}\int_{\mathbf{U}(1)} \mathcal{J}_0^\rho(a \cdot U(b))\, dU &= \int_0^{2\pi} J_0\left(\rho\,\big||z| - e^{i\theta}|w|\big|\right) \frac{d\theta}{2\pi} \\ &= J_0(\rho|z|)J_0(\rho|w|) = \mathcal{J}_0^\rho(a)\mathcal{J}_0^\rho(b).\end{aligned}$$

Let $n \geq 2$. Then, reasoning as in case 1, we have:

$$I = \int_{\mathbf{U}(n)} \mathcal{J}^\rho_{n-1}(a \cdot U(b)) \frac{dU}{(n-1)!2^{n-1}} = \int_0^1 (1-r^2)^{n-2} Q(r) 2r\, dr,$$

where

$$Q(r) = \int_0^{2\pi} \frac{J_{n-1}(\rho\sqrt{|z|^2 + 2|z|\cdot|w|\cdot r\cos\theta + |w|^2})}{(\rho\sqrt{|z|^2 + 2|z|\cdot|w|\cdot r\cos\theta + |w|^2})^{n-1}} \frac{d\theta}{2\pi}.$$

By using the so-called Lommel's expansion

$$(t+s)^{-\frac{\nu}{2}} J_\nu(\sqrt{t+s}) = \sum_{m=0}^{\infty} \left(-\frac{s}{2}\right)^m \frac{t^{-\frac{\nu+m}{2}}}{m!} J_{\nu+m}(\sqrt{t})$$

(see Watson [W, p.140]), integrating term by term we obtain that

$$Q(r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \rho^{2m} |z|^m |w|^m r^m c_m \frac{J_{n-1+m}(\rho\sqrt{|z|^2+|w|^2})}{(\rho\sqrt{|z|^2+|w|^2})^{n-1+m}},$$

where

$$c_m = \int_0^{2\pi} \cos^m\theta \frac{d\theta}{2\pi} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \frac{1\cdot 3\cdots(m-1)}{2\cdot 4\cdots m}, & \text{if } m \text{ is even.}\end{cases}$$

(See [GR, 3.621.3, p.369].) Thus

$$Q(r) = \sum_{\ell=0}^{\infty} \frac{1\cdot 3\cdots(2\ell-1)}{2\cdot 4\cdots(2\ell)} \frac{J_{n-1+2\ell}(\rho\sqrt{|z|^2+|w|^2})}{(\rho\sqrt{|z|^2+|w|^2})^{n-1+2\ell}} \frac{\rho^{4\ell}|z|^{2\ell}|w|^{2\ell}r^{2\ell}}{(2\ell)!},$$

so

$$I = \sum_{\ell=0}^{\infty} b_{n,\ell} \frac{J_{n-1+2\ell}(\rho\sqrt{|z|^2+|w|^2})}{(\rho\sqrt{|z|^2+|w|^2})^{n-1+2\ell}} \frac{\rho^{4\ell}|z|^{2\ell}|w|^{2\ell}r^{2\ell}}{(2\ell)!},$$

where

$$\begin{aligned} b_{n,\ell} &= \frac{1\cdot 3\cdots(m-1)}{2\cdot 4\cdots m}(n-1)\int_0^1 (1-r^2)^{n-2} r^{2\ell} 2r\,dr \\ &= \frac{1\cdot 3\cdots(m-1)}{2\cdot 4\cdots m}\frac{(n-1)!\ell!}{(n+\ell-1)!(2\ell)!} \\ &= \frac{(n-1)!}{2^{2\ell}\ell!(n+\ell-1)!} \end{aligned}$$

And using again Lommel's expansion we get

$$I = (n-1)! \sum_{\ell,m=0}^{\infty} \left(-\frac{\rho^2|w|^2}{2}\right)^m \frac{J_{n-1+2\ell+m}(\rho|z|)}{m!(\rho|z|)^{n-1+2\ell+m}} \frac{\rho^{4\ell}|z|^{2\ell}|w|^{2\ell}}{2^{2\ell}\ell!(n+\ell-1)!}.$$

Then taking into account the Taylor's expansion of J_ν, i.e.

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(\nu+m)!}\left(\frac{x}{2}\right)^{2m+\nu},$$

we have that

$$\begin{aligned} I &= \frac{(n-1)!}{2^{n-1}} \sum_{j,\ell,m=0}^{\infty} \frac{(-1)^{j+\ell}\left(\frac{\rho|z|}{2}\right)^{2(j+\ell)} (-1)^{m+\ell}\left(\frac{\rho|w|}{2}\right)^{2(m+\ell)}}{m!j!(n-1+2\ell+m+j)!\ell!(n+\ell-1)!} \\ &= \frac{(n-1)!}{2^{n-1}} \sum_{\nu=0}^{\infty} \left(\sum_{\mu=0}^{\infty} \frac{(-1)^\mu c(\mu,\nu,\ell)}{\mu!(\mu+n-1)!}\left(\frac{\rho|z|}{2}\right)^{2\mu}\right) \frac{(-1)^\nu}{\nu!(\nu+n-1)!}\left(\frac{\rho|w|}{2}\right)^{2\nu}, \end{aligned}$$

where

$$\begin{aligned} c(\mu,\nu,\ell) &= \sum_{\ell=0}^{\min\{\mu,\nu\}} \frac{\nu!(\nu+n-1)!\mu!(\mu+n-1)!}{(\nu-\ell)!(\mu-\ell)!(\mu+\nu+n-1)!\ell!(n+\ell-1)!} \\ &= \binom{\mu+\nu+n-1}{\nu+n-1}^{-1} \cdot \sum_{\ell=0}^{\min\{\mu,\nu\}} \binom{\nu}{\ell}\binom{\mu+n-1}{\ell+n-1} = 1. \end{aligned}$$

A simple proof of the last identity can be done using the binomial expansion in the following way:

$$\begin{aligned}\sum_{\ell=0}^{\min\{\mu,\nu\}} \binom{\nu}{\ell}\binom{\mu+n-1}{\ell+n-1} &= \int_0^{2\pi} (1+e^{it})^{\nu}\overline{e^{-i(n-1)t}(1+e^{it})^{\mu+n-1}}\frac{dt}{2\pi} \\ &= \int_0^{2\pi} (1+e^{it})^{\nu} e^{i(n-1)t}(1+e^{-it})^{\mu+n-1}\frac{dt}{2\pi} \\ &= \int_0^{2\pi} (1+e^{it})^{\nu} e^{-i\mu t}(1+e^{it})^{\mu+n-1}\frac{dt}{2\pi} \\ &= \int_0^{2\pi} (1+e^{it})^{\mu+\nu+n-1} e^{-i\mu t}\frac{dt}{2\pi} \\ &= \binom{\mu+\nu+n-1}{\mu} = \binom{\mu+\nu+n-1}{\nu+n-1}.\end{aligned}$$

Hence we conclude that

$$I = \frac{(n-1)!}{2^{n-1}} \frac{J_{n-1}(\rho|z|)}{\left(\frac{\rho|z|}{2}\right)^{n-1}} \frac{J_{n-1}(\rho|w|)}{\left(\frac{\rho|w|}{2}\right)^{n-1}},$$

which ends the proof of the desired functional equation.

Corollary 4.5 *The bounded $\mathbb{T}^n$-spherical functions on $\mathbb{H}^n$ are the following functions:*

$$\begin{aligned}\psi^{\lambda}_{\nu,\nu}(z,t) &= e^{2\pi i\lambda t} w^{\lambda}_{\nu,\nu}(z), \qquad (\lambda,\nu) \in \mathbb{R}^* \times (\mathbb{Z}_+)^n \\ \mathcal{J}_0^{\rho}(z,t) &= J_0(\rho_1|z_1|)\cdots J_0(\rho_n|z_n|), \qquad \rho \in \mathbb{R}^n_+.\end{aligned}$$

Proof By induction on n.

For $n=1$, $\mathbb{T}^n = \mathbf{U}(n) = \mathbf{U}(1)$ so, in this case, the statement of the corollary is just that of the above theorem.

Assume $n \geq 2$. Let ψ be a bounded $\mathbb{T}^n$-spherical function on $\mathbb{H}^n$. The first argument used in the proof of the theorem shows that ψ has the form:

$$\psi(z,t) = \psi(a) = e^{2\pi i\lambda t}\psi_0(z),$$

for some $\lambda \in \mathbb{R}$.

Case 1: $\lambda \neq 0$.

Let us fix $z_n \in \mathbb{C}$. Then the function

$$z' \mapsto \psi_0(z', z_n), \qquad z' = (z_1, \ldots, z_{n-1})$$

is an eigenfunction of operators $\square_1, \dots, \square_{n-1}$ and by the induction hypothesis we have

$$\psi_0(z', z_n) = c(z_n) \cdot w^{\lambda}_{\nu',\nu'}(z')$$

for some $\nu' \in (\mathbb{Z}_+)^{n-1}$. On the other hand, $\psi_0(z', z_n)$ is an eigenfunction of the operator $\square_n$, so

$$c(z_n) = c \cdot w^{\lambda}_{\nu_n,\nu_n}(z_n).$$

Consider the following set:

$$D = \{z \in \mathbb{C}^n : w^{\lambda}_{\nu',\nu'}(z') \neq 0 \text{ and } w^{\lambda}_{\nu_n,\nu_n}(z_n) \neq 0\}.$$

Thus for $z \in D$ we have

$$\psi_0(z) = c \cdot w^{\lambda}_{\nu,\nu}(z),$$

where $\nu = (\nu', \nu_n) \in (\mathbb{Z}_+)^n$. Hence, since D is dense in $\mathbb{C}^n$, $\psi_0(z) = c \cdot w^{\lambda}_{\nu,\nu}(z)$ for all $z \in \mathbb{C}^n$. From the condition $\psi_0(0) = 1$ we have $c = 1$.

Case 2: $\lambda = 0$.

The same reasoning by induction gives us

$$\psi_0(z) = \prod_{k=1}^{n} J_0(\rho_k |z_k|).$$

Conversely, we have to show that the functions $\psi^{\lambda}_{\nu,\nu}$ and $\mathfrak{J}^{\rho}_0$ are bounded $\mathbb{T}^n$-spherical functions on $\mathbb{H}^n$, that is, they are bounded functions on $\mathbb{H}^n$ which satisfy the identity $\psi(0) = 1$ and the functional equation

$$\int_{\mathbb{T}^n} \psi(a \cdot \sigma b)\, d\sigma = \psi(a)\, \psi(b), \qquad \text{for every } a, b \in \mathbb{H}^n.$$

Since

$$\mathcal{J}^{\rho}_0(z, t) = \prod_{j=1}^{n} \mathcal{J}^{\rho_j}_0(z_j, t),$$

and every $\mathcal{J}^{\rho_j}_0$ is a bounded $\mathbb{T}^1$-function on $\mathbb{H}^1$, it is clear that $\mathcal{J}^{\rho}_0$ is a bounded $\mathbb{T}^n$-spherical function on $\mathbb{H}^n$.

It is also clear that $\psi^{\lambda}_{\nu,\nu}$ is bounded and $\psi^{\lambda}_{\nu,\nu}(0) = 1$, so let us check that $\psi^{\lambda}_{\nu,\nu}$ satisfies the above functional equation.

Let $a = (z, t)$ and $b = (w, s)$ be arbitrary points in $\mathbb{H}^n$. Then

$$\psi^{\lambda}_{\nu,\nu}(a \cdot \sigma b) = e^{2\pi i \lambda (t+s)} \prod_{j=1}^{n} \psi^{\lambda}_{\nu_j,\nu_j}(a_j \cdot \sigma_j b_j) \qquad \text{for every } \sigma \in \mathbb{T}^n,$$

where a_j and b_j are the following points in $\mathbb{H}^1$:

$$a_j = (z_j, 0), \qquad b_j = (w_j, 0), \qquad j = 1, \ldots, n.$$

Since every $\psi^{\lambda}_{\nu_j,\nu_j}$ satisfies the above functional equation for $n = 1$, so does $\psi^{\lambda}_{\nu,\nu}$:

$$\begin{aligned}\int_{\mathbb{T}^n} \psi^{\lambda}_{\nu,\nu}(a \cdot \sigma b)\, d\sigma &= e^{2\pi i\lambda(t+s)} \prod_{j=1}^{n} \int_{\mathbb{T}} \psi^{\lambda}_{\nu_j,\nu_j}(a_j \cdot \sigma_j b_j)\, d\sigma_j \\ &= e^{2\pi i\lambda(t+s)} \prod_{j=1}^{n} \psi^{\lambda}_{\nu_j,\nu_j}(a_j)\psi^{\lambda}_{\nu_j,\nu_j}(b_j) \\ &= \psi^{\lambda}_{\nu,\nu}(a)\psi^{\lambda}_{\nu,\nu}(b).\end{aligned}$$

□

Remark In a recent paper, Benson–Jenkins–Ratcliff [BJR] have also identified $\mathbb{T}(n)$ and $\mathbf{U}(n)$ spherical functions using a different method. The $\mathbf{U}(n)$ spherical functions have also been studied by Thangavelu [Th1].

4.4. Now we are in a position to describe the maximal ideal spaces $\mathcal{M} = \mathcal{M}(L^1_{\mathbf{0}}(\mathbb{H}^n))$ and $\mathcal{M}^* = \mathcal{M}(L^1_*(\mathbb{H}^n))$ of the group algebras $L^1_{\mathbf{0}}(\mathbb{H}^n)$ and $L^1_*(\mathbb{H}^n)$, respectively.

According to Theorem 3.2 and Corollary 4.5, $\mathcal{M}$ is the union of the disjoint sets

$$\mathcal{M}_1 = \{\, m_{\lambda,\nu} : (\lambda, \nu) \in \mathbb{R}^* \times (\mathbb{Z}_+)^n \,\}$$

and

$$\mathcal{M}_2 = \{\, m_{\rho} : \rho \in (\mathbb{R}_+)^n \,\},$$

where

$$\begin{aligned} m_{\lambda,\nu}(f) &= \int_{\mathbb{H}^n} f(\xi)\overline{\psi^{\lambda}_{\nu,\nu}(\xi)}\, d\xi, && \text{for } f \in L^1_{\mathbf{0}}(\mathbb{H}^n), \\ m_{\rho}(f) &= \int_{\mathbb{H}^n} f(\xi)\mathfrak{J}^{\rho}_0(\xi)\, d\xi, && \text{for } f \in L^1_{\mathbf{0}}(\mathbb{H}^n). \end{aligned}$$

Thus identifying $\mathcal{M}$ with the disjoint union $(\mathbb{R}^* \times (\mathbb{Z}_+)^n) \cup (\mathbb{R}_+)^n$, the Gelfand transform $\tilde{f}$ of $f \in L^1_{\mathbf{0}}(\mathbb{H}^n)$ is defined by

$$\begin{aligned} \tilde{f}(\lambda, \nu) &:= m_{\lambda,\nu}(f), && \text{for } (\lambda, \nu) \in \mathbb{R}^* \times (\mathbb{Z}_+)^n, \\ \tilde{f}(\rho) &:= m_{\rho}(f), && \text{for } \rho \in (\mathbb{R}_+)^n. \end{aligned}$$

Similarly, Theorems 3.2 and 4.4 show that $\mathcal{M}^*$ is the union of the disjoint sets

$$\mathcal{M}^*_1 = \{ m^*_{\lambda,\nu} : (\lambda, \nu) \in \mathbb{R}^* \times \mathbb{Z}_+ \}$$

and

$$\mathcal{M}_2^* = \{m_\rho^* : \rho \in \mathbb{R}_+\},$$

where

$$m_{\lambda,\nu}^*(f) = \int_{\mathbb{H}^n} f(\xi)\overline{\Psi_{\nu,\nu}^\lambda(\xi)}\, d\xi, \qquad \text{for } f \in L_0^1(\mathbb{H}^n),$$

$$m_\rho^*(f) = \int_{\mathbb{H}^n} f(\xi)\mathcal{J}_{n-1}^\rho(\xi)\, d\xi, \qquad \text{for } f \in L_0^1(\mathbb{H}^n).$$

Thus identifying $\mathcal{M}^*$ with the disjoint union $(\mathbb{R}^* \times \mathbb{Z}_+) \cup \mathbb{R}_+$, the Gelfand transform f^* of $f \in L_0^1(\mathbb{H}^n)$ is defined by

$$f^*(\lambda,\nu) := m_{\lambda,\nu}^*(f), \qquad \text{for } (\lambda,\nu) \in \mathbb{R}^* \times \mathbb{Z}_+,$$

$$f^*(\rho) := m_\rho^*(f), \qquad \text{for } \rho \in \mathbb{R}_+.$$

Hulanicki and Ricci show ([HR], see Faraut–Harzallah [FH] for detailed proof) that the Wiener Tauberian theorem holds for the algebras $L_*^1(\mathbb{H}^n)$ and $L_{\mathbf{0}}^1(\mathbb{H}^n)$, i.e., every proper closed ideal in these algebras contained in some maximal regular ideal. Taking into account the concrete realization of the Gelfand spectrum described above the Tauberian theorem takes the following forms:

Theorem 4.6 *Let J be a closed ideal in $L_{\mathbf{0}}^1(\mathbb{H}^n)$ and suppose that*
(1) *For any $(\lambda,\nu) \in \mathbb{R}^* \times (\mathbb{Z}_+)^n$ there exists some $f \in J$ such that*

$$\tilde{f}(\lambda,\nu) \neq 0.$$

(2) *For any $\rho \in (\mathbb{R}_+)^n$ there exists some $f \in J$ such that*

$$\tilde{f}(\rho) \neq 0.$$

Then $J = L_{\mathbf{0}}^1(\mathbb{H}^n)$.

Theorem 4.7 *Let J be a closed ideal in $L_*^1(\mathbb{H}^n)$ and suppose that*
(1) *For any $(\lambda,\nu) \in \mathbb{R}^* \times \mathbb{Z}_+$ there exists some $f \in J$ such that*

$$f^*(\lambda,\nu) \neq 0.$$

(2) *For any $\rho \in \mathbb{R}_+$ there exists some $f \in J$ such that*

$$f^*(\rho) \neq 0.$$

Then $J = L_*^1(\mathbb{H}^n)$.

5. Convolution equations systems with 0 and 0-homogeneous compactly supported Radon measures as coefficients

In this section we shall apply the results of the previous sections to the problem of uniqueness of bounded solutions of convolution equations systems on $\mathbb{H}^n$ which have 0 or **0**-homogeneous compactly supported Radon measures as coefficients.

First we need some auxiliary results.

Lemma 5.1 *Let $\varphi \in C^2(\mathbb{H}^n) \cap L^1(\mathbb{H}^n)$ such that*

$$\frac{\partial^2 \varphi}{\partial t^2} \in L^1(\mathbb{H}^n),$$

and $\varphi(z, \cdot)$ is compactly supported on $\mathbb{R}$, for every $z \in \mathbb{C}^n$.

Consider the function ψ defined by

$$\psi(z,t) = \int_{-\infty}^{\infty} e^{2\pi i \lambda t} \varphi(z, \lambda)\, d\lambda, \qquad \text{for a.e. } (z,t) \in \mathbb{H}^n.$$

Then $\psi \in L^1(\mathbb{H}^n)$ and $\hat{\psi} = \varphi$.

Proof Observe that $\psi(z,t) = \hat{\varphi}(z,-t)$, for every $(z,t) \in \mathbb{H}^n$. Thus the fact that $\psi \in L^1(\mathbb{H}^n)$ is equivalent to $\hat{\varphi} \in L^1(\mathbb{H}^n)$. To prove that we integrate two times by parts and we obtain

$$\int_{-\infty}^{\infty} e^{-2\pi i \lambda t} \frac{\partial^2 \varphi}{\partial t^2}(z,t)\, dt = (-2\pi i \lambda)^2 \int_{-\infty}^{\infty} e^{-2\pi i \lambda t} \varphi(z,t)\, dt = -(2\pi\lambda)^2 \hat{\varphi}(z,\lambda),$$

so

$$(2\pi\lambda)^2 |\hat{\varphi}(z,\lambda)| \le \int_{-\infty}^{\infty} \left| \frac{\partial^2 \varphi}{\partial t^2}(z,t) \right| dt, \qquad \text{for every } (z,\lambda) \in \mathbb{H}^n.$$

Therefore

$$\int_{-\infty}^{\infty} |\hat{\varphi}(z,\lambda)|\, d\lambda \le 2 \left\{ \int_{-\infty}^{\infty} |\varphi(z,t)|\, dt + \left(\int_1^{\infty} \frac{d\lambda}{(2\pi\lambda)^2} \right) \left(\int_{-\infty}^{\infty} \left| \frac{\partial^2 \varphi}{\partial t^2} \right| dt \right) \right\},$$

and hence

$$\int_{\mathbb{H}^n} |\hat{\varphi}(\xi)|\, d\xi \le 2 \left(\|\varphi\|_{L^1(\mathbb{H}^n)} + \frac{1}{(2\pi)^2} \left\| \frac{\partial^2 \varphi}{\partial t^2} \right\|_{L^1(\mathbb{H}^n)} \right) < +\infty.$$

Finally, since φ and $\hat{\varphi}$ are integrable on $\mathbb{H}^n$, the inversion formula shows that

$$\hat{\psi}(z,\lambda) = \int_{-\infty}^{\infty} e^{2\pi i \lambda t} \psi(z,t)\, dt = \int_{-\infty}^{\infty} e^{2\pi i \lambda t} \hat{\varphi}(z,t)\, dt = \varphi(z,\lambda),$$

for every $(z,\lambda) \in \mathbb{H}^n$. □

The Gelfand transform $\tilde{T}$ of a $\mathbf{0}$-homogeneous compactly supported Radon measure T on $\mathbb{H}^n$ is defined by

$$\begin{aligned} \tilde{T}(\lambda,\nu) &= \langle T, \overline{\psi^{\lambda}_{\nu,\nu}}\rangle, &&\text{for every } (\lambda,\nu)\in\mathbb{R}^*\times(\mathbb{Z}_+)^n,\\ \tilde{T}(\rho) &= \langle T, \mathcal{J}^{\rho}_0\rangle, &&\text{for every } \rho\in(\mathbb{R}_+)^n. \end{aligned}$$

Lemma 5.2 *Let T be a $\mathbf{0}$-homogeneous compactly supported Radon measure on $\mathbb{H}^n$, and let $\eta\in L^1_{\mathbf{0}}(\mathbb{H}^n)$. Then $T*\eta\in L^1_{\mathbf{0}}(\mathbb{H}^n)$ and $\widetilde{T*\eta}=\tilde{T}\cdot\tilde{\eta}$.*

Proof It is clear that $T*\eta\in L^1(\mathbb{H}^n)$, so let us see it is $\mathbf{0}$-homogeneous. In fact, since T and η are $\mathbf{0}$-homogeneous we have that

$$\begin{aligned} (T*\eta)(\sigma\zeta) &= \int_{\mathbb{H}^n}\eta(\xi^{-1}\cdot\sigma\zeta)\,dT(\xi)\\ &= \int_{\mathbb{H}^n}\eta((\sigma\xi)^{-1}\cdot\sigma\zeta)\,dT(\xi)\\ &= \int_{\mathbb{H}^n}\eta(\sigma(\xi^{-1}\cdot\zeta))\,dT(\xi)\\ &= (T*\eta)(\zeta), \end{aligned}$$

for every $\sigma\in\mathbb{T}^n$ and $\zeta\in\mathbb{H}^n$.

Now observe that the identity $\widetilde{T*\eta}=\tilde{T}\cdot\tilde{\eta}$ means that

$$\int_{\mathbb{H}^n}(T*\eta)(\zeta)\overline{\psi(\zeta)}\,d\zeta = \langle T,\overline{\psi}\rangle\cdot\int_{\mathbb{H}^n}\eta(\xi)\cdot\overline{\psi(\xi)}\,d\xi, \tag{5.1}$$

for every bounded $\mathbb{T}^n$-spherical function ψ of $\mathbb{H}^n$. So let ψ be such a function, and observe that

$$\begin{aligned} \int_{\mathbb{H}^n}(T*\eta)(\zeta)\overline{\psi(\zeta)}\,d\zeta &= \left((T*\eta)*\check{\overline{\psi}}\right)(0) = \left(T*(\eta*\check{\overline{\psi}})\right)(0)\\ &= \int_{\mathbb{H}^n}(\eta*\check{\overline{\psi}})(\zeta^{-1})\,dT(\zeta). \end{aligned}$$

(Here we use the following notation: if φ is a function on $\mathbb{H}^n$, then $\check{\varphi}$ denotes the function on $\mathbb{H}^n$ defined by $\check{\varphi}(\zeta)=\varphi(\zeta^{-1})$, for every $\zeta\in\mathbb{H}^n$.)

But, since η is $\mathbf{0}$-homogeneous, we have that

$$(\eta*\check{\overline{\psi}})(\zeta^{-1}) = \int_{\mathbb{H}^n}\eta(\xi)\overline{\psi(\zeta\cdot\xi)}\,d\xi = \int_{\mathbb{H}^n}\eta(\xi)\overline{\psi(\zeta\cdot\sigma\xi)}\,d\xi,$$

for every $\zeta \in \mathbb{H}^n$ and $\sigma \in \mathbb{T}^n$. Therefore, using the characteristic functional equation of the $\mathbb{T}^n$-spherical functions, we obtain that

$$\begin{aligned}(\eta * \check{\bar{\psi}})(\zeta^{-1}) &= \int_{\mathbb{T}^n} \left(\int_{\mathbb{H}^n} \eta(\xi) \overline{\psi(\zeta \cdot \sigma\xi)} \right) d\sigma \\ &= \int_{\mathbb{H}^n} \eta(\xi) \overline{\left(\int_{\mathbb{T}^n} \psi(\zeta \cdot \sigma\xi)\, d\sigma \right)} d\xi \\ &= \overline{\psi(\zeta)} \int_{\mathbb{H}^n} \eta(\xi) \cdot \overline{\psi(\xi)}\, d\xi.\end{aligned}$$

Hence we conclude that equation (5.1) holds, and we have finished the proof of the lemma. □

Similarly, the Gelfand transform T^* of a 0-homogeneous compactly supported Radon measure T on $\mathbb{H}^n$ is defined by

$$\begin{aligned} T^*(\lambda, \nu) &= \langle T, \overline{\Psi^{\lambda}_{\nu,\nu}} \rangle, \qquad \text{for every } (\lambda, \nu) \in \mathbb{R}^* \times \mathbb{N}, \\ T^*(\rho) &= \langle T, \mathcal{J}^{\rho}_{n-1} \rangle, \qquad \text{for every } \rho \in \mathbb{R}_+. \end{aligned}$$

Then we have the following lemma, whose proof we omit since it is just a copy of the above one in the **U** context.

Lemma 5.3 *Let T be a 0-homogeneous compactly supported Radon measure on $\mathbb{H}^n$, and let $\eta \in L^1_0(\mathbb{H}^n)$. Then $T * \eta \in L^1_0(\mathbb{H}^n)$ and $(T * \eta)^* = T^* \cdot \eta^*$.*

Now we may state the main results of this section.

Theorem 5.4 *Let $\mathcal{R}$ be a family of **0**-homogeneous compactly supported Radon measures on $\mathbb{H}^n$. Assume that $\mathcal{R}$ satisfies the following two conditions:*

(1) *For any $(\lambda, \nu) \in \mathbb{R}^* \times (\mathbb{Z}_+)^n$, there exists some $T \in \mathcal{R}$ such that $\tilde{T}(\lambda, \nu) \neq 0$.*

(2) *For any $\rho \in (\mathbb{R}_+)^n$, there exists some $T \in \mathcal{R}$ such that $\tilde{T}(\rho) \neq 0$.*

Let f be a bounded continuous function on $\mathbb{H}^n$ such that

$$f * T = 0, \qquad \textit{for every } T \in \mathcal{R}. \tag{5.2}$$

Then $f \equiv 0$.

If one of the conditions (1), (2) fails to hold, then there exists a bounded continuous function $f \not\equiv 0$ satisfying (5.2).

Proof Condition (5.2) clearly implies that

$$f * (T * \eta) = 0, \qquad \text{for every } T \in \mathcal{R} \text{ and } \eta \in L^1_0(\mathbb{H}^n).$$

So the closed ideal J in $L^1_0(\mathbb{H}^n)$ generated by the set

$$\mathcal{R} * L^1_0(\mathbb{H}^n) = \{T * \eta : T \in \mathcal{R}, \eta \in L^1_0(\mathbb{H}^n)\},$$

which is contained in $L^1_0(\mathbb{H}^n)$ by Lemma 5.2, satisfies $f * J = 0$. Thus to show that $f \equiv 0$ it is sufficient to prove that $J = L^1_0(\mathbb{H}^n)$. Hence we want to verify the conditions of the Tauberian theorems 4.7 and 4.8.

By Lemma 5.2, it is clear that our hypothesis (1) and (2) above are equivalent to the corresponding conditions in Theorem 4.7 if we show that:

(1′) For any $(\lambda', \nu') \in \mathbb{R}^* \times (\mathbb{Z}_+)^n$, there exists some $\eta \in L^1_0(\mathbb{H}^n)$ such that $\tilde{\eta}(\lambda', \nu') \neq 0$.

(2′) For any $\rho \in (\mathbb{R}_+)^n$, there exists some $\eta \in L^1_0$ such that $\tilde{\eta}(\rho) \neq 0$.

Let $(\lambda', \nu') \in \mathbb{R}^* \times (\mathbb{Z}_+)^n$, and take $\varphi \in C^\infty(\mathbb{R})$ with compact support in the interval $(\frac{|\lambda'|}{2}, 2|\lambda'|)$ and verifying $\varphi(|\lambda'|) \neq 0$. Then, by Lemma 5.1, the function η defined by

$$\eta(z,t) = \int_{-\infty}^{\infty} e^{2\pi i\lambda t}\varphi(|\lambda|) w^\lambda_{\nu',\nu'}(z)\, d\lambda, \qquad \text{for a.e. } (z,t) \in \mathbb{H}^n,$$

belongs to $L^1_0(\mathbb{H}^n)$, and

$$\hat{\eta}^\lambda = \varphi(|\lambda|) \cdot w^\lambda_{\nu',\nu'}, \qquad \text{for every } \lambda \in \mathbb{R}.$$

Therefore

$$\tilde{\eta}(\lambda', \nu') = \int_{\mathbb{C}^n} \hat{\eta}^{\lambda'}(z) \cdot \overline{w^{\lambda'}_{\nu',\nu'}(z)}\, dV(z) = \varphi(|\lambda'|)\|w^{\lambda'}_{\nu',\nu'}\|^2_{L^2(\mathbb{C}^n)} \neq 0,$$

and we have just shown (1′).

In order to prove (2′) we only have to note that the function

$$\eta(z,t) = e^{-\pi(|z|^2+t^2)},$$

which obviously belongs to $L^1_0(\mathbb{H}^n)$, satisfies:

$$\begin{aligned}
\tilde{\eta}(\rho) &= \left(\int_{-\infty}^{\infty} e^{-\pi t^2}\, dt\right) \prod_{j=1}^n \int_{\mathbb{C}^n} e^{-\pi|z_j|^2} J_0(\rho_j|z_j|)\, dV(z_j) \\
&= \prod_{j=1}^n 2\pi \int_0^\infty J_0(2\pi r_j \frac{\rho_j}{2\pi}) e^{-\pi r_j^2} r_j dr_j \\
&= \prod_{j=1}^n \int_{\mathbb{C}^n} e^{-\pi|z_j|^2} e^{-2\pi i \mathrm{Re}\left(z_j \frac{\rho_j}{2\pi}\right)}\, dV(z_j) \\
&= \prod_{j=1}^n e^{-\frac{1}{4\pi}\rho_j^2} \neq 0,
\end{aligned}$$

for every $\rho \in (\mathbb{R}_+)^n$.

Now assume that (1) or (2) fails. That means there is a bounded $\mathbb{T}^n$-spherical function f on $\mathbb{H}^n$ (e.g., $f = \psi^\lambda_{\nu,\nu}$ or $f = \mathcal{J}^\rho_0$) such that

$$\langle T, \bar{f} \rangle = 0, \qquad \text{for every } T \in \mathcal{R}.$$

Then, since every $T \in \mathcal{R}$ is $\mathbf{0}$-homogeneous, we have:

$$\begin{aligned}(f * T)(\zeta) &= \int_{\mathbb{H}^n} f(\zeta \cdot (\sigma\xi)^{-1})\, dT(\xi) \\ &= \int_{\mathbb{H}^n} f(\zeta \cdot \sigma\xi^{-1})\, dT(\xi),\end{aligned}$$

for every $\zeta \in \mathbb{H}^n$, $\sigma \in \mathbb{T}^n$ and $T \in \mathcal{R}$. Hence, using the characteristic functional equation of the $\mathbb{T}^n$-spherical functions, we conclude that

$$\begin{aligned}(f * T)(\zeta) &= \int_{\mathbb{T}^n} \left(\int_{\mathbb{H}^n} f(\zeta \cdot \sigma\xi^{-1})\, dT(\xi) \right) d\sigma \\ &= \int_{\mathbb{H}^n} \left(\int_{\mathbb{T}^n} f(\zeta \cdot \sigma\xi^{-1})\, d\sigma \right) dT(\xi) \\ &= f(\zeta) \int_{\mathbb{H}^n} f(\xi^{-1})\, dT(\xi) \\ &= f(\zeta) \int_{\mathbb{H}^n} \overline{f(\xi)}\, dT(\xi) \\ &= f(\zeta) \langle T, \bar{f} \rangle = 0,\end{aligned}$$

for every $T \in \mathcal{R}$, and the proof of the theorem is complete. □

Theorem 5.5 *Let $\mathcal{R}$ be a family of 0-homogeneous compactly supported Radon measures on $\mathbb{H}^n$. Suppose that $\mathcal{R}$ satisfies the following two conditions:*

(1) *For any $(\lambda, \nu) \in \mathbb{R}^* \times \mathbb{Z}_+$, there exists some $T \in \mathcal{R}$ such that $T^*(\lambda, \nu) \neq 0$.*

(2) *for any $\rho \in (\mathbb{R}_+)^n$, there exists some $T \in \mathcal{R}$ such that $T^*(\rho) \neq 0$.*

Let f be a bounded continuous function on $\mathbb{H}^n$ such that

$$f * T = 0, \qquad \text{for every } T \in \mathcal{R}. \tag{5.3}$$

Then $f \equiv 0$.

If one of the conditions (1), (2) fails to hold, then there exists a bounded continuous function $f \not\equiv 0$ with the property (5.3).

Proof Arguing as in the proof of Theorem 5.4 (now we use Theorem 4.8 instead of Theorem 4.7) it turns out that to prove the first part of Theorem 5.5 we only have to check that:

(1′) For any $(\lambda', \nu') \in \mathbb{R}^* \times \mathbb{Z}_+$, there exists some $\eta \in L^1_*(\mathbb{H}^n)$ such that $\eta^*(\lambda', \nu') \neq 0$.

(2′) For any $\rho \in \mathbb{R}_+$, there exists some $\eta \in L^1_*(\mathbb{H}^n)$ such that $\eta^*(\rho) \neq 0$.

Let $(\lambda', \nu') \in \mathbb{R}^* \times \mathbb{Z}_+$, and take $\varphi \in C^\infty(\mathbb{R})$ with compact support in the interval $(\frac{|\lambda'|}{2}, 2|\lambda'|)$ and verifying $\varphi(|\lambda'|) \neq 0$. Then, by Lemma 5.1, the function η defined by

$$\eta(z,t) = \int_{-\infty}^{\infty} e^{2\pi i\lambda t}\varphi(|\lambda|)v^\lambda_{\mathbf{0},\nu'}(z)\,d\lambda, \qquad \text{for a.e. } (z,t) \in \mathbb{H}^n,$$

belongs to $L^1_*(\mathbb{H}^n)$, and

$$\hat{\eta}^\lambda(z) = \varphi(|\lambda|)\cdot v^\lambda_{\mathbf{0},\nu'}, \qquad \text{for every } \lambda \in \mathbb{R}.$$

Therefore

$$\begin{aligned}\eta^*(\lambda', \nu') &= \binom{\nu' + n - 1}{\nu'}^{-1} \int_{\mathbb{C}^n} \hat{\eta}^{\lambda'}(z)\cdot \overline{v^{\lambda'}_{\mathbf{0},\nu'}(z)}\,dV(z)\\ &= \binom{\nu' + n - 1}{\nu'}^{-1} \varphi(|\lambda'|)\cdot \|v^{\lambda'}_{\mathbf{0},\nu'}\|^2_{L^2(\mathbb{C}^n)} \neq 0,\end{aligned}$$

and we have just shown (1′).

In order to prove (2′) we only have to note that the function

$$\eta(z,t) = e^{-\pi(|z|^2+t^2)},$$

which obviously belongs to $L^1_*(\mathbb{H}^n)$, satisfies:

$$\begin{aligned}\eta^*(\rho) &= (n-1)!2^{n-1}\left(\int_{-\infty}^{\infty} e^{-\pi t^2}\,dt\right)\int_{\mathbb{C}^n} e^{-\pi|z|^2}\frac{J_{n-1}(\rho|z|)}{(\rho|z|)^{n-1}}\,dV(z)\\ &= 2\pi\left(\frac{2\pi}{\rho}\right)^{n-1}\int_0^\infty J_{n-1}(2\pi r\frac{\rho}{2\pi})e^{-\pi r^2}r^n dr\\ &= \int_{\mathbb{C}^n} e^{-\pi|z|^2}e^{-2\pi i \operatorname{Re}(\frac{\rho}{2\pi}\bar{z}_1)}dV(z)\\ &= e^{-\frac{\rho^2}{4\pi}} \neq 0,\end{aligned}$$

for every $\rho \in \mathbb{R}_+$.

The proof of the second part is just a copy of that one in Theorem 5.4, where now the group $\mathbf{U}$ plays the role played by $\mathbb{T}^n$ in the cited proof. □

6. Pompeiu type theorems in $\mathbb{H}^n$

We are going to prove four Pompeiu type theorems in this section. We shall discuss the proofs of these results in two sub-sections. First of all, let us introduce some notation that will be useful in the present theorem and the next one.

If $\mathcal{F}$ is a family of functions which are defined on the interval $(0, +\infty)$, we denote by $\mathcal{Q}(\mathcal{F})$ the following set:

$$\mathcal{Q}(\mathcal{F}) = \{ \frac{s}{t} : s, t > 0, \ f(s) = g(t) = 0, \ \text{for some} f, g \in \mathcal{F} \}.$$

When $\mathcal{F}$ is composed of only one function, *i.e.* $\mathcal{F} = \{f\}$, we denote $\mathcal{Q}(\mathcal{F})$ simply by $\mathcal{Q}(f)$.

6.1. Integration over tori and spheres

Let $r > 0$. We denote by $T(r)$ the square-type tori in $\mathbb{C}^n$ centered at the origin:

$$T(r) = \{ (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| = r, \ j = 1 \dots, n \}.$$

Then we have the following theorem:

Theorem 6.1 *Let f be a bounded continuous function on $\mathbb{H}^n$ which satisfies*

$$\text{(6.1)} \qquad \int_{T(r_k)} (L_a f)(z, 0) \, d\sigma_{r_k}(z) = 0, \qquad \text{for every } a \in \mathbb{H}^n \text{ and } k = 1, \dots, N,$$

for N square-type tori. Here σ_r is the area measure of $T(r)$.

Suppose that the following conditions hold: For $1 \le k \le N$,

(1) *the functions*

$$\ell_k^{(\mathbf{0})}(\lambda; \nu) = \prod_{i=1}^{n} L_{\nu_i}^{(0)}(4\pi|\lambda| r_k^2),$$

have no common zero for $(\lambda; \nu) \in \mathbb{R}_+ \times (\mathbb{Z}_+)^n$;

(2) *the functions*

$$\mathcal{J}_k^{(\mathbf{0})}(\rho) = \prod_{i=1}^{n} J_0(\rho_i r_k)$$

have no common zero $\rho \in (\mathbb{R}_+)^n$.

Then $f \equiv 0$.

Conversely, if one of the conditions (1) or (2) fails to hold, then there is a bounded continuous function $f \not\equiv 0$ on $\mathbb{H}^n$ satisfying (6.1).

Proof Let T be the compactly supported **0**-homogeneous Radon measure on $\mathbb{H}^n$ defined by

$$(6.2) \qquad \langle T, \varphi \rangle = \frac{1}{(2\pi)^n r^n} \int_{T(r)} \varphi(z, 0)\, d\sigma_r(z), \qquad \text{for every } \varphi \in C(\mathbb{H}^n).$$

Then (6.1) can be written as the convolution equations $f * T_k = 0$, $k = 1, \ldots, N$. On the other hand,

$$(6.3) \qquad \tilde{T}_k(\lambda, \nu) = e^{2\pi\lambda n r_k^2} \prod_{i=1}^{n} L_{\nu_i}^{(0)}(4\pi|\lambda| r_k^2), \qquad \text{for every } (\lambda, \nu) \in \mathbb{R}^* \times (\mathbb{Z}_+)^n.$$

$$(6.4) \qquad \tilde{T}_k(\rho) = \prod_{i=0}^{n} J_0(\rho_i r_k) \qquad \text{for every } \rho \in (\mathbb{R}_+)^n.$$

So it is evident that the first assertion of the theorem is just a consequence of Theorem 5.4. In fact, from (6.3) and (6.4), hypotheses (1) and (2) are equivalent to the corresponding conditions of Theorem 5.4 for the family of **0**-homogeneous compactly supported Radon measures $\mathcal{R} = \{T_1, \ldots, T_N\}$.

Conversely, suppose that the functions $\ell_k^{(\mathbf{0})}$, $k = 1, \ldots, N$ have a common zero $(\lambda; \nu) \in \mathbb{R}_+ \times (\mathbb{Z}_+)^n$. We can take

$$f(z, t) = \psi_{\nu,\nu}^{\lambda}(z, t) = e^{2\pi i \lambda t} w_{\nu,\nu}^{\lambda}(z).$$

Then $f \not\equiv 0$ but, by (6.3), satisfies $f * T_k = 0$ for $k = 1, \ldots, N$.

If the functions $\mathcal{J}_k^{(\mathbf{0})}$, $k = 1, \ldots, N$ have a common zero $\rho \in (\mathbb{R}_+)^n$, we can take

$$f(z, t) = \prod_{i=1}^{n} J_0(\rho_i |z_i|).$$

Taking into account (6.4), we know that $f * T_k = 0$ for $k = 1, \ldots, N$ but $f \not\equiv 0$. Hence we prove the second assertion of the theroem and the proof of the theorem is therefore complete. □

Remarks (1) The condition (2) of Theorem 6.1 requires that $N > n$. Indeed, if $n \leq N$ let us take $s_1, \ldots, s_N > 0$ such that $J_0(s_k) = 0$. Put $\rho_i = \frac{s_i}{r_i}$ for $1 \leq i \leq N$ and define ρ_i arbitrarily for $N < i \leq n$. Then

$$\mathcal{J}_1^{(\mathbf{0})}(\rho) = \cdots = \mathcal{J}_N^{(\mathbf{0})}(\rho) = 0$$

and we can see that for $N \leq n$ the condition (2) is never satisfied.

(2) One can give elementary necessary and sufficient conditions for the radii r_j so that the conditions (1) and (2) of Theorem 6.1 hold in case $N = n + 1$. These conditions are directly in terms of quotients of the radii r_j. They involve avoiding certain quotients of zeroes of Laguerre polynomials and Bessel functions. We restrict ourselves to write them down in the simplest case $n = 1$, $N = 2$.

(1′)

$$\frac{r_1^2}{r_2^2} \notin \bigcup_{\nu \in \mathbb{Z}_+} \mathcal{Q}(L_\nu^{(0)})$$

(2′)

$$\frac{r_i}{r_j} \notin \mathcal{Q}(J_0).$$

(3) Let $r_1, \ldots, r_n > 0$. The conclusions for Theorem 6.1 and Theorem 6.4 below hold for tori in $\mathbb{C}^n$ centered at the origin and with polyradius $\vec{r} = (r_1, \ldots, r_n)$, i.e.

$$T(\vec{r}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| = r_j,\ j = 1, \ldots, n\}.$$

Using square-type tori here just simplifies the notations

For $r > 0$ we denote by $S_n(r)$ the sphere in $\mathbb{C}^n$ centered at the origin and with radius r, i.e.

$$S_n(r) = \{z \in \mathbb{C}^n : |z| = r\}.$$

We have the following result:

Theorem 6.2 *Let f be a bounded continuous function on $\mathbb{H}^n$ which satisfies*

$$\int_{S_n(r)} (L_a f)(z, 0)\, d\sigma_r(z) = 0, \qquad \text{for every } a \in \mathbb{H}^n, \tag{6.5}$$

for two radii r_1, r_2. Here σ_r is the area measure of $S_n(r)$.

Assume that the above radii r_1 and r_2, satisfy the following two conditions:

(1) $\left(\frac{r_1}{r_2}\right)^2 \notin \bigcup_{\nu \in \mathbb{Z}_+} \mathcal{Q}(L_\nu^{(n-1)})$.

(2) $\frac{r_1}{r_2} \notin \mathcal{Q}\left(\frac{J_{n-1}(t)}{t^{n-1}}\right)$.

Then $f \equiv 0$.

Conversely, if one of the conditions (1) or (2) fails to hold, then there is a bounded continuous function $f \not\equiv 0$ on $\mathbb{H}^n$ satisfying (6.5).

Proof Let T be the compactly supported 0-homogeneous Radon measure on $\mathbb{H}^n$ defined by

$$\langle T, \varphi \rangle = \int_{S(r)} \varphi(z, 0) \frac{d\sigma_r(z)}{\omega_{2n-1} r^{2n-1}}, \qquad \text{for every } \varphi \in C(\mathbb{H}^n), \tag{6.6}$$

where ω_{2n-1} is the area of the unit sphere S^{2n-1} of $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Let T_k be the Radon measure on $\mathbb{H}^n$ defined by (6.6) with $r = r_k$, for $k = 1, 2$. Then our conditions on f can be written as the following system of convolution equations:

$$f * T_1 = f * T_2 = 0.$$

According to Theorem 4.4, for $k = 1, 2$,

(6.7)

$$T_k^*(\lambda, \nu) = \binom{\nu + n - 1}{\nu}^{-1} e^{-2\pi|\lambda|r_k^2} L_\nu^{(n-1)}(4\pi|\lambda|r_k^2), \quad \text{for every } (\lambda, \nu) \in \mathbb{R}^* \times \mathbb{N}.$$

(6.8)
$$T_k^*(\rho) = (n-1)!2^{n-1} \frac{J_{n-1}(\rho r_k)}{(\rho r_k)^{n-1}}, \quad \text{for every } \rho \in \mathbb{R}_+.$$

It is easy to see from (6.7) and (6.8), that the conditions (1) and (2) are exactly equivalent to the absence of common zeroes of the functions $T_k^*(\lambda, \nu)$, $k = 1, 2$ and $T_k^*(\rho)$, $k = 1, 2$. So, the first assertion is the consequence of Theorem 5.5.

Conversely, if one of the conditions (1) or (2) fails to hold, the functions then have common zeroes and we can construct a function $f \not\equiv 0$, satisfying (6.5) by means of spherical functions, like in Theorem 6.1. Thus the proof of the theorem is complete. □

6.2. Integration over balls and polydisks

For $r > 0$ denote by $B_n(r)$ the ball in $\mathbb{C}^n$ centered at the origin and with radius r, i.e.

$$B_n(r) = \{ z \in \mathbb{C}^n : |z| < r \}.$$

For $\nu \in (\mathbb{Z}_+)^n$ we consider the function $\mathcal{L}_\nu^{(n-1)}$ defined by

$$\mathcal{L}_\nu^{(n-1)}(x) = \int_0^x e^{-\frac{t}{2}} t^{n-1} L_\nu^{(n-1)}(t)\, dt \qquad \text{for every } x \geq 0.$$

Note that by integrating by parts it is possible to compute explicitly the function $\mathcal{L}_\nu^{(n-1)}$. It has the following form:

$$\mathcal{L}_\nu^{(n-1)}(x) = (-1)^\nu 2^n \frac{(n+\nu)!}{\nu!} + e^{-\frac{x}{2}} P_{\nu,n}(x),$$

where $P_{\nu,n}(x)$ is a polynomial of degree $\nu + n - 1$.

Then we obtain the following theorem:

Theorem 6.3 *Let f be a bounded continuous function on $\mathbb{H}^n$ which satisfies*

$$(6.9)\qquad \int_{B_n(r_k)} (L_a f)(z,0)\,dV(z) = 0, \qquad \textit{for every } a \in \mathbb{H}^n,$$

for two radii $r_k = r_1, r_2$. Here dV is the volume element of $B_n(r)$.

Assume that those radii r_1 and r_2, satisfy the following two conditions:

(1) $\left(\frac{r_1}{r_2}\right)^2 \notin \bigcup_{\nu\in\mathbb{Z}_+} \mathcal{Q}(\mathcal{L}_\nu^{(n-1)})$.

(2) $\frac{r_1}{r_2} \notin \mathcal{Q}\left(\frac{J_n(t)}{t^n}\right)$.

Then $f \equiv 0$.

Conversely, if one of the conditions (1) or (2) fails to hold, then there is a bounded continuous function $f \not\equiv 0$ on $\mathbb{H}^n$ satisfying (6.9).

Remark The set $\cup_{\nu\in\mathbb{N}}\mathcal{Q}(\mathcal{L}_\nu^{(n-1)})$ is non-empty since, at least when $\nu \in \mathbb{N}$ is odd, the function $\mathcal{L}_\nu^{(n-1)}$ has some positive zero.

In order to show that, recall that the ν zeroes of the generalized Laguerre polynomial $L_\nu^{(n-1)}$ are positive and simple (see [E1, p.204, §10.17]). Let $0 < x_1 < \ldots < x_\nu$ be such zeroes. Since the coefficient of x^ν in $L_\nu^{(n-1)}(x)$ is $\frac{(-1)^\nu}{\nu!}$, we have that

$$L_\nu^{(n-1)}(x) = \frac{(-1)^\nu}{\nu!}(x - x_1)\cdots(x - x_\nu).$$

Thus it is clear that $L_\nu^{(n-1)}$ is positive on the interval $(0, x_1)$, so $\mathcal{L}_\nu^{(n-1)}$ is also positive on that interval. On the other hand, by [E1, p.191(32)],

$$\lim_{x\to+\infty} \mathcal{L}_\nu^{(n-1)}(x) = \int_0^\infty e^{-\frac{t}{2}} t^{n-1} L_\nu^{(n-1)}(t)\,dt = (-1)^\nu 2^n \frac{(n+\nu)!}{\nu!},$$

which is negative if ν is odd, and therefore, by continuity, we conclude that $\mathcal{L}_\nu^{(n-1)}$ must have some positive zero.

Proof Let T be the compactly supported 0-homogeneous Radon measure on $\mathbb{H}^n$ defined by

$$(6.10)\qquad \langle T, \varphi\rangle = \int_{B_n(r)} \varphi(z,0)\,dV(z), \qquad \text{for every } \varphi \in C(\mathbb{H}^n).$$

Then (6.9) can be written as the convolution equation $f * T = 0$. In order to apply Theorem 5.5 let us compute the Gelfand transform T^* of T.

For $(\lambda, \nu) \in \mathbb{R}^* \times \mathbb{N}$ we have:

$$T^*(\lambda,\nu) = \binom{\nu+n-1}{\nu}^{-1} \omega_{2n-1} \int_0^r e^{-2\pi|\lambda|t^2} L_\nu^{(n-1)}(4\pi|\lambda|t^2) t^{2n-1} dt$$
$$= \binom{\nu+n-1}{\nu}^{-1} \frac{\omega_{2n-1}}{2(4\pi|\lambda|)^n} \int_0^{4\pi|\lambda|r^2} e^{-\frac{t}{2}} L_\nu^{(n-1)}(t^2) t^{n-1} dt.$$

(Here as usual ω_{2n-1} denotes the area of the unit sphere of $\mathbb{C}^n$.)

Therefore

(6.11)
$$T^*(\lambda,\nu) = \binom{\nu+n-1}{\nu}^{-1} \frac{\omega_{2n-1}}{2(4\pi|\lambda|)^n} \mathfrak{L}_\nu^{(n-1)}(4\pi|\lambda|r^2), \quad \text{for every } (\lambda,\nu) \in \mathbb{R}^* \times \mathbb{N}.$$

On the other side, for $\rho > 0$ we have that

$$T^*(\rho) = (n-1)!2^{n-1}\omega_{2n-1} \int_0^r \frac{J_{n-1}(\rho t)}{(\rho t)^{n-1}} t^{2n-1} dt$$
$$= (n-1)!2^{n-1}\frac{\omega_{2n-1}}{\rho^{2n}} \int_0^{\rho r} t^n J_{n-1}(t)\, dt.$$

So, taking into account that

$$\frac{d}{dt}(t^n J_n(t)) = t^n J_{n-1}(t)$$

(see [E2, p.11(50)]), we obtain that

$$T^*(\rho) = (n-1)!2^{n-1}\omega_{2n-1} r^{2n} \frac{J_n(\rho r)}{(\rho r)^n} \quad \text{for every } \rho > 0. \tag{6.12}$$

The remainder of the proof is the same as the previous one. □

Let $r > 0$. We denote by $\Delta(r)$ the (open) polydisk in $\mathbb{C}^n$ centered at the origin:

$$\Delta(r) = \{ (z_1,\dots,z_n) \in \mathbb{C}^n : |z_j| < r,\ j = 1\dots,n \}.$$

Let T_k, $1 \le k \le N$, be the compactly supported $\mathbf{0}$-homogeneous Radon measure on $\mathbb{H}^n$ defined by

$$\langle T_k, \varphi\rangle = \int_{\Delta(r_k)} \varphi(z,0)\, dV(z), \quad \text{for every } \varphi \in C(\mathbb{H}^n). \tag{6.14}$$

Then the condition

$$\int_{\Delta(r_k)} (L_a f)(z,0)\, dV(z) = 0, \quad \text{for every } a \in \mathbb{H}^n, \tag{6.15}$$

can be written as the convolution equations $f * T_k = 0$. On the other hand, observe that

$$\psi^{\lambda}_{\nu,\nu}(z,0) = \prod_{i=1}^{n} \psi^{\lambda}_{\nu_i,\nu_i}(z_i,0) \qquad \text{for every } z \in \mathbb{C}^n,$$

so

$$\tilde{T}_k(\lambda,\nu) = \prod_{i=1}^{n} \tilde{T}^1_k(\lambda,\nu_i), \qquad \text{for every } (\lambda,\nu) \in \mathbb{R}^* \times (\mathbb{Z}_+)^n$$

$$\tilde{T}(\rho) = \prod_{i=1}^{n} \tilde{T}^1_k(\rho_i), \qquad \text{for every } \rho \in (\mathbb{R}_+)^n.$$

But T^1_k is just the Radon measure on $\mathbb{H}^1$ considered in the proof of Theorem 6.5 for $n = 1$, so its Gelfand transform $\widetilde{T^1_k} = (T^1_k)^*$ was calculated there. Thus we conclude that for

$$(6.16) \qquad \tilde{T}_k(\lambda,\nu) = \left(\frac{\pi}{4\pi|\lambda|}\right)^n \prod_{i=1}^{n} \mathcal{L}^{(0)}_{\nu_i}(4\pi|\lambda|r_k^2),$$

for every $(\lambda,\nu) \in \mathbb{R}^* \times (\mathbb{Z}_+)^n$, and

$$(6.17) \qquad \tilde{T}_k(\rho) = (2\pi)^n \prod_{i=1}^{n} r_k^2 \frac{J_1(\rho_i r_k)}{\rho_i r_k},$$

for every $\rho \in (\mathbb{R}_+)^n$. (Here we use the usual convention that the value of $\frac{J_1(z)}{z}$ at $z = 0$ is equal to $\frac{1}{2}$.)

Using the same argument as Theorem 6.3, we have the following theorem:

Theorem 6.4 *Let f be a bounded continuous function on $\mathbb{H}^n$ which satisfies (6.15) for N polydisks $D(r_1),\ldots,D(r_N)$ with $N > n$.*

Suppose that the following conditions hold: For $1 \le k \le N$,

(1) *the functions*

$$(\lambda,\nu) \mapsto \prod_{i=1}^{n} \mathcal{L}^{(0)}_{\nu_i}(4\pi|\lambda|r_k^2),$$

have no common zero for $(\lambda;\nu) \in \mathbb{R}_+ \times (\mathbb{Z}_+)^n$;

(2) *the functions*

$$\rho \mapsto \prod_{i=1}^{n} r_k^2 \frac{J_1(\rho_i r_k)}{\rho_i r_k},$$

have no common zero $\rho \in (\mathbb{R}_+)^n$.

Then $f \equiv 0$.

Conversely, if one of the conditions (1) or (2) fails to hold, then there is a bounded continuous function $f \not\equiv 0$ on $\mathbb{H}^n$ satisfying (6.14).

Remark The Theorems 6.1 to 6.4 hold with the condition $f \in L^\infty(\mathbb{H}^n)$ replaced by $f \in L^p(\mathbb{H}^n)$, $1 \le p \le \infty$. Let us prove this statement.

Let $\psi \in C_c^\infty(\mathbb{H}^n)$ be a positive radial function such that $\int_{\mathbb{H}^n} \psi(z,t)dV(z)dt = 1$ and $\{\psi_k\}_{k=1}^\infty$ the corresponding approximation of the identity. Then $\psi_k \in L^q(\mathbb{H}^n)$, $1 \le q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, hence

$$f * \psi_k = f_k \in L^p(\mathbb{H}^n) \cap L^\infty(\mathbb{H}^n).$$

As f_k will integrate to zero along the same sets as f does. We derive $f_k \equiv 0$ from the corresponding theorems in section 6 for all k. Then, letting $k \to \infty$, we obtain $f = 0$.

7. The interpretation from the point of view of the Weyl functional calculus

In this section we shall show that the uniqueness conditions in section 6 (the Pompeiu type theorems for the case of spheres) can be formulated in terms of "zeroes" of operator-valued Bessel functions, considered as functions of the position and impulse operators in the Heisenberg group.

7.1 Let us recall some basic facts, concerning the Weyl representation and Weyl calculus (see Taylor [T, section 1] or Geller [G, Chapter 6]).

Denote by $X = (X_1, \dots, X_n)$ the usual position operator:

$$X_j u(x) = x_j u(x), \qquad x = (x_1, \dots, x_n),$$

and by $D = (D_1, \dots, D_n)$ the impulse operator:

$$D_j u(x) = \frac{1}{i}\frac{\partial u}{\partial x_j}(x),$$

i.e., infinitesimal generators of the group

$$u(x) \mapsto u(x+p), \qquad p \in \mathbb{R}^n$$

of translations and group of multiplications

$$u(x) \mapsto e^{ix \cdot p} u(x), \qquad p \in \mathbb{R}^n,$$

respectively. These operators are defined in $L^2(\mathbb{R}^n)$ and connected by the well-known Weyl commutator relations:

$$[X_k, D_j] = i\delta_{kj} I.$$

For a point $(z,t) \in \mathbb{H}^n$ we introduce $x, y \in \mathbb{R}^n$ by

$$z = x + iy.$$

The Weyl representation of the Heisenberg group includes two parts:

$$(1) \qquad \pi_{\pm\lambda}(x,y,t) = e^{2\pi i(\pm\lambda^{\frac{1}{2}}x\cdot X + \lambda^{\frac{1}{2}}y\cdot D \pm \lambda t)}, \qquad \lambda > 0,$$

$$(2) \qquad \pi_{(\xi,\eta)}(x,y,t) = e^{2\pi i(x\cdot\xi + y\cdot\eta)}, \qquad (\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n.$$

The Fourier transform generated by these representations, is the following mapping which transform functions on $\mathbb{H}^n$ in operators:

$$(1) \qquad \pi_{\pm\lambda}(f) = \int_{\mathbb{R}^{2n+1}} f(x,y,t)\pi_{\pm\lambda}(x,y,t)\,dxdydt,$$

$$(2) \qquad \pi_{(\xi,\eta)}(f) = \int_{\mathbb{R}^{2n}} f(x,y,t)\pi_{(\xi,\eta)}(x,y,t)\,dxdydt.$$

In case (2) we have the Euclidean inverse Fourier transform of the function f, computed at the point

$$(\xi,\eta,0) \in \mathbb{R}^{2n+1},$$

or, in our notations,

$$\pi_{(\xi,\eta)}(f) = \mathcal{F}_{2n+1}(f)(-\xi,-\eta,0).$$

This function is considered as operator of multiplication.

Now we concentrate on the case (1). By definition

$$\begin{aligned} \pi_{\pm\lambda}(f) &= \int_{\mathbb{R}^{2n+1}} f(x,y,t)\pi_{\pm\lambda}(x,y,t)\,dxdydt \\ &= \int_{\mathbb{R}^{2n+1}} f(x,y,t)e^{2\pi i(\pm\lambda^{\frac{1}{2}}x\cdot X + \lambda^{\frac{1}{2}}y\cdot D \pm \lambda t)}, \qquad \lambda > 0. \end{aligned}$$

According to the functional operator calculus, this is

$$(7.1) \qquad \pi_{\pm\lambda}(f) = \mathcal{F}_{2n+1}(f)(\mp\lambda^{\frac{1}{2}}X, -\lambda^{\frac{1}{2}}D, \pm\lambda).$$

The operator $a(X,D)$ is defined by the Weyl calculus as:

$$a(X,D) = \int_{\mathbb{R}^{2n}} \mathcal{F}_{2n+1}a(x,y)e^{2\pi i(x\cdot X + y\cdot D)}dxdy.$$

We can apply the construction above to compactly supported Radon measures, not only to functions. So, let $f \in L^1(\mathbb{H}^n)$ or compactly supported Radon measure on $\mathbb{H}^n$. Let us also assume that f is $\mathbf{U}(n)$-invariant, i.e.

$$f(U(z),t) = f(z,t), \qquad \text{for all } U \in \mathbf{U}(n).$$

Then the Weyl–Fourier transform

$$\mathcal{F}_{2n+1}(f)(\pm\lambda^{\frac{1}{2}}X, \pm\lambda^{\frac{1}{2}}D, \pm\lambda)$$

is a function $F_\lambda(H)$ of harmonic oscillator Hamiltonian H (see [T, Chapter 1, Prop. 7.7]):

$$H = -\triangle + |x|^2.$$

The function $\mathcal{F}_{2n+1}(f)(\xi, \eta, \lambda)$ is the Fourier transform in $\mathbb{R}^{2n}$ of the function $\widehat{f}^\lambda$, which is the Fourier transform by t-variable only. Then the connection between $\mathcal{F}_{2n+1}(f)$ and $F_\lambda(H)$ is as follows (see [T, Chapter 1, Prop. 7.9] and [Th1]):

$$\begin{aligned} \pi_{\pm\lambda}(f) &= \mathcal{F}_{2n+1}(f) \\ &= c\sum_{j=0}^{\infty} F_\lambda(2j+1)\Psi_j(|\lambda|^{\frac{1}{2}}X, |\lambda|^{\frac{1}{2}}D), \end{aligned} \tag{7.2}$$

where

$$\Psi_j(x,y) = (-1)^j e^{-2\pi(|x|^2+|y|^2)} L_j^{(n-1)}\left(4\pi(|x|^2+|y|^2)\right),$$

the constant c depends on n only. Note that at this point Laguerre functions come into play.

Let us apply this construction to our concrete case. The Euclidean Fourier transform of the area measure σ_r on the sphere $S(r)$ is the Bessel function

$$\begin{aligned} (\mathcal{F}_{2n+1}(\sigma_r))(x,y,t) &= \frac{1}{(2\pi)^n} \frac{J_{n-1}(2\pi r(|x|^2+|y|^2)^{\frac{1}{2}})}{(r(|x|^2+|y|^2)^{\frac{1}{2}})^{\frac{n}{2}}} \\ &= j_{n-1}(2\pi r(|x|^2+|y|^2)^{\frac{1}{2}}) \end{aligned}$$

where $j_k(x) = \frac{J_k(x)}{x^k}$. Therefore, we have by (7.1):

$$\pi_{\pm\lambda}(\sigma_r) = j_{n-1}\left(4\pi r|\lambda|^{\frac{1}{2}}(X^2+D^2)^{\frac{1}{2}}\right). \tag{7.3}$$

What is the function F_λ in the decomposition (7.2)? We can recognize it, for instance, from the Fourier–Laguerre expansion

$$J_{n-1}(xy) = \sum_{k=1}^{\infty} \gamma_k(x) e^{-\frac{y^2}{2}} y^{\frac{n-1}{2}} L_k^{(n-1)}(y). \tag{7.4}$$

The coefficients $\gamma_k(x)$ can be obtained from the following formula (see [L, p. 83 (4.20.3)]):

$$e^{-\frac{x}{2}}x^{\frac{n-1}{2}}L_k^{(n-1)}(x) = \frac{(-1)^k}{2}\int_0^\infty J_{n-1}(\sqrt{xy})e^{-\frac{y}{2}}y^{\frac{n-1}{2}}L_k^{(n-1)}(y)dy,$$

so that

$$\gamma_k(x) = 2(-1)^k\frac{k!}{(n+k-1)!}e^{-\frac{x}{2}}x^{\frac{n-1}{2}}L_k^{(n-1)}(x).$$

Now, substituting $\gamma_k(x)$ in (7.4), letting $x = t^2$, $y = s^2$ and dividing both sides by $(ts)^{n-1}$, we finally obtain:

$$j_{n-1}(ts) = 2\sum_{j=1}^{\infty}(-1)^j e^{-\frac{t^2}{2}}L_j^{(n-1)}(t^2)\cdot e^{-\frac{s^2}{2}}L_j^{(n-1)}(s^2). \tag{7.5}$$

Comparing formulas (7.2), (7.3) with (7.5) where we set $t^2 = 4\pi(|x|^2+|y|^2)$ and $s^2 = r^2|\lambda|$, we conclude that one can take as F_λ any (nice) function taking at the odd integers the values

$$F_\lambda(2j+1) = \frac{2j!}{c(n+j-1)!}e^{-\lambda r^2}L_j^{(n-1)}(|\lambda|r^2).$$

Obviously, the operator $F(H)$ has the spectrum $\mathrm{sp}(F(H)) = F(\mathrm{sp}(H))$. Since $\mathrm{sp}(H) = \{2j+1, j\in\mathbb{Z}_+\}$, then $\mathrm{sp}(F(H)) = \{F(2j+1), j\in\mathbb{Z}_+\}$. Thus, we conclude that eigenvalues of the operator $\pi_{\pm\lambda}(\sigma_r)$ are

$$c_j(\lambda, r) = \frac{2j!}{c(n+j-1)!}L_j^{(n-1)}(4\pi|\lambda|r^2), \qquad j = 1, 2, \ldots.$$

As for eigenfunctions, they coincide with the eigenfunctions of the Hamiltonian H (Hermite functions).

Now we can see that condition (1) for radii in the Theorem 6.2 simply means that the operator Bessel functions

$$\pi_{\pm\lambda}(\sigma_{r_k}) = j_{n-1}\left(4\pi r_k|\lambda|^{\frac{1}{2}}(X^2+D^2)^{\frac{1}{2}}\right), \qquad k = 1, 2,$$

have eigenvalues $c_j(\lambda, r_1)$, $c_j(\lambda, r_2)$ (corresponding the same eigenfunctions), which do not vanish simultaneously at the same point $\lambda \neq 0$. This is equivalent to the following: the intersection of the kernels

$$\ker\left\{j_{n-1}\left(4\pi r_1|\lambda|^{\frac{1}{2}}(X^2+D^2)^{\frac{1}{2}}\right)\right\}\cap\ker\left\{j_{n-1}\left(4\pi r_2|\lambda|^{\frac{1}{2}}(X^2+D^2)^{\frac{1}{2}}\right)\right\} = \{0\} \tag{7.3}$$

for all $\lambda \neq 0$. In the limit case $\lambda \to 0$ we come to the classical scalar-valued Bessel functions $j_{n-1}\left(r_k(x^2+y^2)^{\frac{1}{2}}\right)$, $k = 1, 2$, and the condition (2) in Theorem 6.2 says that operators of multiplication, corresponding to these functions, also have no common nontrivial kernel. Thus, both condition (1) and (2) can be unified in the condition (7.3) satisfied for all real λ. Thus the condition (1) is "quantization" of the condition (2) and the condition (2) is a particular (limit) case of the condition (1) as $\lambda \to 0$. The parameter λ plays role of Planck constant. This may not be so surprising, since the Heisenberg group itself can be considered as "quantization" of the Euclidean space.

Remark After we finished this paper, we received a preprint from S. Thangavelu [Th2]. Inspired by a result of Strichartz [S], he has shown that one radius condition is enough to prove the injectivity of the Pompeiu transform for $f \in L^p(\mathbb{H}^n)$, $1 \leq p < \infty$, i.e., $f * \sigma_r = 0$ implies $f = 0$. However, we are mainly interested in $L^\infty(\mathbb{H}^n)$ functions in this paper. As we pointed out in our earlier paper [ABCP], one radius condition is not enough to prove the injectivity of the Pompeiu transform for this class. To handle the $L^\infty(\mathbb{H}^n)$ class, the one-dimensional representation plays an essential role. In fact, we can also obtain a one radius theorem for $L^p(\mathbb{H}^n)$ functions, $1 \leq p < \infty$, by a careful analysis of our proof. We refer the reader to our recent paper [ABC].

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(Received March 1, 1992)