BOUNDARY SMOOTHNESS PROPERTIES OF LIPα ANALYTIC FUNCTIONS

By

D. J. LORD* AND A. G. O'FARRELL†

Abstract. Let U be an open set and $b \in \text{bdy}(U)$. Let $0 < \alpha < 1$. Let $A(U)$ denote the space of Lipα functions that are analytic on U, and $a(U)$ the subspace lip$\alpha \cap A(U)$. The space $a(U \cup \{b\})$, consisting of the functions that are analytic near b, is dense in $a(U)$. Let k be a natural number. We say that $a(U)$ admits a k-th order continuous point derivation (cpd) at b if the functional $f \mapsto f^{(k)}(b)$ is continuous on $a(U \cup \{b\})$, with respect to the Lipα norm.

Theorem *$a(u)$ admits a k-th order cpd at b if and only if*

$$\sum_{n=1}^{\infty} 2^{(k+1)n} M_*^{1+\alpha}(A_n(b) \sim U) < +\infty.$$

Here M_*^{β} denotes β-dimensional lower Hausdorff content, and $A_n(b)$ denotes the annulus

$$\{z \in \mathbb{C} : |z-b| \in [2^{-n-1}, 2^{-n}]\}.$$

There is a weak-star topology on $A(U)$, and the space $A(U \cup \{b\})$ is weak-star dense in $A(U)$. We say that $A(U)$ admits a k-th order weak-star cpd at b if the functional $f \mapsto f^{(k)}(b)$ is weak-star continuous on $A(U \cup \{b\})$.

Theorem *$A(u)$ admits a k-th order weak-star cpd at b if and only if*

$$\sum_{n=1}^{\infty} 2^{(k+1)n} M^{1+\alpha}(A_n(b) \sim U) < +\infty.$$

This time, M^{β} denotes ordinary β-dimensional Hausdorff content.

1. Introduction

Let $0 < \alpha < 1$. For $E \subset \mathbb{C}$ and $f : E \to \mathbb{C}$ let

$$\|f\|'_{\text{Lip}\alpha(E)} = \sup\left\{\frac{|f(z)-f(w)|}{|z-w|^{\alpha}} : z \neq w\right\}.$$

We call $\|f\|'_{\text{Lip}\alpha(E)}$ the Lip$\alpha(E)$ *seminorm of f*. We denote

$$\text{Lip}\alpha(E) = \{f \in \mathbb{C}^E : \|f\|'_{\text{Lip}\alpha} < +\infty\}.$$

* Supported by EOLAS grant BR/89/125.

† Supported by EOLAS grant SC/90/070.

This is a Banach space when endowed with the norm

$$\|f\|_{\mathrm{Lip}\alpha} = |f(b)| + \|f\|'_{\mathrm{Lip}\alpha},$$

where b is any fixed point of E. We abbreviate Lip$\alpha(\mathbb{C})$ to Lipα. The subspace lip$\alpha \subset$ Lipα consists of those $f \in$ Lipα such that

$$\lim_{\delta \downarrow 0} \sup_{0<|z-w|<\delta} \frac{|f(z) - f(w)|}{|z-w|^\alpha} = 0.$$

For open sets $U \subset \mathbb{C}$ we denote

$$A(U) = \{f \in \mathrm{Lip}\alpha : \bar{\partial} f = 0 \text{ on } U\},$$

$$a(U) = \{f \in \mathrm{lip}\alpha : \bar{\partial} f = 0 \text{ on } U\}.$$

Here $\bar{\partial} f$ denotes the distributional $\bar{\partial}$–derivative

$$\bar{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

In view of Weyl's Lemma, "$\bar{\partial} f = 0$ on U" is a way of saying that the restriction $f|U$ is an analytic function.

This paper is about the extent to which the functions belonging to $A(U)$ or $a(U)$ may be better-behaved at points of bdy$U$ than are typical elements of Lipα or lipα. Specifically, we consider the question of the existence of bounded point derivations. We will explain this concept shortly. First, we review some classical facts.

Suppose b is an isolated point of bdy$U$. Then, since the elements $f \in A(U)$ are bounded and analytic on a deleted neighbourhood of b, it follows that they extend analytically across b, and since they are continuous, they are already analytic on $U \cup \{b\}$.

Similarly, if a line segment I forms a relatively-open subset of $\mathbb{C} \sim U$, then each function $f \in A(U)$ extends analytically across I.

These facts may be rephrased in terms of the concept of $\bar{\partial}$-Lipα-null set:

A compact $K \subset \mathbb{C}$ is said to be $\bar{\partial}$-Lipα-*null* if

$$A(U \sim K) = A(U)$$

whenever $U \subset \mathbb{C}$ is open.

Singletons and line segments are $\bar{\partial}$-Lipα-null.

Not every compact set K having no interior is $\bar{\partial}$-Lip-null. For instance, if K has positive area, then the function

$$f(z) = \frac{-1}{\pi} \int_K \frac{d\xi d\eta}{z - (\xi + i\eta)}$$

(the Cauchy transform of area restricted to K) belongs to each Lipα ($\alpha < 1$), and is analytic on $\mathbb{C} \sim K$, and nonconstant. Dolženko characterised the $\bar{\partial}$-Lipα-null compact sets in terms of Hausdorff contents.

A *measure function* is a monotone nondecreasing function $h : [0, +\infty) \to [0, +\infty)$. The *Hausdorff content* M_h associated to a measure function h is defined by

$$M_h(E) = \inf_{\mathcal{S}} \sum_{B \in \mathcal{S}} h(\text{diam} B),$$

whenever $E \subset \mathbb{C}$, where $\mathcal{S}$ runs over all countable coverings of E by balls (or, equivalently, open balls, or closed balls, or arbitrary sets). When $h(r) = r^\beta$, we denote $M_h = M^\beta$.

Dolženko's result is that a compact set K is $\bar{\partial}$-Lipα-null if and only if $M^{1+\alpha}(K) = 0$ [3].

A similar result holds for lipα. The *lower β-dimensional Hausdorff content* of E is

$$M_*^\beta(E) = \sup_h M_h(E)$$

where h runs over all measure functions such that $h(r) \le r^\beta$ and $r^{-\beta} h(r) \to 0$ as $r \downarrow 0$. We say that K is $\bar{\partial}$-lipα-null if

$$a(U \sim K) = a(U), \qquad \forall U \text{ open}.$$

The result [11] is that K is $\bar{\partial}$-lipα-null if and only if $M_*^{1+\alpha}(K) = 0$.

For example, if C is the usual middle-thirds Cantor set, then the square Cantor set $C \times C$ has

$$M^{1+\alpha}(C \times C) = 0 \Leftrightarrow \alpha > \log_3 4/3,$$
$$M_*^{1+\alpha}(C \times C) = 0 \Leftrightarrow \alpha \ge \log_3 4/3,$$

so $C \times C$ is $\bar{\partial}$-Lipα-null if and only if $\alpha > \log_3 4/3$ and $\bar{\partial}$-lipα-null if and only if $\alpha \ge \log_3 4/3$.

Obviously, if all functions $f \in A(U)$ extend analytically across a boundary point a, then they are as smooth as can be. But it may happen that limited smoothness occurs even at points which are not of this type.

For an arbitrary set $E \subset \mathbb{C}$, let

$$A(E) = \bigcup\{A(U) : U \text{ open}, E \subset U\},$$
$$a(E) = \bigcup\{a(U) : U \text{ open}, E \subset U\}.$$

The spaces $A(E)$, $a(E)$ are closed subspaces of Lipα in case E is open.

Lemma 1.1 *Let $b \in \mathbb{C}$ and $U \subset \mathbb{C}$ be open. Then $a(U \cup \{b\})$ is dense in $a(U)$.*

Accepting this for the moment, we note that for $k \in N$ the maps

$$\delta_b^k : f \mapsto f^{(k)}(b)$$

are well-defined linear functionals on $A(U \cup \{b\})$.

Definition We say that $a(U)$ *admits a k-th order continuous point derivation* (cpd) *at b* if δ_b^k extends to a continuous linear functional on $a(U)$.

Equivalently, $a(U)$ admits a k-th order cpd at b if and only if there exists a constant $\kappa > 0$ such that

$$|f^{(k)}(b)| \leq \kappa \|f\|_{\text{Lip}\alpha}$$

whenever $f \in a(U \cup \{b\})$.

We denote

$$A_n(b) = \{z \in \mathbb{C} : 2^{-n} < |z - b| \leq 2^{-n-1}\}.$$

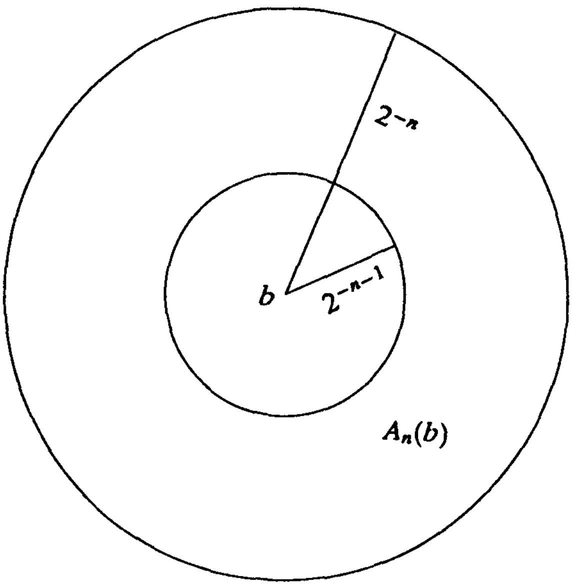

Fig. 1.

Our first main result is:

Theorem 1.2 *Let $0 < \alpha < 1$, $U \subset \mathbb{C}$ be open, $b \in \mathbb{C}$, and $k \in N$. Then $a(U)$ admits a k-th order cpd at b if and only if*

$$\sum_{n=1}^{+\infty} 2^{(k+1)n} M_{*}^{1+\alpha}(A_n(b) \sim U) < +\infty.$$

The second main result is a similar theorem for $A(U)$. It involves *weak-star* continuous cpd's.

Let b be a point of the interior of some (large) closed disk D. The restriction space

$$\text{Lip}\alpha(D) = \{f|D : f \in \text{Lip}\alpha\}$$

is a Banach space (indeed, a Banach algebra) with the quotient norm. Similarly, for $\text{lip}\alpha(D)$. De Leeuw showed that

$$\text{lip}\alpha(D)^{**}$$

is isometrically isomorphic to $\text{Lip}\alpha(D)$. Thus $\text{Lip}\alpha(D)$ acquires a weak-star topology, as the dual of $\text{lip}\alpha(D)^{*}$. When we refer to weak-star topological concepts in the sequel, we intend that the topology be of this kind, for some suitably large D.

Lemma 1.3 *Let $b \in \mathbb{C}$ and $U \subset \mathbb{C}$ be open. Then $A(U \cup \{b\})$ is weak-star dense in $A(U)$.*

We say that $A(U)$ admits a k-th *order weak-star cpd at b* if δ_b^k extends to a weak-star continuous linear functional on $\text{Lip}\alpha(D)$. Whether this happens or not does not depend on the choice of D (with $b \in \text{int}(D)$). The condition may be expressed in terms of the De Leeuw representation (see 2.7 below): $A(U)$ admits a k-th order weak-star continuous point derivation at b if and only if there exists a finite-total-variation Borel-regular measure ρ on $\mathbb{C} \times \mathbb{C}$, having no mass on the diagonal, such that

$$f^{(k)}(b) = \int_{\mathbb{C}\times\mathbb{C}} \frac{(f(z) - f(w))d\rho(z, w)}{|z - w|^{\alpha}}$$

whenever $f \in A(U \cup \{b\})$.

Theorem 1.4 *Let $0 < \alpha < 1$, $U \subset \mathbb{C}$ be open, $b \in \mathbb{C}$, and $k \in N$. Then $A(U)$ admits a k-th order weak-star continuous point derivation at b if and only if*

$$\sum_{n=1}^{+\infty} 2^{(k+1)n} M^{1+\alpha}(A_n(b) \sim U) < +\infty.$$

These results are in a line of development which goes back to Wiener's work [16] on the problem of regular boundary points for the Dirichlet problem. Series involving capacities of intersections with annuli are called Wiener series. Wiener series have been used a lot, to characterise various kinds of thinness, and in connection with analytic functions, for instance in Melnikov's characterisation of peak points for the algebra $R(X)$ [7], and in [6], [5], [8–10,12], [15]. Like most Wiener series characterizations, the present results are quite concrete, and allow us to determine whether or not a cpd exists at any boundary point of a sufficiently-precisely described set U.

We prove the results in Section 2, and give a few illustrative examples in Section 3.

2. Proof of Results

2.1. We begin by recalling extension theorems for Lipschitz functions. Each real-valued Lip1 function on a subset of a metric space may be extended to the whole space, so that it remains Lip1 with the same constant. This is Kirszbraun's theorem [4]. Applying this to $\mathbb{C}$, with the metric

$$\rho(x,y) = |x-y|^{\alpha}$$

(remember that $0 < \alpha < 1$), we see that each real-valued Lipα function on a subset of $\mathbb{C}$ has an extension having the same Lipα seminorm. Applying this to the real and imaginary parts, we see that each complex-valued Lipα function f on a subset X of $\mathbb{C}$ has a Lipα extension $\tilde{f}$ to $\mathbb{C}$ having at most twice the seminorm:

$$\|\tilde{f}\|_{\mathrm{Lip}\alpha(\mathbb{C})} \le 2 \cdot \|f\|_{\mathrm{Lip}\alpha(X)}.$$

There is a similar result for lipα functions. Given a lipα function on a subset $E \subset \mathbb{C}$, we may choose a nondecreasing concave function $\omega(r)$ such that ω dominates the modulus of continuity of f and has $r^{-\alpha}\omega(r) \to 0$ as $r \downarrow 0$. Moreover, we can arrange that $r^{-\alpha}\omega(r)$ is bounded by the Lipα seminorm of f. Applying Kirszbraun's theorem to $\mathbb{C}$ with the metric

$$\rho(x,y) = \omega(|x-y|),$$

we find that f has an extension in lipα that has at most double the Lipα seminorm of f.

2.2. Next, we recall **Frostman's Lemma**. According to this, if a Borel set $E \subset \mathbb{R}^d$ has positive M_h content, for some measure function h, then it is possible to find a positive measure ν, with compact support contained in E, such that the total mass of ν exceeds $M_h(E)$, and such that ν has *growth* $h(r)$, in the sense that

$$\nu\mathbb{B}\,(b,r) \le \kappa h(r)$$

whenever $b \in \mathbb{C}$ and $r > 0$. Here, the constant κ may be taken to depend only on the dimension d. See [1].

If $M_*^\beta(E) > 0$, then the measure ν may be chosen to have compact support in E, with

$$\nu(\mathbb{C}) \ge M_*^\beta(E),$$

$$\nu\,(\mathbb{B}(b,r)) \le \kappa \cdot r^\beta, \qquad \forall b \in \mathbb{C} \text{ and } r > 0,$$

and

$$\frac{\sup_b \nu(\mathbb{B}\,(b,r))}{r^\beta} \to 0 \quad \text{as } r \downarrow 0.$$

2.3. The Lipα seminorm of a product is controlled by the Lipα seminorm and the sup norm of the factors:

$$\|fg\|'_{\mathrm{Lip}\alpha} \le \|f\|'_{\mathrm{Lip}\alpha} \cdot \|g\|_{\mathrm{L}^\infty} + \|g\|'_{\mathrm{Lip}\alpha} \cdot \|f\|_{\mathrm{L}^\infty}.$$

Also, the Lipα seminorm of a differentiable function on a bounded set is controlled by the sup norm of the derivative:

$$\|g\|'_{\mathrm{Lip}(\alpha,X)} \le \|Dg\|_{\mathrm{L}^\infty} \cdot \mathrm{diam}(X)^{1-\alpha}.$$

Putting these facts together, we obtain the following estimate for the Lipα seminorm of $f(z)/(z-b)^k$ on annuli about the point b:

Lemma 2.3 *Let f belong to* Lipα *and let $b \in \mathbb{C}$. There is a constant κ_k, independent of $n \in \mathbb{N}$, such that*

$$\left\| z \mapsto \frac{f(z)}{(z-b)^k} \right\|'_{\mathrm{Lip}\alpha(A_n(b))} \le \kappa \cdot 2^{kn} \cdot \|f\|_{\mathrm{Lip}\alpha(A_n(b))}.$$

(Note that f may be assumed to have a zero on A_n, without loss in generality.)

2.4. We recall the Decay Estimates for Lipschitz holomorphic functions.

Decay Lemma *Let $0 < \alpha < 1$, let $K \subset \mathbb{C}$ be compact and let $f \in$* Lip$\alpha(\mathbb{C})$ *be analytic off K and vanish at ∞. Then there are constants $\kappa > 0$ and $\kappa_k > 0$, depending on α but not on K or f, such that the following estimates hold for $z \notin K$:*

(1) $$\|f\|_{\mathrm{L}^\infty} \le \kappa \cdot \|f\|'_{\mathrm{Lip}\alpha} \cdot M^{1+\alpha}(K)^{\frac{\alpha}{1+\alpha}},$$

(2) $$|f(z)| \le \frac{\kappa \cdot \|f\|'_{\mathrm{Lip}\alpha} \cdot M^{1+\alpha}(K)}{\mathrm{dist}(z,K)},$$

and if $k \in \mathbb{N}$ then

$$|f^{(k)}(z)| \leq \frac{\kappa_k \cdot \|f\|'_{\mathrm{Lip}\alpha} \cdot M^{1+\alpha}(K)}{\mathrm{dist}(z,K)^{k+1}}. \tag{3}$$

See [11, section 12] and [13, section 7] (The argument for (3) is a routine extension of the argument for (1) that is given in the former paper.)

2.5. We recall the T-invariance properties of Lipα.

The Vitushkin localisation operator, T_ϕ, is defined by

$$(T_\phi)f = C(\phi \cdot \bar{\partial} f),$$

for all distributions f and all test functions ϕ, where C denotes the Cauchy transform:

$$Cg = \frac{-1}{\pi z} * g,$$

for all distributions g having compact support. In view of the fact that C inverts $\bar{\partial}$ on the compactly-supported distributions, we see that $T_\phi f$ is analytic off the support of ϕ and off the support of $\bar{\partial} f$. The spaces Lipα and lipα (like all translation-symmetric concrete spaces) are mapped into themselves by the action of T_ϕ [14]. But they also have the additional property of *nice* T-invariance. This means that they are mapped equicontinuously by the sequence of operators $\{T_{\phi_n}\}$ whenever $\{\phi_n\}$ is a *standard pincher*. To be precise:

Lemma *Let ϕ_n ($n = 1, 2, 3, \ldots$) be a C^∞ function having compact support, and be such that*
(1) sptϕ_n *is a subset of the ball* $\mathbb{B}(0, 1/n)$,
(2) $\sup_n \|\phi_n\|_{L^\infty} \leq \kappa_1$, *where* $\kappa_1 < +\infty$ *is a constant independent of* n.
(3) $\|\nabla\phi_n\|_{L^\infty} \leq \kappa_2 \cdot n$, *where* $\kappa_2 < +\infty$ *is a constant independent of* n.
Let $0 < \alpha < 1$. *Then there is a constant* $\kappa(\kappa_1, \kappa_2, \alpha) > 0$ *such that*

$$\|T_{\phi_n} f\|_{\mathrm{Lip}\alpha(\mathbb{C})} \leq \kappa \|f\|_{\mathrm{Lip}\alpha(\mathbb{B}(0,1/n))}, \qquad \forall n.$$

This follows from Lemma (4.1) in [13] (see (2.2) of that paper for the definition of the quantity $N(\phi)$ there referred to).

2.6. We now prove a Quantitative Cauchy Theorem for Lipschitz functions.

Theorem (a) *Let Γ be a piecewise analytic curve bounding a region $\Omega \in \mathbb{C}$, and suppose that Γ is free of outward-pointing cusps. Let $0 < \alpha < 1$. Then there exists a constant $\kappa(\Gamma, \alpha) > 0$ such that*

$$\left|\int_\Gamma f(z)dz\right| \leq \kappa \cdot M^{1+\alpha}(\Omega \cap S)$$

whenever $\|f\|'_{\alpha,\mathbb{C}} \leq 1$, *$S$ is closed, and f is analytic on $\Omega \sim S$.*

(b) *The constant $\kappa(\Gamma, \alpha)$ depends only on the equivalence class of Γ under the action of the conformal group of $\mathbb{C}$. In other words, it is the same for any curve obtained from Γ by rigid motion and dilation.*

(c) *A similar result holds with* Lipα *replaced by* lipα *and $M^{1+\alpha}$ replaced by $M_*^{1+\alpha}$.*

Versions of this theorem have appeared previously. For the case when Γ is a circle, it was essentially proven by Dolzhenko (cf. [Garnett 1972, p.65, Lemma 2.2]). As we shall see, the general case is not far from the circle case.

Proof. 1. It is enough to prove the result for the case when Γ is a simple closed Jordan curve, i.e. bdyΓ is connected. Once this case is proven, the general case follows on cutting up Ω into a finite number of pieces and using the subadditivity of $M^{1+\alpha}$.

2. For the same reason, it is enough to prove the case in which Γ consists of three analytic arcs, making a curvilinear triangle in which none of the vertices is an outward-pointing cusp. We may also assume that S is a subset of $\Omega \cup \Gamma$.

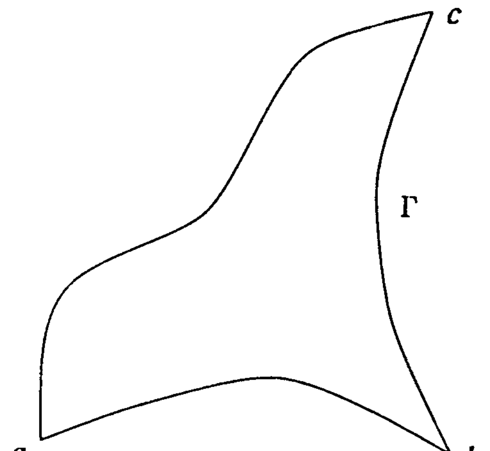

Fig. 2.

3. Using a mixture of rectilinear triangles and curvilinear triangles partly bounded by pieces of Γ, we can cover S by a finite number of patches P_n such that (1) each P_n is bounded by a piecewise-analytic curvilinear triangle,

$$M^{1+\alpha}(S) \geq \text{const} \cdot \sum_n (\text{diam} P_n)^{1+\alpha}, \tag{2}$$

and (3) perimeter(P_n)$\leq$ const $\cdot$ diam(P_n). This depends upon the fact that there are no outward cusps in Γ. The constants depend on the shape of Γ but do not change if Γ is rescaled, translated, or rotated.

4. Then, using Cauchy's Theorem, and denoting by c_n any chosen point of P_n, we get

$$\left|\int_\Gamma f(z)dz\right| \le \sum_n \left|\int_{\mathrm{bdy}P_n} (f(z) - f(c_n))dz\right|$$
$$\le \text{const} \sum_n \operatorname{diam}(P_n) \cdot \operatorname{diam}(P_n)^\alpha$$
$$\le \kappa \cdot M^{1+\alpha}(S).$$

□

2.7. We recall the De Leeuw representation of lipα^* [2]. If $T \in \mathrm{lip}\alpha(B)^*$, then it can be represented as follows. There is a Borel-regular measure ρ on $B \times B$, having no mass on the diagonal, having finite total variation, and such that

$$Tf = \int \frac{(f(z) - f(w)d\rho(z,w)}{|z-w|^\alpha}$$

whenever $f \in \mathrm{lip}\alpha(B)$. The proof is a simple application of the Hahn–Banach theorem and the F. Riesz representation theorem. Using this representation the double-dual action of a function $f \in \mathrm{Lip}\alpha(B)$ on T *is given by the same formula.* (In essence, that is De Leeuw's proof that Lipα is the double dual of lipα.)

2.8. We now prove Lemma 1.1 and Lemma 1.3.

Fix $f \in A(U)$.

Choose C^∞ functions ϕ_n having compact support, such that $\phi_n = 1$ near b, $0 \le \phi_n \le 1$, $\mathrm{spt}\phi_n \subset \mathbb{B}(0, 1/n) = B_n$, and

$$\|\nabla\phi_n\|_{L^\infty} \le 8n.$$

Form $g_n = T_{\phi_n} f$. Then g_n belongs to $A(U)$, and is also analytic on the complement of B_n. Moreover, by Lemma 2.5, $\|g_n\|_{\mathrm{Lip}\alpha} \le$const.

By the Decay Lemma,

$$|g_n(z)| \le \frac{\kappa}{n^{1+\alpha}|z-b|}$$

whenever $|z - b| > 1/n$, and thus by a simple argument

$$|g_n(z)| \le \kappa/n^\alpha, \qquad \forall z \in \mathbb{C}.$$

Since $\|g_n\|_{\mathrm{Lip}\alpha}$ is bounded, we deduce from the De Leeuw representation and the Lebesgue dominated convergence theorem that $g_n \to 0$ weak-star in Lipα. Thus $f - g_n \to f$ weak-star, and $f - g_n \in A(U \cup \{b\})$. This proves Lemma 1.3.

If $f \in a(U)$, we may use the fact that $T_\phi 1 = 0$ to get the estimate

$$\|T_{\phi_n} f\|_{\mathrm{Lip}\alpha} \le \text{const} \cdot \|f - f(b)\|_{\mathrm{Lip}\alpha(B_n)} \to 0,$$

and so conclude that in that case $g_n \to 0$ in Lipα norm. Thus we obtain Lemma 1.1. □

2.9. Now we prove the first part of Theorem 1.2: If the series converges, then there is a k-th order cpd on $a(U)$ at the point b.

We may suppose that $b = 0$.

Let $\gamma_n = M_*^{1+\alpha}(A_n(b) \sim U)$. Suppose that

$$\sum_n 2^{(k+1)n}\gamma_n < +\infty.$$

Let $f \in A(U \cup \{0\})$. Choose N such that f is analytic on $\mathbb{B}(0, 2^{-N})$. Then by the Cauchy integral formula,

$$\begin{aligned} f^{(k)}(0) &= \frac{1}{2\pi i}\int_{|z|=2^{-N}} \frac{f(z)dz}{z^{k+1}} \\ &= \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)dz}{z^{k+1}} - \sum_{n=1}^{N-1} \frac{1}{2\pi i}\int_{\text{bdy}A_n} \frac{f(z)dz}{z^{k+1}} \\ &= \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)dz}{z^{k+1}} - \sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{\text{bdy}A_n} \frac{f(z)dz}{z^{k+1}}, \end{aligned}$$

and hence, using the quantitative Cauchy theorem, and Lemma 2.3, we get

$$\begin{aligned} \left|f^{(k)}(0)\right| &\le \frac{1}{2\pi}\int_{|z|=1} \frac{|f(z)|\cdot|dz|}{|z|^{k+1}} + \sum_{n=1}^{\infty} \frac{1}{2\pi}\left|\int_{\text{bdy}A_n} \frac{f(z)dz}{z^{k+1}}\right| \\ &\le \sup_{\mathbb{B}(0,1)} |f(z)| + \sum_{n=1}^{\infty} \text{const}\cdot 2^{(k+1)n}\gamma_n \|f\|_{\text{Lip}\alpha}. \end{aligned}$$

This shows that there is a k-th order cpd at b on $a(U)$, as required. □

2.10. Next, we prove the other direction of Theorem 1.2.

Let γ_n be as in the last subsection, and suppose that

$$\sum_n 2^{(k+1)n}\gamma_n = +\infty.$$

Choose $\epsilon_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} 2^{(k+1)n}\epsilon_n\gamma_n = +\infty,$$

and $2^{(k+1)n}\epsilon_n\gamma_n \le 1$ for all n.

Applying Frostman's Lemma, we obtain, for each n, a positive measure ν_n on $A_n \sim U$ such that

$$\nu_n \mathbb{B}(z,r) \le \epsilon_n \cdot r^{1+\alpha}, \qquad \forall z \in \mathbb{C}, \forall r > 0,$$

$$\sup_z \frac{\nu_n \mathbb{B}(z,r)}{r^{1+\alpha}} \to 0 \quad \text{as } r \downarrow 0,$$

and

$$\|\nu_n\| = \kappa \epsilon_n \gamma_n.$$

Let

$$f_n(z) = \int \left(\frac{\zeta}{|\zeta|}\right)^{k+1} \frac{d\nu_n(\zeta)}{\zeta - z}.$$

Then $\|f_n\|_{\mathrm{Lip}\alpha} \le \kappa\epsilon_n, f_n \in a(U), f_n$ is analytic off A_n, and

$$f_n^{(k)}(0) = k! \int \frac{d\nu_n(\zeta)}{|\zeta|^{k+1}}$$

lies between fixed constant multiples of $2^{(k+1)n}\epsilon_n\gamma_n$.

For each n, choose $p \ge n$ such that

$$\sum_n^p 2^{(k+1)m} \epsilon_m \gamma_m \in [1,2].$$

Let

$$g_n(z) = \sum_{m=n}^p f_m(z).$$

From the Decay Lemma, we get

$$|f_m(z)| \le \text{const} \cdot \epsilon_m \cdot 2^{-\alpha m},$$
$$|f_m(z)| \le \frac{\text{const} \cdot \epsilon_m \gamma_m}{\text{dist}(z, A_m)},$$
$$|f_m'(z)| \le \frac{\text{const} \cdot \epsilon_m \gamma_m}{\text{dist}(z, A_m)^2}.$$

Fix $z, w \in C$ with $z \ne w$. Fix m with $n \le m \le p$. The following five cases cover all the possibilities:

Case 1: z or w belongs to A_{m-1} or A_m or A_{m+1}.

Case 2: $|z| > 2^{-m+1}$ and $|w| > 2^{-m+1}$, and $|z - w| < 2^{-m}$.

Case 3: $|z| < 2^{-m-2}$ and $|w| < 2^{-m-2}$.

Case 4: $|z| > 2^{-m+1}$ and $|w| > 2^{-m+1}$, and $|z - w| \ge 2^{-m}$.

Case 5: $|z| < 2^{-m-2}$ and $|w| > 2^{-m+1}$.

In Case 1, we have

$$\frac{|f_m(z) - f_m(w)|}{|z-w|^\alpha} \le \|f_m\|_{\mathrm{Lip}\alpha} \le \text{const} \cdot \epsilon_n.$$

In Case 2 or Case 3, we may connect z to w using an arc Γ of length at most $\pi|z-w|$ such that

$$|\zeta - t| \ge 2^{-m-2}, \quad \forall \zeta \in \Gamma \quad \forall t \in A_m.$$

Thus integrating the decay estimate for $f_m'(\zeta)$ gives

$$\frac{|f_m(z) - f_m(w)|}{|z-w|^\alpha} \le \kappa \cdot 2^{2m} \epsilon_m \gamma_m \cdot 2^{(\alpha-1)n}.$$

In Case 4 or Case 5, we get

$$\begin{aligned}\frac{|f_m(z) - f_m(w)|}{|z-w|^\alpha} &\le \frac{|f_m(z)| + |f_m(w)|}{2^{-\alpha(m+2)}} \\ &\le \text{const} \cdot 2^{2m} \epsilon_m \gamma_m \cdot 2^{(\alpha-1)n}.\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{|g_n(z) - g_n(w)|}{|z-w|^\alpha} &\le 6\kappa \cdot \epsilon_n + \kappa \cdot 2^{(\alpha-1)n} \sum_{m=n}^{p} 2^{2m} \epsilon_m \gamma_m \\ &\le \kappa \cdot (\epsilon_n + 2^{(\alpha-1)n}).\end{aligned}$$

Thus the functions g_n have Lipα norms tending to zero, and yet $g_n^{(k)}(0)$ is bounded away from zero, whence there is no cpd at 0.

This completes the proof of Theorem 1.2. □

2.11. There is a weak-star continous point derivation on $A(U)$ at b if and only if there is an estimate

$$|f^{(k)}(0)| \le \text{const} \cdot \|f\|_{\mathrm{Lip}\alpha},$$

valid for $f \in A(U \cup \{b\})$. Thus the argument just given needs practically no change to prove Theorem 1.4. □

3. Examples

3.1. If bdy$U$ is smooth near b, then there is no cpd at b. More generally, if there is a sector

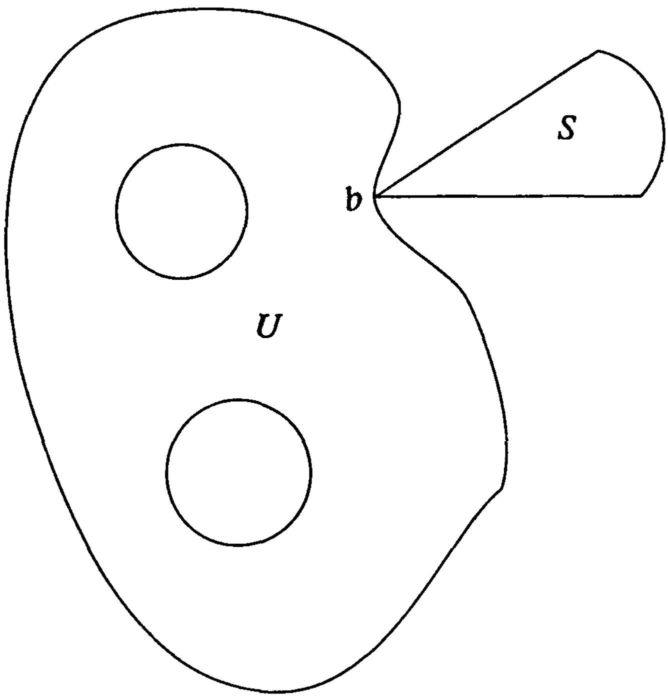

Fig. 3.

$$S = \{z \in \mathbb{C} : 0 \leq |z - b| \leq \delta, |\phi - \arg z| \leq \epsilon\}$$

with $\delta > 0$ and $\epsilon > 0$, that lies outside U, then there is no cpd at b. For, given such a sector, and $n > \log_2 \delta$, the set $A_n(b) \sim U$ will contain a disk of radius $\eta 2^{-n}$ for some $\eta > 0$, independent of n. Then

$$M_*^{1+\alpha}(A_n \sim U) \geq (\eta 2^{-n})^{1+\alpha},$$

so the series

$$\sum_n 4^n M_*^{1+\alpha}(A_n \sim U)$$

diverges.

3.2. Let $\omega : [0, +\infty) \to [0, +\infty)$ be a monotone increasing function, and let U be the set

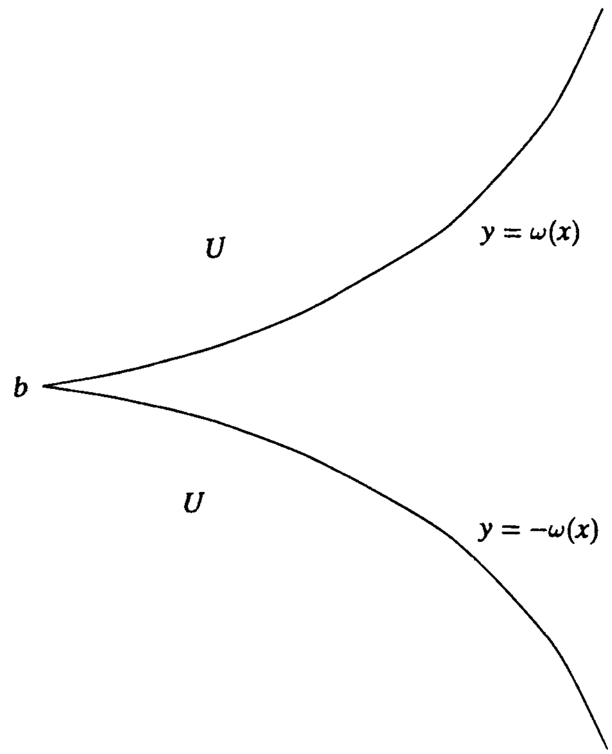

Fig. 4.

$$U = \{z = x + iy : |y| \leq \omega(x) \text{ if } x > 0\}.$$

If $\omega(r)/r \to 0$ as $r \downarrow 0$, then $\mathbb{C} \sim U$ has a cusp at 0. In this situation, there are positive constants κ_1 and κ_2 such that $A_n(0) \sim U$ contains a ball of radius $\kappa_1\omega(2^{-n})$ and is contained in a ball of radius $\kappa_2\omega(2^{-n})$. Thus there is a cpd of order k on $a(U)$ at 0 if and only if

$$\sum_n 2^{(k+1)n}\omega(2^{-n})^{1+\alpha} < +\infty.$$

For instance, if the region is that outside the cubic cusp

$$U = \{x + iy : |y| > x^{3/2}\},$$

then there is a first-order cpd on a(U) as soon as $\alpha > 1/3$, but there is a second-order cpd for no $\alpha < 1$.

For an exponential cusp, there are cpd's of every order for every $\alpha > 0$.

For this kind of set the *same* condition characterises the existence of weak-star continuous cpd's on $A(U)$.

3.3. Let $U = \mathbb{C} \sim \Gamma$, where Γ is the von Koch snowflake curve, and let b be any point of Γ. We have $M^\beta(\Gamma) = 0$ if and only if $\beta > \log_3 4$ and $M^\beta_*(\Gamma) = 0$ if and only if $\beta \geq \log_3 4$. Moreover, because of the self-similarity properties of Γ, there are positive constants κ_1 and κ_2 such that (letting $d = \log_3 4$)

$$\kappa_1 r^d \leq M^d(\mathbb{B}(z,r) \cap \Gamma) \leq \kappa_2 r^d,$$

whenever $z \in \Gamma$ and $0 < r < \operatorname{diam}\Gamma$. Using these facts and the theorems proved above, we readily see that the following are equivalent:
(1) $A(U)$ admits a weak-star continuous first-order cpd at b;
(2) Γ is $\bar\partial$-Lipα-null;
(3) $\alpha > \log_3 4$.

Also, the following are equivalent:
(1′) $a(U)$ admits a continuous first-order cpd at b;
(2′) Γ is $\bar\partial$-lipα-null;
(3′) $\alpha \geq \log_3 4$.

3.4. Let U be a *road-runner set*

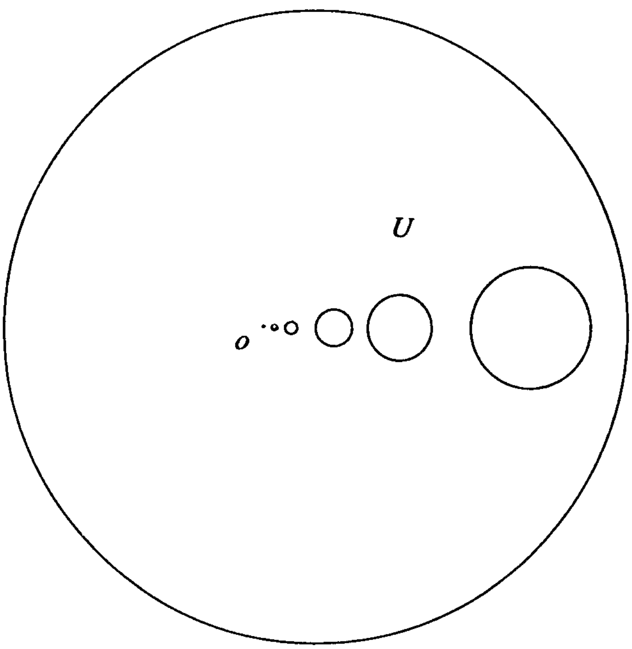

Fig. 5.

$$U = \mathbb{U}(0,1) \sim \bigcup_{n=1}^{\infty} \mathbb{B}(a_n, r_n),$$

where $a_n > 0$, $r_n > 0$, and $0 < a_{n+1} + r_{n+1} < a_n - r_n < 1$, for each n. Then application of the above theorems shows that $a(U)$ admits a k-th order cpd at 0 if and only if $A(U)$ admits a k-th order weak-star cpd, and that both happen if and only if

$$\sum_{n=1}^{\infty} \frac{r_n}{a_n^{k+1}} < +\infty.$$

ACKNOWLEDGEMENT

The second author is grateful for the hospitality of CIMAC, La Laguna, Tenerife, at which some of this work was done.

REFERENCES

[1] L. Carleson, *Selected Problems on Exceptional Sets*, Van Nostrand, New York, 1969.

[2] K. De Leeuw, *Banach spaces of Lipschitz functions*, Studia Math. **21** (1962), 55–66.

[3] E. P. Dolzhenko, *On removal of singularities of analytic functions*, Uspehi Mat. Nauk **18**, no.4 (112) (1963), 135–142; English transl., Amer. Math. Soc. Transl. (2) **97** (1971), 33–41.

[4] H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1969.

[5] T. W. Gamelin and J. Garnett, *Distinguished homomorphisms and fibre algebras*, Amer. J. Math. **92** (1970), 455–474.

[6] A. Hallstrom, *On bounded point derivations and analytic capacity*, J. Functional Analysis **4** (1969), 153–165.

[7] M. S. Melnikov, *Estimates of the Cauchy integral along an analytic curve*, Mat. Sb. **71** (113) (1966), 503–514; English transl., Amer. Math. Soc. Transl. (2) **80** (1969), 243–255.

[8] A. G. O'Farrell, *Density of parts of algebras on the plane*, Trans. Amer. Math. Soc. **196** (1974), 403–414.

[9] A. G. O'Farrell, *Analytic capacity, Hölder conditions, and τ-spikes*, Trans. Amer. Math. Soc. **196** (1974), 415–424.

[10] A. G. O'Farrell, *Equiconvergence of derivations*, Pacific J. Math. **53** (1974), 539–554.

[11] A. G. O'Farrell, *Hausdorff content and rational approximation in fractional Lipschitz norms*, Trans. Amer. Math. Soc. **228** (1977), 187–206.

[12] A. G. O'Farrell, *Analytic capacity and equicontinuity*, Bull. Lond. Math. Soc. **10** (1978), 276–279.

[13] A. G. O'Farrell, *Estimates for capacities, and approximation in Lipschitz norms*, J. Reine Angew. Math. **311/2** (1979), 101–115.

[14] A. G. O'Farrell, *T-invariance*, Proceedings R.I.A. **92A**(1992), 185–203.

[15] J. Wang, *Modulus of approximate continuity for $R(X)$*, Math. Scand. **34** (1974), 219–225.

[16] N. Wiener, *The Dirichlet problem*, M.I.T. J. Math. Phys. **3** (1924), 127–146.

MAYNOOTH COLLEGE
CO. KILDARE
IRELAND

(Received December 22, 1991 and in revised form July 15, 1992)