ON REFLEXIVE SPACES HAVING THE METRIC APPROXIMATION PROPERTY

BY

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ABSTRACT

It is proved that some conjectures concerning non-separable reflexive Banach spaces are true for reflexive spaces having the metric approximation property.

Let Γ be an abstract set and let $c_0(\Gamma)$ be the Banach space consisting of the real (or complex) valued functions on Γ which vanish at infinity, i.e., those functions f: $\Gamma \to$ scalars such that $\{\gamma; \gamma \in \Gamma, |f(\gamma)| < \varepsilon\}$ is finite for every $\varepsilon > 0$. In the paper [2] H. H. Corson and the author study the properties of w compact subsets of $c_0(\Gamma)$. It is conjectured there that every w compact subset of a Banach space is homeomorphic to a w compact subset of $c_0(\Gamma)$ for a suitable Γ, and thus that the results proved in [2] (and also in [1] for w compact subsets of $c_0(\Gamma)$ are valid for an arbitrary w compact subset of a Banach space.

The purpose of this note is to prove a result which gives a further reason to believe that the above mentioned conjecture is true. Our result here connects this conjecture with a conjecture of Grothendieck.(1)

In [6] Grothendieck introduced and studied the notion of the metric approximation property (M.A.P.). A Banach space X is said to have the M.A.P. if for every finite subset $\{x_i\}_{i=1}^n$ of X and every $\varepsilon > 0$ there is an operator T of norm 1 from X into a finite dimensional subspace of X such that $\| Tx_i - x_i \| \leqq \varepsilon$ for every i. Grothendieck showed that the common Banach spaces have the M.A.P. and raised the question whether every Banach spacee has the M.A.P. He conjectures (cf. problem 1 at the end of the Memoir [6]) that there are Banach spaces which do not have M.A.P. but that every reflexive Banach space has the M.A.P. He showed [6, pp. 178-185] that for reflexive spaces the M.A.P. is equivalent to the formally weaker topological approximation property and thus, for example, every separable reflexive space with a basis has the M.A.P. As far as we know no progress has been made toward the solution of these problems of Grothendieck.

The main result of the present note is the

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(1) See the note added in proof at the end of the paper.

THEOREM. *Let X be a reflexive Banach space having the* M.A.P. *Then there exists a set Γ and a bounded linear operator $T: X \to c_0(\Gamma)$ such that $Tx = 0$ (if and) only if $x = 0$.*

It follows immediately from this theorem that every w compact subset of a reflexive Banach space having the M.A.P. is affinely homeomorphic to a w compact subset of $c_0(\Gamma)$ for a suitable Γ. We state here some further results which follow from the theorem.

COROLLARY 1. *Let X be a reflexive Banach space having the* M.A.P. *Then X has an equivalent norm which is strictly convex.*

A norm $\| \cdot \|$ in a Banach space is called strictly convex if $\| x + y \| = \| x \| + \| y \|$ implies that $\lambda x = \mu y$ for some $\lambda, \mu \geqq 0$. For the known results concerning the existence of strictly convex norms in Banach spaces we refer to [3] and [4]. It is stated in [4] that Klee has conjectured that every reflexive Banach space has an equivalent strictly convex norm. Thus Corollary 1 gives a partial answer to the conjecture of Klee. Corollary 1 follows from the theorem by using the result of Day [4] that $c_0(\Gamma)$ admits an equivalent strictly convex norm for every Γ. By passing to the dual we get

COROLLARY 2. *Let X be a reflexive Banach space having the* M.A.P. *Then X has an equivalent norm which is smooth.*

A norm $\| \cdot \|$ is called smooth if for every x with $\| x \| = 1$ there is a unique $f \in X^*$ with $\| f \| = f(x) = 1$. For reflexive spaces it is known [cf. 4] that X is strictly convex if and only if X^* is smooth. Since X has the M.A.P. if and only if X^* has the M.A.P. (cf. [6] and also Lemma 1 below which is a somewhat stronger assertion) Corollary 2 is a consequence of Corollary 1.

COROLLARY 3. *Let X be a reflexive Banach space having the* M.A.P. *Then every closed convex and bounded subset of X is the closed convex hull of its exposed points.*

A point x of a subset K of X is called an exposed point of K if there is an $f \in X^*$ such that $f(x) > f(y)$ for every $y \neq x$ in K. Corollary 3 is an immediate consequence of the theorem and the results of [2] and [7]. In this connection cf. also [9].

COROLLARY 4. *Let X be a reflexive Banach space having the* M.A.P. *Then every w compact subset of X has a dense subset consisting of G_δ points (in the w topology).*

A point x of a set $K \subset X$ is called a G_δ point in the w topology if there is a sequence $\{O_n\}_{n=1}^{\infty}$ of w open subsets of X such that $\{x\} = \bigcap_{n=1}^{\infty} O_n \cap K$. Again, Corollary 4 follows immediately from the theorem and the results of [2].

COROLLARY 5. *Let X be a reflexive Banach space having the* M.A.P. *Then the norm is Gateaux differentiable at the norm dense subset of $\{x; \| x \| = 1\}$.*

The norm is said to be Gateaux differentiable at a point x if for every $y \in X$,

$\lim_{h\to 0}(\| x+hy \| - \| x \|)/h$ exists. Corollary 5 is a consequence of Corollary 2 and the results of [7].

Further consequences of the theorem proved here can be obtained by combining it with the results of [1]. We omit the details.

The proof of the theorem is based on some lemmas.

LEMMA 1. *Let X be a reflexive Banach space having the* M.A.P. *Let $\{x_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^m$ be finite sets in X and X^* respectively. Then for every $\varepsilon > 0$ there exists an operator T from X into itself having a finite dimensional range such that*

$$\| T \| \leqq 1, \ \| Tx_i - x_i \| \leqq \varepsilon \ \ i = 1, \ 2, \cdots, \ n, \ \textit{and} \ \| T^*f_j - f_j \| \leqq \varepsilon \ j = 1, \cdots, m.$$

Proof. The assumption that X has the M.A.P. can be expressed also by saying that there is a net $\{T_\alpha\}_{\alpha\in A}$ of operators with norm $\leqq 1$, each having a finite dimensional range, from X into itself such that $\{T_\alpha\}$ converges in the strong operator topology to the identity operator (i.e. $\lim_\alpha T_\alpha\, x = x$ for every $x \in X$). Since for every $f \in X^*$ and $x \in X$ $\lim_\alpha f(T_\alpha x) = f(x)$ it follows that the net $\{T_\alpha^*\}$ converges in the weak operator topology to the identity operator of X^*. Hence (see e.g. [5, page 477]), there is a net $\{T_\beta^*\}_{\beta\in B}$ of operators in X^* converging to the identity operator in the strong operator topology such that

(i) Every T_β^* is a convex combination of a finite number of members of the net $\{T_\alpha^*\}$.

(ii) For every $\alpha_0 \in A$ there is a $\beta_0 \in B$ such that if $\beta \succ \beta_0$ then in the representation of T_β^* given by (i) appear only T_α^* with $\alpha > \alpha_0$.

It follows that the nets $\{T_\beta\}_{\beta\in B}$ and $\{T_\beta^*\}_{\beta\in B}$ converge in the strong operator topology to the respective identity operators and this proves the lemma.

Before we state the next lemma we would like to recall the definition of density character. The density character of a Banach space X is the smallest cardinal number $\mathfrak{m}$ for which there is a dense subset of X whose cardinality is $\mathfrak{m}$.

LEMMA 2. *Let X be a reflexive space having the* M.A.P. *and let $\mathfrak{m}$ be a cardinal number. Let Y and Z be subspaces of X and X^* respectively having density character $\leqq \mathfrak{m}$. Then there is a subspace W of X and a projection P from X onto W such that*

(i) $W \supset Y$.

(ii) *The density character of* $W \leqq \mathfrak{m}$.

(iii) $\| P \| = 1$.

(iv) $P^*f = f$ *for every* $f \in Z$.

Proof. Assume first that $\mathfrak{m} = \aleph_0$, i.e. that Y and Z are separable. Let $\{y_i\}_{i=1}^\infty$ be a dense set in Y and $\{f_i\}_{i=1}^\infty$ be dense in Z. By Lemma 1 we can construct inductively a sequence $\{T_n\}_{n=1}^\infty$ of operators from X into itself such that

1. Every T_n has a finite-dimensional range.
2. $\| T_n \| \leqq 1$, for every n.
3. $\| T_n y_i - y_i \| \leqq 1/n$, $i = 1,2,\cdots, n$.
4. $\| T_n^* f_i - f_i \| \leqq 1/n$, $i = 1,2,\cdots, n$.
5. $\| T_n x - x \| \leqq 1/n$ if $\| x \| \leqq 1$ and $x \in \bigcup_{k=1}^{n-1} T_k X$.

By a well known and simple consequence of Tychonoff's theorem the unit cell of the space of all operators from X into itself is compact in the weak operators topology. Hence the sequence $\{T_n\}_{n=1}^{\infty}$ has a limiting point P. It is easily seen that P is a projection on $W = \overline{\bigcup_{n=1}^{\infty} T_n X}$ and that (i) – (v) are satisfied. This proves the lemma for $\mathfrak{m} = \aleph_0$.

We continue by transfinite induction. Assume that the lemma holds for all cardinals $< \mathfrak{m}$. Let Ω be the well ordered set of all ordinal numbers whose cardinality is $< \mathfrak{m}$. Then we can find subspaces $\{Y_\alpha\}_{\alpha \in \Omega}$ of Y and $\{Z_\alpha\}_{\alpha \in \Omega}$ of Z such that $Y_\alpha \subset Y_\beta$, $Z_\alpha \subset Z_\beta$ for $\alpha < \beta$, $Y = \overline{\bigcup_{\alpha \in \Omega} Y_\alpha}$, $Z = \overline{\bigcup_{\alpha \in \Omega} Z_\alpha}$ and such that density character of every Y_α and Z_α is at most the cardinality of α for infinite α. By the induction hypothesis we can construct inductively for every $\alpha \in \Omega$ a projection P_α of norm $\leqq 1$ from X on a subspace W_α containing $Y_\alpha \cup \bigcup_{\beta < \alpha} W_\beta$ such that the restriction of P_α^* to Z_α is the identity and such that the density character of W_α is at most cardinality of α for infinite α. Let P be the limit (in the w operator topology) of a converging subnet of $\{P_\alpha\}_{\alpha \in \Omega}$. Then P is a projection from X onto $W = \overline{\bigcup_{\alpha \in \Omega} W_\alpha}$ which has the required properties.

The next lemma is well known and goes back to Lorch [8]. We include a proof for the sake of completeness.

Lemma 3. *Let X be a reflexive space, let α_0 be an ordinal number and let Ω be the well ordered set of all the ordinals $\leqq \alpha_0$. Assume that for every $\alpha \in \Omega$ there is a projection operator P_α from X into itself such that $\| P_\alpha \| \leqq 1$, and $P_\alpha P_\beta = P_{\min(\alpha,\beta)}$ for every α and β. Then for every $x \in X$ and every $\varepsilon > 0$ the set $\{\alpha; \| P_{\alpha+1} x - P_\alpha x \| \geqq \varepsilon\}$ is finite.*

Proof. Assume that there is a sequence of ordinals $\{\alpha_i\}_{i=1}^{\infty}$ such that $\alpha_{i+1} > \alpha_i + 1$ and $\| P_{\alpha_i+1} x_0 - P_{\alpha_i} x_0 \| \geqq \varepsilon$ for some $\varepsilon > 0$ and some $x_0 \in X$. Put $P_{2i-1} = P_{\alpha_i}$, and $P_{2i} = P_{\alpha_i+1}$, $i = 1,2,\cdots,$. Clearly $P_i P_j = P_{\min(i,j)}$ for every i and j. Let P_∞ be a limiting point of the sequence $\{P_i\}_{i=1}^{\infty}$ in the weak operator topology. P_∞ is a projection from X onto $Y = \overline{\bigcup_{i=1}^{\infty} P_i X}$, and $P_i P_\infty = P_i$ for every i. Hence, since $\| P_j x - x \| = 0$ if $x \in P_i X$ and $j > i$, we get that $\lim_{j\to\infty} \| P_j x - x \| = 0$ for every $x \in Y$. Hence

$$\lim_{j\to\infty} \| P_j x_0 - P_\infty x_0 \| = \lim_{j\to\infty} \| P_j P_\infty x_0 - P_\infty x_0 \| = 0$$

and this contradicts the assumption that $\| P_{2i} x_0 - P_{2i-1} x_0 \| \geqq \varepsilon$ for every i.

Proof of the Theorem. The proof is by transfinite induction. The theorem is trivial if X is separable. Let $\mathfrak{m}$ be a cardinal number and assume that the theorem holds for all reflexive Banach spaces having the M.A.P. whose density is $< \mathfrak{m}$. Let X be a reflexive space having the M.A.P. whose density character is $\mathfrak{m}$. Let Y be a subspace of X whose density character is $\mathfrak{n} < \mathfrak{m}$, and assume that there is a projection P of norm 1 from X onto Y. Let $y_0 \in X \sim Y$. By Lemma 2 there is a subspace Y_1 of X containing $Y \cup \{y_0\}$ and a projection P_1 of norm 1 from X onto Y_1 such that the density character of Y_1 is $\mathfrak{n}$ and $P_1^* f = f$ for every $f \in P^* X^*$. It follows that $P_1^* P^* = P^*$ and thus $P_1 P = PP_1 = P$.

Let Ω be the well ordered set of all ordinal numbers whose cardinality is smaller than $\mathfrak{m}$. By the remark we just made we can construct inductively for every $\alpha \in \Omega$ a subspace Y_α of X and a projection P_α from X onto Y_α such that

1. $\| P_\alpha \| = 1$ for every α.
2. $P_\alpha P_\beta = P_{\min(\alpha,\beta)}$ for every α and β.
3. For every limiting ordinal α, P_α is the limit in the w operator topology of a subnet of the net $\{P_\beta\}_{\beta<\alpha}$.
4. $X = \overline{\bigcup_{\alpha\in\Omega} Y_\alpha}$.
5. The density character of Y is at most the cardinality of α for infinite α.

Since there is a projection from X onto Y_α it follows that the Y_α have the M.A.P. By the induction hypothesis there is for every $\alpha \in \Omega$ a set Γ_α and a one to one linear operator T_α from Y_α into $c_0(\Gamma_\alpha)$ with $\| T_\alpha \| \leqq 1$. Let $\Gamma = \bigcup_{\alpha\in\Omega} \Gamma_{\alpha+1}$ and define the operator T from X into $c_0(\Gamma)$ by putting

$$T x(\gamma) = T_{\alpha+1}(P_{\alpha+1} x - P_\alpha x)(\gamma) \text{ if } \gamma \in \Gamma_{\alpha+1} .$$

That $Tx \in c_0(\Gamma)$ follows from Lemma 3. That T is one to one follows from requirements 3 and 4 above (we assume here, as we clearly can, that $P_1 = P_2$). This concludes the proof of the theorem.

The present note is devoted to questions concerning non-separable reflexive spaces. For separable spaces the theorem and its corollaries are well known and hold also if we do not assume that the spaces have the M.A.P. One of the consequences of the results proved here is that the question of the validity of the theorem and its corollaries for every reflexive space can be reduced to a question concerning separable spaces: "Does every separable reflexive space have the M.A.P."? Indeed, we have the following simple

PROPOSITION. *Let X be a reflexive Banach space and assume that every separable subspace of X has the M.A.P. Then X has the M.A.P.*

Proof. Let $\{x_i\}_{i=1}^n$ be a finite subset of X and let $\varepsilon > 0$. It follows easily from the asssumptions that there is an integer k, a set $\{Y_\alpha\}$ directed by inclusion of separable subspace of X, and operators $T_\alpha : Y_\alpha \to Y_\alpha$ such that:

(i) $X = \cup_\alpha Y_\alpha$.

(ii) $Y_\alpha \supset \{x_i\}_{i=1}$.

(iii) $\| T_\alpha x_i - x_i \| \leqq \varepsilon, \quad i = 1, 2, \cdots, n.$

(iv) $\dim (T_\alpha Y_\alpha) = k$, $\| T_\alpha \| \leqq 1$, for every α.

By a simple compactness argument we deduce that there is an operator $T : X \to X$ such that $\| T \| \leqq 1$, $\dim (TX) \leqq k$, and $\| Tx_i - x_i \| \leqq \varepsilon$ for $1 \leqq i \leqq n$. Hence X has the M.A.P.

Added in proof. The theorem of the present note can be proved without assuming the M.A.P.Thus the theorem and its corollaries hold for ever reflexive Banach space. The proof of this fact is based on the proof presented in this paper and is given in the note "On non-separable reflexive spaces", which will appear in the Bulletin of the American Mahtematical Society.

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