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NON-ERGODIC INTERVAL EXCHANGE TRANSFORMATIONS

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ABSTRACT

We construct interval exchange transformations on four intervals satisfying a strong irrationality condition and having exactly two ergodic invariant probability measures. This shows that although Kronecker's theorem remains true for interval exchange transformations, the Weyl equidistribution theorem is false even under the strongest irrationality assumptions.

Given a probability vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ and a permutation τ of the integers $1, \cdots, n$, a transformation T of the unit interval is obtained by cutting the interval up into n pieces of lengths α and then interchanging the pieces using the permutation τ. This transformation T is called the (α, τ) interval exchange transformation. Interval exchange transformations were introduced in [1] and the following results were shown:

1) The transformation T is minimal (i.e. for every x, the orbit of x is dense in the unit interval) if and only if no finite union of intervals is T-invariant (except of course the empty union and the whole interval).

2) If the orbits of the discontinuity points of T are infinite and distinct, then T is minimal.

3) If τ does not map any segment $\{1, 2, \cdots, k\}$ to itself (except for $k = n$) and if $\alpha_1, \cdots, \alpha_{n-1}$ are rationally independent, then the condition in (2) is satisfied and T is minimal.

If we take $n = 2$ and $\tau(1) = 2$, $\tau(2) = 1$, then T is easily recognized as the map $Tx = x + \alpha_2 \bmod 1$, i.e. T is a "rotation on the circle". Hence the above results generalize the well-known Kronecker theorem on irrational rotations of the circle. Another interesting result in the case of rotations is the Weyl equidistribution theorem, which says that any orbit of an irrational rotation on the circle is

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uniformly distributed. Its proof can be found in any book on elementary number theory. It was conjectured in [1] that the generalization of the Weyl theorem to interval exchange transformations is valid. The question was, more precisely: If an interval exchange transformation is minimal, are then all orbits uniformly distributed in the unit interval? The answer is positive in the cases $n = 2$ (Weyl theorem) and $n = 3$, the latter since it can be reduced to $n = 2$ by looking at the induced transformation on a suitable subinterval.

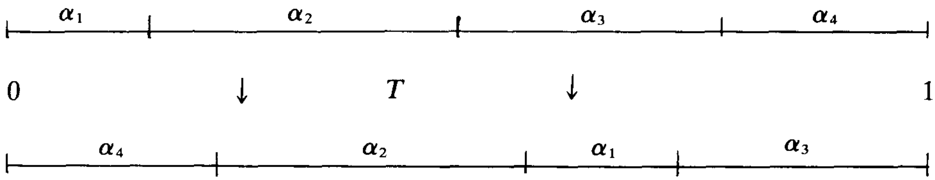
More recently, H. B. Keynes and D. Newton [2] found a counter-example to the conjecture. In [2] they construct an interval exchange transformation with $n = 5$ which has precisely two ergodic invariant probability measures and which satisfies the condition in (2). Their idea consists of constructing an interval exchange on three intervals satisfying (2) but having an eigenvalue -1. Its square is then an interval exchange on five intervals which still satisfies (2) but which is not ergodic. The two ergodic measures thus obtained are both absolutely continuous with respect to Lebesgue measure. (Note that Lebesgue measure is always invariant for an interval exchange transformation and must be a convex combination of the ergodic measures. It was also shown in [1] that there are at most n ergodic measures for an interval exchange transformation on n intervals.) This class of examples does not satisfy the irrationality condition (3), since it is a square of an interval exchange on three intervals.

In the following, the study of counterexamples is continued, using methods basically different from those of [2]. We construct for $n = 4$ interval exchange transformations satisfying the irrationality condition (3) and having exactly two ergodic invariant probability measures. Moreover, either both ergodic measures are absolutely continuous with respect to Lebesgue measure, or one is Lebesgue measure and the other is singular, and we show that both possibilities can occur. The first is in some sense more frequent than the second.

These results close the gaps concerning the counterexamples. A question remaining to be answered is the following: Do almost all interval exchange transformations have the equidistribution property?

1. Induced transformations for an interval exchange transformation

Define τ on $\{1, 2, 3, 4\}$ by setting $\tau(1) = 3$, $\tau(2) = 2$, $\tau(3) = 4$, $\tau(4) = 1$. Let m and n be fixed integers greater than one. We imagine an interval exchange transformation as shown in the following diagram:



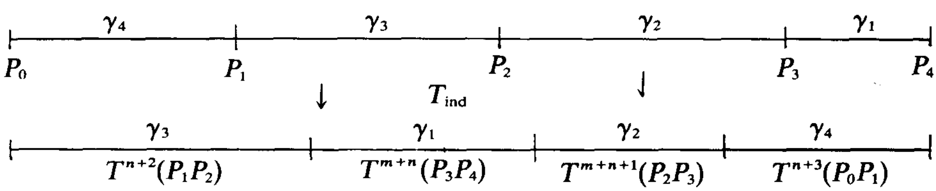
We require that the lengths α_1, α_2, α_3, α_4 be chosen so that the following conditions are satisfied:

1) The interval α_4 is larger than the interval α_1, so that a part of α_4 (the left part) is mapped onto α_1 by T, and the right part is mapped onto the beginning of α_2. Denote by P_0 and P_4 the left and right end points of α_4, and choose P_2 in the interval α_4 such that P_0P_2 is mapped to α_1 and P_2P_4 to the beginning of α_2.

2) Consider the interval P_2P_4. It is mapped onto the beginning of α_2, and then, under successive applications of T, is mapped into consecutive intervals of length $\alpha_4 - \alpha_1$ contained in α_2 until at some point a part (right part) is mapped into α_3, while the left part remains in α_2. Let P_3 denote the point of P_2P_4 corresponding to this division. We require that the intervals $T(P_2P_4)$, $T^2(P_2P_4), \cdots, T^{m-1}(P_2P_4)$ be the successive intervals entirely in α_2, whereas $T^m(P_2P_3)$ is at the right end of α_2 and $T^m(P_3P_4)$ is at the left end of α_3. Here m is the integer fixed at the beginning.

3) Notice now that under the conditions (1) and (2), a beginning (left) part of the interval α_3 is formed by the successive intervals $T^m(P_3P_4)$, $T^{m+1}(P_2P_3)$ and $T^2(P_0P_2)$. Call this union of intervals I. I is an interval of length α_4, which is mapped under successive applications of T into consecutive intervals of length α_4 contained in α_3, until at some point a right part is mapped into α_4 while the remaining left part stays in α_3. If we denote by P_1 the point of α_4 corresponding to this division, then we require that P_1 lies between P_0 and P_2, and that the intervals $I, TI, \cdots, T^{n-1}I$ be the consecutive intervals lying entirely in α_3, whereas T^nI is cut up as above by the division point between α_3 and α_4. Here n is the integer fixed originally.

Under conditions (1), (2) and (3) we may recognize the form of the induced transformation on the interval α_4. Let γ_4, γ_3, γ_2, γ_1 denote the lengths of the subinterval P_0P_1, P_1P_2, P_2P_3, P_3P_4 of α_4 respectively (the reason for the order inversion will become clear presently). Then the induced transformation has the following picture:



If the reader now turns this journal upside down, he will recognize from the diagram that the induced transformation, when conjugated with this isomorphism, has the same permutation τ as the interval exchange we started out with (this is the reason for the order inversion, and will permit us to iterate the procedure we are now describing).

Assuming that it is possible to find numbers with the above properties, we now calculate the α_i's from the γ_i's. We have

$$(*)\qquad \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ m-1 & m & 0 & 0 \\ n & n & n-1 & n \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$$

since the interval P_3P_4, of length γ_1, spends time $m-1$ in α_2 and n in α_3 before returning to α_4, etc. Denote now the matrix in (*) by $A_{m,n}$, and the open positive cone in R^4 by P. Then $A_{m,n}$ maps P into P, and it is easy to see that $\det(A_{m,n}) = 1$. Thus for any $\alpha = (\alpha_1, \cdots, \alpha_4) \in A_{m,n}(P)$ there is a unique solution $\gamma = (\gamma_1, \cdots, \gamma_4)$ which belongs to P. It is then trivial to check that conditions (1), (2) and (3) are satisfied for α with the given values of m and n.

Suppose now that we are given an infinite sequence $(m_k, n_k)_{k=1}^{\infty}$ of pairs of integers each greater than one. Define

$$\Omega_k = A_{m_1,n_1} A_{m_2,n_2} \cdots A_{m_k,n_k}(P).$$

Then $A_{m_k,n_k}(P) \subseteq P$ implies that $\Omega_{k-1} \subseteq \Omega_k$ and a compactness argument (note that any product of two successive A's has all positive entries) shows that $\Omega = \bigcap_{k=1}^{\infty} \Omega_k$ is non-empty. In fact, $\Omega = \bigcap_{k=1}^{\infty} A_{m_1,n_1} \cdots A_{m_k,n_k}(\bar{P})$ which is an infinite intersection of decreasing compact sets in projective space. Let $\alpha^{(1)} = \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Omega$ be any element of this set with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, and let $T = T_1$ denote the interval exchange transformation based on $\alpha^{(1)}$ and τ. For any $m, n \geqq 2$ denote by $\tilde{A}_{m,n}$ the mapping on $\bar{\tilde{P}}$, with

$$\tilde{P} = \left\{ x = (x_1, x_2, x_3, x_4)\colon x_i > 0,\ 1 \leqq i \leqq 4,\ \sum_{i=1}^{4} x_i = 1 \right\},$$

defined by

$$\tilde{A}_{m,n}(x) = \frac{1}{|A_{m,n}(x)|} \cdot A_{m,n}(x),$$

where $|y| = \Sigma_{i=1}^{4} y_i$. By the construction of Ω, for each $k > 1$ the point

$$(**) \qquad \alpha^{(k)} = \tilde{A}^{-1}_{m_{k-1},n_{k-1}} \cdots \tilde{A}^{-1}_{m_1,n_1}\alpha^{(1)}$$

belongs to P, and defines an interval exchange transformation T_k based on $\alpha^{(k)}$ and τ. We thus obtain the following theorem.

THEOREM 1. *Let $(m_k, n_k)_{k=1}^{\infty}$ be an infinite sequence of pairs of integers each greater than one. Then there exists an $\alpha \in \tilde{P}$ and a decreasing sequence $I_2, I_3, \cdots$ of subintervals of the unit interval such that if T is the (α, τ) interval exchange, then the transformation $T_k = T_{I_k}$ induced on I_k by T is an $(\alpha^{(k)}, \tau)$-interval exchange (with order reversed for even k), where $\alpha^{(k)}$ is given by the formula (**) for each $k \geqq 2$.*

2. Properties of products of the mappings $\tilde{A}_{m_k,n_k}$

We set $\tilde{A}_k = \tilde{A}_{m_k,n_k}$ for $k \geqq 1$. In order to prove non-ergodicity of the transformations defined in §1 we shall need some estimates on the products $\tilde{B}_k = \tilde{A}_1\tilde{A}_2\cdots\tilde{A}_k$.

LEMMA 2. *For any $x \in \bar{\tilde{P}}$, $(\tilde{A}_k x)_i \leqq 1/(n_k+1)$ for $i = 1$ and $i = 4$.*

PROOF. If $i = 1$ or $i = 4$, then $(A_k x)_i \leqq 1$ and

$$|A_k x| = n_k + 1 + m_k x_2 + (m_k - 1)x_1 + x_4 \geqq n_k + 1.$$

LEMMA 3. *Suppose that for all $k \geqq 1$ we have $m_k/(n_{k+1}+1) \leqq 1/2$. Let $e_3 = (0,0,1,0)$. Then for each $k \geqq 1$ we have*

$$(\tilde{B}_k e_3)_3 \geqq 1 - \frac{3}{n_1+1}.$$

PROOF. Fix $k \geqq 1$ and define

$$x^{(k)} = \tilde{A}_k e_3, \qquad x^{(k-1)} = \tilde{A}_{k-1}x^{(k)}, \cdots, x^{(j)} = \tilde{A}_j x^{(j+1)}, \cdots, x^{(1)} = \tilde{A}_1 x^{(2)}.$$

Then $x^{(1)} = \tilde{B}_k e_3$ and we prove the inequality by induction on j. It suffices to show for each j that $x_2^{(j)} \leqq 1/(n_j+1)$, in view of Lemma 1. This is certainly true for $x^{(k+1)} = e_3$. Suppose it is true for $j+1$: $x_2^{(j+1)} \leqq 1/(n_{j+1}+1)$. Then

$$x_2^{(j)} = \frac{(m_j-1)x_1^{(j+1)} + m_j x_2^{(j+1)}}{n_j + 1 + m_j x_2^{(j+1)} + (m_j-1)x_1^{(j+1)} + x_4^{(j+1)}} \leqq \frac{2m_j}{(n_{j+1}+1)(n_j+1)} \leqq \frac{1}{n_j+1},$$

as required.

LEMMA 4. *Suppose that for all $k \geqq 1$ we have $m_k/(n_k+1) \geqq 3$. Then for each k and for $e_2 = (0,1,0,0)$ we have*

$$(\tilde{B}_k e_2)_2 \geqq \frac{1}{3}.$$

PROOF. We follow the same line as in the proof of Lemma 3, with the $x^{(k)}$ defined analogously. The induction hypothesis is $x_2^{(j+1)} \geqq 1/3$. It follows that

$$x_2^{(j)} \geqq \frac{m_j x_2^{(j+1)}}{2m_j x_2^{(j+1)} + n_j + 1} \geqq \frac{1}{2 + \frac{1}{3 \cdot \frac{1}{3}}} = \frac{1}{3}.$$

3. Ergodicity and irrationality

It is now possible to prove the following result.

THEOREM 5. *Let $(m_k, n_k)_{k=1}^{\infty}$ be a sequence satisfying the conditions*
i) $3(n_k + 1) \leqq m_k \leqq 1/2(n_{k+1} + 1)$ *for all* $k \geqq 1$, *and*
ii) $n_1 \geqq 9$.
Then for any $\alpha \in \Omega \cap \tilde{P}$, the interval exchange transformation described in Theorem 1 is not uniquely ergodic.

PROOF. Let $\{I_k\}_{k=2}^{\infty}$ be the sequence of intervals described in Theorem 1. For each k, let $I_k^{(j)}$ $(1 \leqq j \leqq 4)$ denote the partition of I_k into the intervals defining the transformation T_k. Then for $j = 2$ or 3, the vector $\tilde{B}_k e_j$ is easily seen to be the relative frequency of visits of any point $x \in I_k^{(j)}$ to the original partition into intervals of lengths $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ up to the first return time of x to I_k. Condition (1) insures the applicability of Lemmas 3 and 4. Thus for $x \in I_k^{(3)}$ the relative frequency of visits to the interval α_3 is at least $1 - 3/(9+1) = 7/10$, and for $x \in I_k^{(2)}$the relative frequency of visits to the interval α_2 is at least 1/3. These estimates hold for all $k \geqq 1$. By choosing a sequence $x_k \in I_k^{(3)}$ as $k \to \infty$ and using a well-known technique, we may obtain an invariant probability measure λ_3 on the unit interval with $\lambda_3(\alpha_3) \geqq 7/10$, and similarly obtain another measure λ_2 with $\lambda_2(\alpha_2) \geqq 1/3$. Since $7/10 + 1/3 > 1$, we have $\lambda_2 \neq \lambda_3$ and T is not uniquely ergodic.

We now conclude our arguments by showing that there exist vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ which are rationally independent in the sense of [1] and whose sequences (m_k, n_k) satisfy the conditions of Theorem 5.

THEOREM 6. *There exists an $\alpha \in \tilde{P}$ such that the $(\alpha\text{-}\tau)$ interval exchange is not uniquely ergodic and such that if $0 = \Sigma_{i=1}^4 k_i \alpha_i$, then $0 = k_1 = \cdots k_4$.*

PROOF. We show that there exists a sequence (m_k, n_k) satisfying the conditions of Theorem 5 such that all $\alpha \in \Omega$ are independent in the above sense. Each 4-tuple of integers (k_1, k_2, k_3, k_4) defines a hyperplane $\{x : \Sigma_{i=1}^4 k_i x_i = 0\}$, if k_i are not all equal to zero. Let $H_1, H_2, \cdots$ be an enumeration of all of these hyperplanes in R^4. The number α must then lie outside of all of these hyperplanes. Suppose that for any hyperplane H_k and any integer N we could find a finite sequence $A_{m'_1,n'_1}, \cdots, A_{m'_j,n'_j}$ of matrices such that the (m'_i, n'_i) satisfy the conditions of Theorem 5 with $n'_1 \geqq N$, and such that

$$\{A_{m'_1,n'_1} \cdots A_{m'_j,n'_j}(P)\} \cap H_k = \{0\}.$$

We then claim that we may choose (m_k, n_k) such that $\Omega \cap H_k = \{0\}$ for all k. To see this, start with choosing a sequence $(m'_1, n'_1), \cdots, (m'_{j_1}, n'_{j_1})$ which works for H_1 and with $n'_1 \geqq 9$. Set $(m_i, n_i) = (m'_i, n'_i)$ for $1 \leqq i \leqq j_1$. Define $N_1 = 2m_{j_1}$, and consider

$$H'_2 = A^{-1}_{m_{j_1},n_{j_1}} \cdots A^{-1}_{m_1,n_1}(H_2).$$

This is again a hyperplane and we can choose a sequence $(m''_1, n''_1), \cdots, (m''_{j_2}, n''_{j_2})$ with $n''_1 \geqq N_1$ which avoids H'_2. Setting

$$(m_{j_1+1}, n_{j_1+1}) = (m''_1, n''_1), \cdots, (m_{j_1+j_2}, n_{j_1+j_2}) = (m''_{j_2}, n''_{j_2}),$$

we see that by our construction $A_{m_1,n_1} \cdots A_{m_{j_2},n_{j_2}}(P)$ avoids both H_1 and H_2, and so forth. The theorem thus follows from the result for one hyperplane. Let H be therefore a hyperplane defined by $k_1, \cdots, k_4$. We distinguish three cases.

Case 1. k_2 is non-zero. Here we choose $n'_1 = N$, and then

$$m'_1 \geqq |k_1| + |k_4| + n_1|k_3|.$$

Then choose (n'_2, m'_2) much larger than m'_1, with say $m'_2 = 4n'_2$. Consider for any $x \in \tilde{P}$ the vector $\tilde{A}_{m'_2,n'_2}x$. It is close to a vector of the form $(0, 1-\alpha, \alpha, 0)$ where α may be any number between 1/5 and 1. Thus $A_{m'_1,n'_1}A_{m'_2,n'_2}x$ looks approximately like $(\alpha, m'_1 - 1 + \alpha, n'_1 - \alpha, 1)$, and

$$k_1\alpha + k_2(m'_1 - 1 + \alpha) + k_3(n'_1 - \alpha) + k_4$$

has for all possible α the same sign, namely that of k_2, because of the choice of m'_1. Thus $A_{m'_1,n'_1}A_{m'_2,n'_2}P \cap H = \{0\}$.

Case 2. $k_2 = 0$ but $k_3 \neq 0$. This case is handled like Case 1, this time choosing $n'_1 \geqq \max(N, |k_4| + |k_1|)$, $m'_1 = 3(n'_1 + 1)$ and (m'_2, n'_2) very large.

Case 3. $k_2 = 0 = k_3$. Suppose that k_1 and k_4 are fixed. Choose any admissible matrix $A_{m'_1,n'_1}$; i.e. $n'_1 \geqq N$ and $m'_1 \geqq 3(n'_1 + 1)$. To say that $k_1x_1 + k_4x_4 = 0$ for a vector $x = A_{m'_1,n'_1}y$ means that

$$k_1(y_3+y_4)+k_4(y_1+y_2+y_3+y_4)=0$$

for the vector y. This yields the hyperplane

$$k_4y_1+k_4y_2+(k_1+k_4)y_3+(k_1+k_4)y_4=0$$

which falls into Case 1 or Case 2 unless $k_1=k_4=0$. This ends the proof of Theorem 6.

Our next goal is to show that, in the transformations constructed above, there are exactly two ergodic measures, and that by a suitable choice of α we may force Lebesgue measure to be one of these.

THEOREM 7. *Suppose that the conditions of Theorem 5 hold. Then the set $\Omega\cap\tilde{P}$ is an interval in $\tilde{P}$. If α is chosen in the interior of this interval, then there are exactly two ergodic probability measures, both absolutely continuous with respect to Lebesgue measure. If α is chosen to be an endpoint of this interval, then there are exactly two ergodic probability measures, one of which is Lebesgue measure and the other singular with respect to Lebesgue measure.*

PROOF. Let $e_1=(1,0,0,0)$ and $e_4=(0,0,0,1)$. We shall show that $\lim_{k\to\infty}\tilde{B}_ke_j$ exists for $1\leqq j\leqq 4$ and that the limits are the same for e_1 and e_2, and for e_3 and e_4. For any norm $|\cdot|$ on R^4 and any two vectors $0\neq x,\ y\in R^4$, we have

$$\left|\frac{x}{|x|}-\frac{y}{|y|}\right|\leqq 2\cdot\min\left(\frac{1}{|x|},\frac{1}{|y|}\right)\cdot|x-y|.$$

Choose a subsequence $k_n\to\infty$ such that $e_j^\infty=\lim_{n\to\infty}\tilde{B}_{k_n}e_j$ exists for each j. Then since each $\tilde{B}_k\tilde{P}$ is a simplex with extremal points $\tilde{B}_ke_j$, we have that Ω is the simplex spanned by the e_j^∞, $1\leqq j\leqq 4$. Now the simplexes $\tilde{B}_{k_n}\tilde{P}$ and $\tilde{B}_{k_{n+1}}\tilde{P}$ are close to each other, and for any k with $k_n<k<k_{n+1}$ we have $\tilde{B}_{k_n}\tilde{P}\supseteq\tilde{B}_k\tilde{P}\supseteq\tilde{B}_{k_{n+1}}\tilde{P}$. This is impossible unless each $\tilde{B}_ke_j$ is close to $\tilde{B}_{k_n}e_j$ and $\tilde{B}_{k_{n+1}}e_j$. Thus $\lim_{k\to\infty}\tilde{B}_ke_j$ exists for $1\leqq j\leqq 4$. Next we show that $\lim_{k\to\infty}\tilde{B}_ke_3=\lim_{k\to\infty}\tilde{B}_ke_4$, using the norm $|x|=\sum_{i=1}^4|x_i|$.

We have

$$|\tilde{B}_ke_3-\tilde{B}_ke_4|=\left|\frac{B_ke_3}{|B_ke_3|}-\frac{B_ke_4}{|B_ke_4|}\right|\leqq 2\,\frac{|B_k(e_4-e_3)|}{\min(|B_ke_3|,|B_ke_4|)}.$$

Now $B_k=B_{k-1}A_k$ and $A_k(e_4-e_3)=e_3$, so that $B_k(e_4-e_3)=B_{k-1}e_3$. Moreover,

$$B_ke_4=B_{k-1}(1,0,n_k,1)\geqq B_{k-1}(0,0,n_k,0)=n_kB_{k-1}e_3,$$

so that

$$|\tilde{B}_k e_3 - \tilde{B}_k e_4| \leqq \frac{2|B_{k-1}e_3|}{n_k |B_{k-1}e_3|} = \frac{2}{n_k} \to 0$$

as $k \to \infty$. This shows that $\lim_{k\to\infty} \tilde{B}_k e_3 = \lim_{k\to\infty} \tilde{B}_k e_4$, and $\lim_{k\to\infty} \tilde{B}_k e_1 = \lim_{k\to\infty} \tilde{B}_k e_2$ is proved in the same manner. It follows that Ω is the interval in $\tilde{P}$ with endpoints $\beta = e_1^\infty = e_2^\infty$ and $\gamma = e_3^\infty = e_4^\infty$. Let α be chosen in this interval and let μ be an invariant probability measure for the interval exchange transformation based on α. Let λ_2 and λ_3 be the probability measures defined in the proof of Theorem 5. Define

$$a_k = \frac{\mu(I_k^{(1)} \cup I_k^{(2)})}{\mu(I_k)} \quad \text{and} \quad b_k = \frac{\mu(I_k^{(3)} \cup I_k^{(4)})}{\mu(I_k)}$$

for each $k \geqq 1$. Then it is easy to see that

$$\lim_{k\to\infty} a_k\lambda_2 + b_k\lambda_3 = \mu$$

in the weak topology. Choosing subsequences of a_k and b_k which converge, we obtain μ as a convex combination of λ_2 and λ_3. This implies that λ_2 and λ_3 are the only ergodic measures (and that they are ergodic). Moreover, the λ_2-measure of the original partition into 4 intervals is given by β, and the λ_3-measure of this partition by γ. Thus λ_2 is Lebesgue measure iff $\alpha = \beta$ and λ_3 is Lebesgue measure iff $\alpha = \gamma$. This completes the proof.

References

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