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THE T-IDEAL GENERATED BY THE STANDARD IDENTITY $s_3[x_1, x_2, x_3]$

BY

AMITAI REGEV

ABSTRACT

Let $K = T_0(s_3)$, $\{c_n\}$ its codimensions, $\{l_n\}$ its colengths and $\{\chi_n\}$ its sequence of co-characters. For $9 \leqq n$, $c_n = 2n - 1$ or $c_n = n(n+1)/2 - 1$, $3 \leqq l_n \leqq 4$ and $\chi_n = [n] + 2[n-1,1] + \alpha[n-2,2] + \beta[2^2, 1^{n-4}]$ where $\alpha + \beta \leqq 1$.

Introduction

In [2], [3] and [4], J. Olsson and the present author demonstrated that the representation theory of the symmetric group can be used for studying certain problems concerning algebras satisfying a polynomial identity (P.I. algebras) over a field F of characteristic zero. This is done by identifying the space V_n of multilinear polynomials in $x_1, \cdots, x_n$ with the group algebra $F[S_n]$ of the symmetric group. The intersection $K_n = K \cap V_n$ of a T-ideal K with $V_n \equiv F[S_n]$ is then a left ideal in V_n (see [2]), and we can write $V_n = K_n \oplus J_n$, where J_n is a left ideal. Although J_n is not unique, its character χ_n is, and χ_n is "the n-th co-character of K". $\{\chi_n\}$ form the sequence of co-characters of K. The codimension $c_n = \dim J_n$ and the length l_n of J_n can be recovered from $\{\chi_n\}$ (see [3], [6]). We saw in [3] that $\{l_n\}$ is closely related to the question of whether or not a T-ideal is T-finitely generated.

This paper continues [4]: we study the T_0-ideal $K = T_0(s_3[x_1, x_2, x_3])$, generated by $s_3 = \Sigma_{\sigma \in S_3}(-1)^\sigma x_{\sigma_1} x_{\sigma_2} x_{\sigma_3}$, and find estimates on its sequences $\{c_n\}$, $\{l_n\}$ and $\{\chi_n\}$. In §1 we prove a general "cancellation" theorem 1.1, and use it in §3 to estimate $c_n = c_n(K)$. In §2, we find three components of $\chi_n = \chi_n(K)$. The information obtained in §§2, 3 is combined in §4 to give a close estimate of χ_n (for $n \geqq 9$).

We feel that some parts of the paper (for instance, §2) can be generalized so as to enable us to study the T-ideals generated by polynomials of higher degrees.

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§1. A cancellation theorem

THEOREM 1.1. *Let* $K = T_0(S_n)$. *If* $f(x_1, \cdots, x_n)x_{n+1} \in K_{n+1}$, *then* $f(x_1, \cdots, x_n) \in K_n$ (*i.e.* $f(x_1, \cdots, x_n) = \alpha \cdot S_n[x_1, \cdots, x_n]$ *for some* $\alpha \in F$).

Here we shall use the same notation that was used in [2], [3], [4]. In particular, we assume that char $F = 0$ and shall currently use the identification of V_n and the group algebra $F[S_n]$.

The proof of Theorem 1 is divided into two major steps. We first show that if $f(x_1, \cdots, x_n) \cdot x_{n+1} \in K_{n+1}$ and $f(x_1, \cdots, x_n) \notin K_n$, then

$$x_n \cdot S_{n-1}[x_1, \cdots, x_{n-1}] \cdot x_{n+1} \in K_{n+1}.$$

Next we show that

$$x_n \cdot S_{n-1}[x_1, \cdots, x_{n-1}] \cdot x_{n+1} \notin K_{n+1}.$$

DEFINITION. Let M be a T-ideal (T_0 or T_1). We say that M has the "right cancellation property" (r.c.p.) if M satisfies the following condition:

For any n, if $g(x_1, \cdots, x_n) \cdot x_{n+1} \in M_{n+1}$, then $g(x_1, \cdots, x_n) \in M_n$.

If M is any set of polynomials, M has r.c.p. if $T_0(M)$ has it.

"Left cancellation property" (l.c.p.) is similarly defined.

REMARK. Any T_1 ideal has both l.c.p. and r.c.p. ([3], prop. 1.1), and therefore Theorem 1 holds trivially when n is even, since then $T_0(S_n) = T_1(S_n)$ ([3], lemma 2.8). We shall therefore assume throughout the rest of this section that n is odd.

THEOREM 1.2. *Let M be a set of polynomials and let x be a variable which does not occur in M. Let $N = T_0(xM)$. If M has r.c.p., then N also has r.c.p. (Similarly for Mx and l.c.p.)*

PROOF. We need the following characterization of the elements of N: $g(x_1, \cdots, x_n) \in N$ if and only if

$$g(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i p_i (x_1, \cdots, \hat{x}_i, \cdots, x_n),$$

where $p_i \in T_0(M)$, $1 \leqq i \leqq n$.

The proof of the above statement was given in [3, theor. 3.1]. It now follows easily that

$$g(x_1, \cdots, x_n) \cdot x_{n+1} \in T_0(xM)$$

if and only if

$$g(x_1, \cdots, x_n) \cdot x_{n+1} = \sum_{i=1}^{n} x_i p_i(\hat{x}_i) \cdot x_{n+1},$$

where $p_i(\hat{x}_i) \cdot x_{n+1} = p_i(x_1, \cdots, \hat{x}_i, \cdots, x_n) \cdot x_{n+1} \in T_0(M)$.

Since M has r.c.p., this implies that $p_i \in T_0(M)$, $1 \leqq i \leqq n$, and therefore

$$g(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i p_i(\hat{x}_i) \in T_0(M),$$

as was to be proved.

COROLLARY 1.3. *Since $T_0(S_{2k}) = T_1(S_{2k})$ and $n-1$ is even, it follows that*

$$T_0(x_n \cdot S_{n-1}[x_1, \cdots, x_{n-1}]) \quad \textit{has r.c.p.}$$

We can now complete the first step by proving

LEMMA 1.4. *Let $K = T_0(S_n[x_1, \cdots, x_n])$. If*

$$f(x_1, \cdots, x_n) \cdot x_{n+1} \in K_{n+1} \quad \textit{and} \quad f(x_1, \cdots, x_n) \notin K_n,$$

then

$$x_n \cdot S_{n-1}[x_1, \cdots, x_{n-1}] \cdot x_{n+1} \in K_{n+1}.$$

PROOF. Let $R = T_0(x_n \cdot S_{n-1}[x_1, \cdots, x_{n-1}])$, then $R \supseteq K$ so that $f(x_1, \cdots, x_n) \cdot x_{n+1} \in R_{n+1}$. By Corollary 1.3, R has r.c.p. and therefore $f(x_1, \cdots, x_n) \in R_n$.

It follows from [3, theor. 3.3] that

$$R_n = V_n(x_n \cdot S_{n-1}[x_1, \cdots, x_{n-1}]) = H_1 \oplus H_2,$$

where H_1, H_2 are minimal left ideals (with characters $[\lambda_1]$, $[\lambda_2]$, $\lambda_1 = (1^n)$, $\lambda_2 = (2, 1^{n-2})$). Let $V_n f + V_n S_n = L_n$ denote the left ideal generated by S_n and f in V_n. If $f \neq \alpha \cdot S_n$ for any $\alpha \in F$, then $V_n S_n \subsetneqq L_n \subseteq R_n = H_1 \oplus H_2$, so that $L_n = H_1 \oplus H_2$ by the minimality of H_1, H_2.

In particular, $x_n \cdot S_{n-1}[x_1, \cdots, x_{n-1}] \in L_n$. We therefore have:

$$\begin{aligned} x_n \cdot S_{n-1}[x_1, \cdots, x_{n-1}] \cdot x_{n+1} &\in V_{n+1} L_n = V_{n+1} S_n + V_{n+1} f \\ &= V_{n+1}(S_n[x_1, \cdots, x_n] \cdot x_{n+1}) + V_{n+1}(f(x_1, \cdots, x_n) \cdot x_{n+1}) \subseteq K_{n+1} \end{aligned}$$

which completes the proof of the lemma.

The proof of Theorem 1.1 will be completed once we show that, in fact, $x_n \cdot S_{n-1}[x_1, \cdots, x_{n-1}] \cdot x_{n+1} \notin K_{n+1}$. To this end we compare the characters of K_{n+1} and of

$$D_{n+1} = V_{n+1}(x_n \cdot S_{n-1}[x_1, \cdots, x_{n-1}] \cdot x_{n+1}).$$

For $\lambda \in \mathrm{Par}(n)$, we denote by $I_\lambda \subseteq V_n$ the minimal 2-sided ideal that corresponds to λ (see [1, chap. IV]). Consider the following partitions of $n+1$:

$$\lambda_1 = (1^{n+1}), \quad \lambda_2 = (2, 1^{n-1}), \quad \lambda_3 = (2^2, 1^{n-3}), \quad \lambda_4 = (3, 1^{n-2}).$$

It was proved in [4] that

$$K_{n+1} = J_1 \oplus J_2 \oplus J_3 \oplus J_4,$$

where $J_i \subseteq I_{\lambda_i}$, $1 \leqq i \leqq 4$, are left ideals, and J_1, J_3, J_4 are minimal.

The character of D_{n+1} is $[\lambda_1] + 2[\lambda_2] + [\lambda_3] + [\lambda_4]$, a fact that follows by twice applying [3, theor. 3.3]. Therefore

$$D_{n+1} = J'_1 \oplus J'_2 \oplus J'_3 \oplus J'_4, \quad J'_i \subseteq I_{\lambda_i}, \quad 1 \leqq i \leqq 4, \quad J'_1, J'_3, J'_4$$

being minimal.

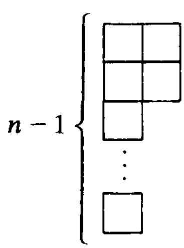
It follows that if

$$x_n \cdot s_{n-1}[x_1, \cdots, x_{n-1}] \cdot x_{n+1} \in K_{n+1},$$

then $J_1 = J'_1$, $J_3 = J'_3$ and $J_4 = J'_4$. We shall derive the contradiction by showing that $J_3 \neq J'_3$. For this purpose we invoke the theory of Young diagrams (see [1]).

The two minimal left ideals J_3 and J'_3 can be computed as follows:

To the partition $\lambda_3 = (2^2, 1^{n-3})$ corresponds the Young diagram



Consider the two tableaux T_1, T_2 based on the above diagram:

$T_1 =$

1	$n+1$
2	n
3	
$\vdots$	
$n-1$	

and

$T_2 =$

2	1
3	$n+1$
4	
$\vdots$	
n	

Let e_i be the "essentially idempotent" ([1], Ch. IV, §1) defined by the tableau T_i, $i = 1, 2$. Note that $T_2 = k_{n+1}T_1$, where

$$k_{n+1} = (1, 2, \cdots, n+1) \in S_{n+1}.$$

It follows that

$$e_2 = k_{n+1}e_1k_{n+1}^{-1} = k_{n+1}e_1k_{n+1}^{n}$$

and, in the same way,

$$e_1 = k_{n+1}^{-1}e_2k_{n+1}.$$

LEMMA 1.5. *With the above notations*

I) $J_3' = V_{n+1}e_2$,

II) $J_3 = V_{n+1}e_1\nu_{n+1}$, *where* $\nu_{n+1} = 1 + k_{n+1} + \cdots + k_{n+1}^{n-1}$.

PROOF. I) By definition, $e_2 = a \cdot b$ where

$$a = (1 + (1, 2))(1 + (3, n+1))$$

and, under the identification of V_{n+1} and $F[S_{n+1}]$ ([2]),

$$b = \Big(\sum_{\sigma \in S_{n-1}(2, \cdots, n)} (\operatorname{sgn} \sigma) \cdot \sigma\Big)(1 - (1, n+1))$$

$$= x_1 \cdot S_{n-1}[x_2, \cdots, x_n] \cdot x_{n+1} - x_{n+1} \cdot S_{n-1}[x_2, \cdots, x_n] \cdot x_1.$$

Obviously, $b \in D_{n+1}$; hence $e_2 = ab \in D_{n+1}$, so that $V_{n+1}e_2 \subseteq D_{n+1}$. But $V_{n+1}e_2$ is a minimal left ideal in I_{λ_3}, hence $V_{n+1}e_2 = J_3'$.

II) A similar computation for e_1 yields that

$$e_1 = c \cdot d \quad \text{where} \quad c = (1 + (1, n+1))(1 + (2, n))$$

and

$$d = S_{n-1}[x_1, \cdots, x_{n-1}] \cdot [x_n, x_{n+1}].$$

Following [4, lemma 3], one can easily show that

$$S_n[x_1, \cdots, x_{n-1}, [x_n, x_{n+1}]] = S_{n-1}[x_1, \cdots, x_{n-1}] \cdot [x_n, x_{n+1}]\, \nu_n,$$

and therefore

$$V_{n+1}c \cdot S_n[x_1, \cdots, x_{n-1}, [x_n, x_{n+1}]] = V_{n+1}e_1\nu_{n+1}.$$

By assumption, n is odd, hence it follows [4, lemma 4] that ν_{n+1} is invertible. But this implies that right multiplication by ν_{n+1} in V_{n+1} maps a minimal left ideal in I_λ to a minimal left ideal in I_λ, for any $\lambda \in \mathrm{Par}(n+1)$. In particular, since $V_{n+1}e_1$ is such an ideal in I_{λ_3}, so is $V_{n+1}e_1\nu_{n+1}$. Since $V_{n+1}e_1\nu_{n+1} \subseteq K_{n+1}$, it follows from what we know about the character of K_{n+1} that $J_3 = V_{n+1}e_1\nu_{n+1}$.

We shall later need

LEMMA 1.6. *Let α_i be the coefficient of k^i_{n+1} in e_1, $0 \leqq i \leqq n$. Then $\alpha_0 = \alpha_2 = 1$ and $\alpha_i = 0$ if $i \neq 0, 2$.*

PROOF. Assume $5 \leqq n$, and write explicitly

$$e_1 = (1 + (1, n+1))(1 + (2, n))\Big(\sum_{\sigma \in S_{n-1}} (\operatorname{sgn}\sigma)\cdot\sigma\Big)(1 - (n, n+1)).$$

Let $\rho \in S_{n+1}$ be a permutation whose coefficient in e_1 is $\neq 0$. Direct computation shows that the following are the only possibilities:

$$\rho(n) = \begin{cases} n+1 \\ n \\ 2 \\ 1 \end{cases} \qquad \rho(n+1) = \begin{cases} n+1 \\ n \\ 2 \\ 1 \end{cases}$$

Since $k^i_{n+1}(n+1) = i$, $\alpha_i = 0$ for $3 \leqq i \leqq n-1$. Since $k^n_{n+1}(n) = n-1$, $\alpha_n = 0$ also. Note also that if $\rho(n) = n+1$ in the above, then $\rho(n+1) \neq 1$, hence $\alpha_1 = 0$.

Obviously, $\alpha_0 = 1$, so that the lemma will be proved once we show that $\alpha_2 = 1$.

Let $\sigma = (1, n+1)(2, n)k^2_{n+1}(n, n+1)$. It is easy to verify that $\sigma(n) = n$, $\sigma(n+1) = n+1$; therefore $\sigma \in S_{n-1}$. Since k^2_{n+1} is an even permutation, σ is odd: $\operatorname{sgn}\sigma = -1$. Write

$$k^2_{n+1} = (1, n+1)(2, n)\sigma(n, n+1);$$

then obviously, k^2_{n+1} appears in e_1, and $\alpha_2 = (\operatorname{sgn}\sigma)(\operatorname{sgn}(n, n+1)) = 1$.

The case $n = 3$ can be done by similar arguments, and is left for the reader.

COROLLARY 1.7. $e_1(1 + k_{n+1} + \cdots + k^n_{n+1}) \neq 0$.

PROOF. Compute the coefficient β_1 of 1 on the left side. If $\rho \in S_{n+1}$ has a non-zero coefficient in e_1 and $\rho k^i_{n+1} = 1$ for some $0 \leqq i \leqq n$, then $\rho = k^{n+1-i}_{n+1}$ and by Lemma 1.6, $\rho = 1$ or $\rho = k^2_{n+1}$—whose coefficients in e_1 are equal to 1. It follows that $\beta_1 = 2 \neq 0$.

THEOREM 1.8. $x_nS_{n-1}[x_1, \cdots, x_{n-1}]\cdot x_{n+1} \notin K_{n+1}$.

PROOF. By previous remarks, the theorem will be proved if we show that $J_3 \neq J'_3$.

Assume that $J_3 = J'_3$. Since $e_2 = k_{n+1}e_1k^n_{n+1}$, we have that $J_3 = J'_3 = V_{n+1}e_2 = V_{n+1}e_1k^n_{n+1}$. Therefore

$$0 \neq J''_3 = V_{n+1}e_1(1 + k_{n+1} + \cdots + k^n_{n+1}) \subseteq V_{n+1}e_1(1 + \cdots + k^{n-1}_{n+1}) + V_{n+1}e_1k^n_{n+1}$$
$$= J_3 + J_3 = J_3.$$

Since J_3 is a minimal left ideal and $0 \neq J''_3 \subseteq J_3$ is obviously a left ideal, $J''_3 = J_3$. Now, $V_{n+1}e_1\nu_{n+1} = J_3$ implies that

$$V_{n+1}e_1(k_{n+1} + \cdots + k^n_{n+1}) = V_{n+1}e_1\nu_{n+1}k_{n+1} = J_3k_{n+1}$$
$$= J'_3k_{n+1} = V_{n+1}(k^{-1}_{n+1}e_2k_{n+1}) = V_{n+1}e_1,$$

so

$$J''_3 = V_{n+1}e_1(1 + \cdots + k^n_{n+1}) \subseteq V_{n+1}e_1 + V_{n+1}e_1(k_{n+1} + \cdots + k^n_{n+1})$$
$$= V_{n+1}e_1,$$

and again we conclude that $J''_3 = V_{n+1}e_1$. The assumption $J_3 = J'_3$ therefore implies that $V_{n+1}e_1 = V_{n+1}e_2$. But this is impossible, since e_1, e_2 are orthogonal "essential" idempotents; $\{1, n+1\}$ appears in the same row in T_1 and the same column in T_2, while $\{1, 2\}$ appears in the same row of T_2 and the same column in T_1 ([1], Ch. IV). The proof of Theorem 1.8, hence also of Theorem 1.1 is now completed.

Remark. Since $K = T_0(s_3)$ is invariant under left-right reflection, it follows that if $x_{n+1}f(x_1, \cdots, x_n) \in K_{n+1}$, then $f(x_1, \cdots, x_n) \in K_n$. It is this form of Theorem 1.1 that we are going to use later.

§2. Let Q be a T-ideal, $Q_n = Q \cap V_n$, then

$$Q'_{n+1} = V_{n+1}Q_nx_{n+1} + V_{n+1}x_{n+1}Q_n \subseteq Q_{n+1}.$$

Our aim in this section is to obtain some information about Q'_{n+1}. It can be shown that

$$V_{n+1}Q_nx_{n+1} \neq V_{n+1}x_{n+1}Q_n$$

unless $Q_n = (0)$ or $Q_n = V_n$. Also, we assume throughout this section that $Q \subseteq C = T([x_1, x_2])$. The notations in this section can be found in [3, §2]. The set of partitions $\mathrm{Par}(n)$ is well-ordered by the lexicographic order

$$(1^n) < (2, 1^{n-2}) < \cdots < (n-1, 1) < (n).$$

Note that if $Q \subseteq C$ and $I_\lambda \cap Q_n \neq (0)$, then $\lambda \lneq (n)$. We shall show that if $\lambda' \in \mathrm{Par}(n+1)$ is of maximal order such that $I_{\lambda'} \cap V_{n+1}Q_n x_{n+1} \neq (0)$, then $I_{\lambda'} \cap V_{n+1}Q_n x_{n+1} \neq I_{\lambda'} \cap V_{n+1}x_{n+1}Q_n$.

Definition. Let $\lambda(a_1,\cdots,a_r)\in \mathrm{Par}(n)$, $a_1 \geqq \cdots \geqq a_r$, $T = T(\lambda) = T_\lambda$ a Young tableau for λ, $\lambda' = (a_1+1, a_2,\cdots,a_r)\in\mathrm{Par}(n+1)$ and denote by $T(\lambda') = T_\lambda^{\boxed{n+1}} = \tilde{T}(\lambda)$ the tableau obtained from $T(\lambda)$ by adjoining an additional box, with $n+1$ in it, to the right upper corner of $T(\lambda)$. For example, let $\lambda = (2^2)\in\mathrm{Par}(4)$,

$$T(\lambda) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

then

$$\tilde{T}(2,2) = T(2,2)^{\boxed{5}} = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$$

If $e_{T(\lambda)}$ is the corresponding idempotent, denote $\tilde{e}_{T(\lambda)} = e_{\tilde{T}(\lambda)}$. Finally, define $\tilde{I}_\lambda \subseteq I_{\lambda'}$ by $\tilde{I}_\lambda = \Sigma_{T(\lambda)} V_{n+1}\tilde{e}_{T(\lambda)}$.

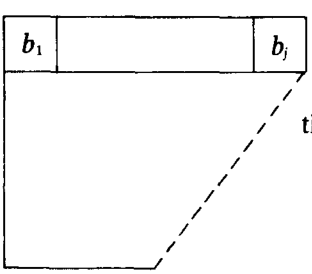
Lemma 2.1. *Let $\lambda \in \mathrm{Par}(n)$, $\lambda' \in \mathrm{Par}(n+1)$ as above and let $J_n \subseteq I_\lambda$ be a left ideal, then*

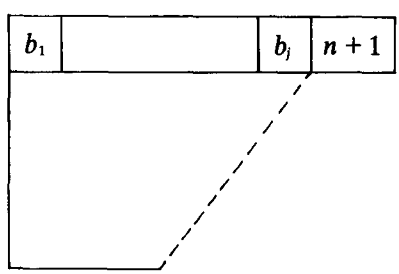
$$V_{n+1}J_n x_{n+1} \cap I_{\lambda'} \subseteq \tilde{I}_\lambda.$$

Proof. Note first that for any set of left ideals $\{L_j\}$ in V_{n+1}, $(\Sigma_j L_j)\cap I_{\lambda'} = \Sigma_j(L_j \cap I_{\lambda'})$. Now $J_n \subseteq I_\lambda = \Sigma_{T(\lambda)} V_n e_{T(\lambda)}$, hence

$$V_{n+1}J_n x_{n+1} \subseteq \sum_{T(\lambda)} (V_{n+1}e_{T(\lambda)}x_{n+1} \cap I_{\lambda'}).$$

By [3, theor. 3.3], $M_{n+1} = V_{n+1}e_{T(\lambda)}x_{n+1} \cap I_{\lambda'}$ is a minimal left ideal, and we show that $M_{n+1} \subseteq \tilde{I}_\lambda$. In fact, $M_{n+1} = V_{n+1}\tilde{e}_{T(\lambda)}$. To see this, let

$T(\lambda) =$ 

then $\tilde{T}(\lambda) =$ 

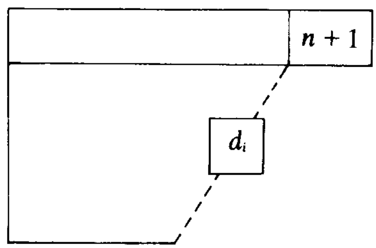
and it follows from basic definitions that

$$\tilde{e}_{T(\lambda)} = \sum_{i=1}^{j+1} (b_i, n+1) e_{T(\lambda)} \cdot x_{n+1} \qquad (b_{j+1} = n+1),$$

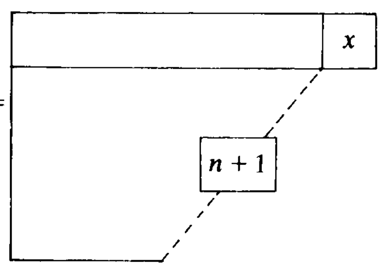
so that $V_{n+1}\tilde{e}_{T(\lambda)} \subseteq V_{n+1} e_{T(\lambda)} x_{n+1} \cap I_{\lambda'} = M_{n+1}$. Since the two sides are minimal left ideals, $M_{n+1} = V_{n+1}\tilde{e}_{T(\lambda)}$. Q.E.D.

LEMMA 2.2 *Let* $2 \leqq n$, $(n) \neq \lambda = (a_1, \cdots, a_r) \in \mathrm{Par}(n)$, $T_1(\lambda), \cdots, T_k(\lambda)$ k *tableaux for* λ *and* $\tilde{e}_i = \tilde{e}_{T_i(\lambda)}$ *the idempotents that correspond to the tableaux* $\tilde{T}_i(\lambda)$, $1 \leqq i \leqq k$. *Finally, let* $\sigma \in S_{n+1}$ *be any* $n+1$ *cycle. Then for each* $1 \leqq i \leqq k$ *there exists* $m = m(i)$ *such that* $e'_i = \sigma^m \tilde{e}_i \sigma^{-m}$ *is "orthogonal" to* $\tilde{e}_1, \cdots, \tilde{e}_k$, *i.e.*, $\tilde{e}_1 e'_i = \cdots = \tilde{e}_i e'_i = \cdots = e_k e'_i = 0$.

PROOF. Since $2 \leqq n$ and $\lambda \neq (n)$, each tableau $T_i(\lambda)$ has more than one row, so we can write

$\tilde{T}_i(\lambda) =$ 

Assume, without loss of generality, that $i = 1$, and let $d_1 = j$ (in $\tilde{T}_1(\lambda)$). Trivially, there exists $m = m(1)$ such that $\sigma^m(j) = n+1$, and therefore

$\sigma^m \tilde{T}_1(\lambda) =$ 

Denote now by T_1^* the tableau obtained from $\sigma^m \tilde{T}_1(\lambda)$ by removing the box $\boxed{n+1}$, and write $T_1^* = T(\lambda^*)$. Clearly, $\lambda^* \in \mathrm{Par}(n)$ and $\lambda^* \underset{\neq}{>} \lambda$. Hence, for each $1 \leqq s \leqq k$ there exist two numbers that occur in one row of T_1^* and one column of $T_s(\lambda)$ (see [1]). This is still true when we re-adjoin the box $\boxed{n+1}$ back to the above tableaux, so that $e_s \cdot e'_1 = 0$ for all $1 \leqq s \leqq k$, where $e'_1 = e_{\sigma^m \tilde{T}_1(\lambda)} = \sigma^m e_{\tilde{T}_1(\lambda)} \sigma^{-m} = \sigma^m \tilde{e}_1 \sigma^{-m}$. Q.E.D.

THEOREM 2.3. *Let* $2 \leqq n$, $(n) \neq \lambda \in \mathrm{Par}(n)$, $(0) \neq J_{n+1} \subseteq \tilde{I}_\lambda$ *a left ideal and* $\sigma \in S_{n+1}$ *an* $n+1$ *cycle, then* $J_{n+1}\sigma \neq J_{n+1}$.

PROOF. Let k be minimal and $T_1(\lambda), \cdots, T_k(\lambda)$ standard tableaux such that $J_{n+1} \subseteq \sum_{i=1}^{k} V_{n+1}\tilde{e}_{T_i(\lambda)}$. Note that the tableaux $\tilde{T}_i(\lambda)$ are also standard, so we may assume $\tilde{e}_i\tilde{e}_1 = 0$ for $2 \leqq i \leqq k$.

Suppose $J_{n+1}\sigma = J_{n+1}$. Hence $J_{n+1}\sigma^m = J_{n+1}$ for all m. By Lemma 2.2 there exists $m = m(1)$ such that $e'_1 = \sigma^m \tilde{e}_1 \sigma^{-m}$ satisfies $\tilde{e}_i e'_1 = 0$, $1 \leqq i \leqq k$, so that $J_{n+1}e'_1 = (0)$. Therefore

$$J_{n+1}\tilde{e}_1\sigma^{-m} = J_{n+1}\sigma^m\tilde{e}_1\sigma^{-m} = J_{n+1}e'_1 = (0)$$

which implies $J_{n+1}\tilde{e}_1 = (0)$.

Let $a \in J_{n+1}$, then $a = a_1\tilde{e}_1 + \cdots + a_k\tilde{e}_k$ and $a\tilde{e}_1 = 0$. Since $\tilde{e}_1^2 = \tilde{e}_1$, $\tilde{e}_2\tilde{e}_1 = \cdots = \tilde{e}_k\tilde{e}_1 = 0$, we have $a_1\tilde{e}_1 = 0$ so that $a = a_2\tilde{e}_2 + \cdots + a_ke_k$ which implies that $J_{n+1} \subseteq \sum_{i=2}^{k} V_{n+1}\tilde{e}_i$, a contradiction to the minimality of k. Q.E.D.

COROLLARY 2.4. *Let* Q *be a* T*-ideal,* $Q_n = V_n \cap Q$,

$$Q_n \to \begin{Bmatrix} \lambda_1, \cdots, \lambda_k \\ m_1, \cdots, m_k \end{Bmatrix} \qquad 2 \leqq n \quad (\textit{see } [3])$$

and assume that $(n) \neq \lambda_1 = (a_1, \cdots, a_r)$ *is maximal among* $\lambda_1, \cdots, \lambda_k$, $m_1 \geqq 1$ *Then*

$$Q_{n+1} \to \begin{Bmatrix} \lambda'_1, \cdots \\ m'_1, \cdots \end{Bmatrix}$$

where $\lambda'_1 = (a_1 + 1, a_2, \cdots, a_r)$ *and* $m'_1 \geqq m_1 + 1$.

PROOF. By [3, theor. 3.3]

$$V_{n+1}Q_nx_{n+1} \to \begin{Bmatrix} \lambda'_1, \cdots, \\ m_1, \cdots \end{Bmatrix}$$

(and the same for $V_{n+1}x_{n+1}Q_n$), so that $I_{\lambda'} \cap V_{n+1}Q_nx_{n+1}$ has length m_1 as a left V_{n+1} module. By Lemma 2.1, $I_{\lambda'} \cap V_{n+1}Q_nx_{n+1} \subseteq \tilde{I}_\lambda$. If $\sigma = (n+1, \cdots, 1) \in S_n$, then $(V_{n+1}Q_nx_{n+1} \cap I_{\lambda'})\sigma = V_{n+1}x_{n+1}Q_n \cap I_{\lambda'}$, and by the last theorem, $V_{n+1}Q_nx_{n+1} \cap I_{\lambda'} \neq V_{n+1}x_{n+1}Q_n \cap I_{\lambda'}$, so that

$$I_{\lambda'} \cap Q_{n+1} \supseteq I_{\lambda'} \cap (V_{n+1}Q_nx_{n+1} + V_{n+1}x_{n+1}Q_n) \supsetneqq I_{\lambda'} \cap V_{n+1}Q_nx_{n+1}$$

as was to be shown. Q.E.D.

APPLICATIONS. Let $R = T_0([x_1, x_2]x_3)$, $L = T_0(x_1[x_2, x_3])$, $Q = L \cap R$, $Q_n = V_n \cap Q = L_n \cap R_n$. The structure of Q_n, $n \geqq 3$, can be determined as follows:

The structure of $R_n (\approx L_n)$ is given by [3, theor. 3.3], which implies that

a) $\lambda = (n-1,1) \in \mathrm{Par}(n)$ is maximal such that $R_n \cap I_\lambda \neq (0)$,

b) the length of $R_n \cap I_{(n-1,1)}$ is $n-2$,

c) if $\lambda \underset{\neq}{>} \mu \in \mathrm{Par}(n)$, then $R_n \supseteq I_\mu$.

The same is true also for L_n, hence, if $\lambda \underset{\neq}{>} \mu \in \mathrm{Par}(n)$, $Q_n = R_n \cap L_n \supseteq I_\mu$. By Lemma 2.1 and Theorem 2.3, $L_n \neq R_n$, so that $L_n \cap I_{(n-1,1)} \neq R_n \cap I_{(n-1,1)}$ and by (b), $L_n + R_n \supseteq I_{(n-1,1)}$. It follows from that, by an easy dimensions argument, that $Q_n \cap I_{(n-1,1)}$ has length $n-3$. In other words, the n-th co-character of Q_n is $[n]+2[n-1,1]$.

Next consider $K = T_0(s_3)$. Since $s_3[x_1, x_2, x_3] \in L, R$, it follows that $K \subseteq Q$.

PROPOSITION 2.5. *With the above notations,* $K_n \cap I_{(n-1,1)} = Q_n \cap I_{(n-1,1)}$.

PROOF. By induction on $n \geqq 3$. If $n = 3$, $K_3 = Q_3$, so assume the equation holds for n and show that it holds for $n+1$. Since $K_{n+1} \subseteq Q_{n+1}$, it is enough to show that $K_{n+1} \cap I_{(n,1)}$ has length $\geqq n-2$, which is the length of $Q_{n+1} \cap I_{(n,1)}$. By induction, $K_n \cap I_{(n-1,1)}$ has length $n-3$. Apply Corollary 2.4 to the T-ideal K to deduce that the length of $K_{n+1} \cap I_{(n,1)}$ is $\geqq n-2$. Q.E.D.

§3. Theorem 1.1 is applied now to study the codimensions $\{c_n\}$ of $K = T_0(s_3)$. The notations can be found in [6]. It was shown there that $c_n = \sum_{k=1}^n c_{k,n}$, where

$$c_{k,n} = \dim \frac{V_n^{(k)} + U_n^{(k)} + K_n}{U_n^{(k)} + K_n},$$

and that $c_{1,n} \leqq \cdots \leqq c_{n,n} \leqq c_{n-1}$.

The relation between cancellation and codimensions is revealed in

PROPOSITION 3.1. *Let Q be any T-ideal and $\{c_n\}$ its codimensions, then $c_{n,n} = c_{n-1}$ if and only if Q_n has the following "n-left cancellation property":*

$$x_n g(x_1, \cdots, x_{n-1}) \in Q_n \quad \textit{implies} \quad g(x_1, \cdots, x_{n-1}) \in Q_{n-1}.$$

PROOF. Left multiplication by x_n induces an isomorphism of V_{n-1} onto $V_n^{(n)}$ which implies that

$$c_{n-1} = \dim \frac{V_{n-1} + Q_{n-1}}{Q_{n-1}} = \dim \frac{V_n^{(n)} + x_n Q_{n-1}}{x_n Q_{n-1}} \geqq \dim \frac{V_n^{(n)} + Q_n}{Q_n} = c_{n,n}.$$

Therefore, $c_{n,n} = c_{n-1}$ if and only if any linear dependence modulo Q_n among the monomials of $V_n^{(n)}$ implies the same dependence modulo $x_n Q_{n-1}$. But that is exactly n-left cancellation, which is therefore equivalent to the condition $c_{n,n} = c_{n-1}$. Q.E.D.

LEMMA 3.2. *Let $\{d_n\}$ be the codimensions of $L = T_0(x_1[x_2, x_3])$. Then for all $1 \leqq k \leqq n$, $d_{k,n} = 1$.*

PROOF. Let $C = T_0([x_1, x_2])$ be the commutator ideal. It is well-known that its codimensions are all equal to 1. Since $x_nC_{n-1} \subseteq L_n$ we have

$$1 = \dim \frac{V_{n-1} + C_{n-1}}{C_{n-1}} = \dim \frac{V_n^{(n)} + x_nC_{n-1}}{x_nC_{n-1}} \geqq \dim \frac{V_n^{(n)} + L_n}{L_n} = d_{n,n}.$$

Hence, $d_{n,n} \leqq 1$. On the other hand, $d_n = n$; hence $d_{k,n} = 1$ for all $1 \leqq k \leqq n$. Q.E.D.

COROLLARY 3.3. *Let $\{c_n\}$ be the codimensions of $K = T_0(s_3)$. Then $1 \leqq c_{k,n}$ for all $1 \leqq k \leqq n$.*

PROOF. Since $K \subseteq L$ we have $c_n \geqq d_n$ as well as $c_{k,n} \geqq d_{k,n} = 1$ for all $1 \leqq k \leqq n$. Q.E.D.

The key result that will enable us to carry on the computation of $\{c_n\}$ is

COROLLARY 3.4. *With the above notations ($K = T_0(s_3)$), $c_{1,4} = c_{2,4} = 1$, $c_{3,4} = 2$ and $c_{4,4} = 5$.*

PROOF. It follows from [4] that $c_4 = 9$. Trivially, $c_3 = 5$, so Theorem 1.1 and Proposition 3.1 imply that $c_3 = c_{4,4} = 5$. Hence $c_{1,4} + c_{2,4} + c_{3,4} = 9 - 5 = 4$ and since $1 \leqq c_{1,4} \leqq c_{2,4} \leqq c_{3,4}$ are integers, the only possibility is $c_{1,4} = c_{2,4} = 1$ and $c_{3,4} = 2$. Q.E.D.

REMARK. Let $\mu = x_1x_{\sigma_2} \cdots x_{\sigma_n} \in V_n^{(1)}$ and assume that $\mu \in U_n^{(1)} + K_n$. By applying any permutation θ on $\{2, \cdots, n\}$ we still have $\mu(x_1, x_{\theta_2}, \cdots, x_{\theta_n}) \in U_n^{(1)} + K_n$, and therefore $V_n^{(1)} \subseteq U_n^{(1)} + K_n$. But this implies that $c_{1,n} = 0$, a contradiction. In other words, for any single monomial $\mu \in V_n^{(1)}$, we have $\mu \notin U_n^{(1)} + K_n$. This can easily be extended to a more general statement: Let $\mu \in V_n^{(k)}$ be a monomial, $1 \leqq k \leqq n$. Then $\mu \notin U_n^{(k)} + K_n$ (to prove this, transpose the indices 1 and k). As a result we have the following statement: Let $\mu_1, \cdots, \mu_t \in V_n^{(k)}$ ($1 \leqq k \leqq n$) be a set of monomials such that each two are linearly dependent modulo $U_n^{(k)} + K_n$. Then each monomial among $\{\mu_1, \cdots, \mu_t\}$ spans all the others modulo $U_n^{(k)} + K_n$. We will find it convenient to write "dep$(U_n^{(k)} + K_n)$" instead of "linearly dependent modulo $U_n^{(k)} + K_n$".

PROPOSITION 3.5. *Let $3 \leqq n$ and $1 \leqq k \leqq n - 2$. Then $c_{k,n} = 1$.*

PROOF. For $n = 3$ this is trivial, while Corollary 3.4 implies it for $n = 4$. By the previous remarks, it is enough to show that $c_{n-2,n} \leqq 1$, and we show it by induction on n, assuming that $c_{n-3,n-1} = 1$.

Let $\mu_1 = x_{n-2}a(x_ix_1)b$, $\mu_2 = x_{n-2}c(x_ix_1)d$ be two monomials in $V_n^{(n-2)}$ that "contain" x_ix_1. The substitution $x_i \to x_ix_1$ and $x_j \to x_{j+1}$, $j \neq 1$, induces a one-to-one linear map $\varphi: V_{n-1} \to V_n$ satisfying:

1) $\mu_1, \mu_2 \in \varphi(V_{n-1}^{(n-3)})$,
2) $\varphi(U_{n-1}^{(n-3)}) \subseteq U_n^{(n-2)}$,
3) $\varphi(K_{n-1}) \subseteq K_n$.

This, together with $c_{n-3,n-1} = 1$ implies that μ_1, μ_2 are dep$(U_n^{(n-2)} + K_n)$.

By a previous remark we can choose arbitrary monomial from the set of monomials $\{\mu = x_{n-2} \cdots x_ix_1 \cdots\}$ such that

$$\mathcal{N}_1 = x_{n-2}x_1 \cdots$$

$$\mathcal{N}_2 = x_{n-2} \cdots x_2x_1$$

$$\vdots$$

$$\mathcal{N}_{n-1} = x_{n-2} \cdots x_nx_1$$

span $V_n^{(n-2)}$ modulo $U_n^{(n-2)} + K_n$.

Now, the substitution $x_j \to x_{j+1}$, $1 \leqq j \leqq n-1$, followed by right multiplication by x_1 induces a one-to-one linear map ψ: $V_{n-1} \to V_n$ which has properties $1', 2', 3'$ similar to $1, 2, 3$ above. In particular:

1′) $\mathcal{N}_2, \cdots, \mathcal{N}_{n-1} \in \psi(V_{n-1}^{(n-3)})$

so that again we conclude that each two from $\mathcal{N}_2, \cdots, \mathcal{N}_{n-1}$ are dep$(U_n^{(n-2)} + K_n)$. Hence $\mathcal{N}_1 = x_{n-2}x_1a$ and $\mathcal{N}_{n-3} = x_{n-2}ax_1$ span $V_n^{(n-2)}$ modulo $U_n^{(n-2)} + K_n$, and a can be chosen conveniently. Choose $a = b \cdot c$ where $b = x_{n-1}$, $c = x_nx_2 \cdots x_{n-3}$. The substitution $x_1 \to x_1$, $x_2 \to x_{n-2}$, $x_3 \to b$, $x_4 \to c$ induces a one-to-one linear map θ: $V_4 \to V_n$, such that

1″) $\mathcal{N}_1, \mathcal{N}_{n-3} \in \theta(V_4^{(2)})$,
2″) $\theta(U_4^{(2)}) \subseteq U_n^{(n-2)}$,
3″) $\theta(K_4) \subseteq K_n$.

This implies that $\mathcal{N}_1, \mathcal{N}_{n-3}$ are dep$(U_n^{(n-2)} + K_n)$, and therefore $c_{n-2,n} \leqq 1$.

Q.E.D.

PROPOSITION 3.6. *Let* $n \geqq 3$, *then* $c_{n-1,n} \leqq 2$.

To prove this proposition, one needs a further knowledge about the linear relations, modulo $U_4^{(3)} + K_4$, among the monomials of $V_4^{(3)}$. Let us begin with

DEFINITION 3.7. Let $\lambda \in \mathrm{Par}(n)$, $I_\lambda \subseteq V_n$ the corresponding 2-sided minimal ideal with $u_\lambda \in I_\lambda$ its unit element, and let $g \in V_n$. Then gu_λ is called "the component of g in I_λ" and g has a trivial λ-component if $gu_\lambda = 0$.

LEMMA 3.8. *The components of $x_2[x_3, x_4]x_1$ and $x_2[x_4, x_1]x_3$ in $I_{(2,2)}$ are linearly independent.*

PROOF. Let $u = u_{(2,2)}$ be the unit element in $I_{(2,2)}$ and suppose there is an $\alpha \in F$ such that:

$$0 = (x_2[x_3, x_4]x_1)u + \alpha(x_2[x_4, x_1]x_3)u = x_2([x_3, x_4]x_1 + \alpha[x_4, x_1]x_3)u.$$

Hence $J_4 = V_4 \cdot x_2([x_3, x_4]x_1 + \alpha[x_4, x_1]x_3)$ intersect $I_{(2,2)}$ trivially. Since $J_4 \approx V_4 \otimes_{V_3} V_3([x_2, x_3]x_1 + \alpha[x_3, x_1]x_2)$, theorem 3.3 in [3] implies: $J_3 = V_3([x_2, x_3]x_1 + \alpha[x_3, x_1]x_2)$ intersect $I_{(2,1)}$ trivially and is therefore contained in $F \cdot s_3$. This would imply that $J_3 = F \cdot s_3$, an obvious contradiction.

As in §2, let $Q = L \cap R$ and let $P_4 = V_4(x_1[x_2, x_3]x_4)$. The character $\chi(P_4)$ is $[1^4] + 2[2, 1^2] + [2, 2] + [3, 1]$. Since $P_4 \subseteq Q_4$ we have $P_4 \cap I_{(3,1)} = Q_4 \cap I_{(3,1)} = K_4 \cap I_{(3,1)}$. Also, by [4], $K_4 \supseteq P_4 \cap I_{(2,1^2)}$, $P_4 \cap I_{(1^4)}$. We can now prove

LEMMA 3.9. *Let $g \in V_4$ and suppose that $g \cdot u_{(4)} = 0$ and $g \cdot u_{(3,1)} \in K_4$, then there exist $\alpha, \beta \in F$ such that*

$$g + \alpha x_2[x_3, x_4]x_1 + \beta x_2[x_4, x_1]x_3 \in K_4.$$

PROOF. Denote $A_4 = K_4 \cap I_{(2,2)}$ and $B_4 = P_4 \cap I_{(2,2)}$. It was shown in §1 that $A_4 \neq B_4$. Since A_4, B_4 are minimal and $I_{(2,2)}$ has length 2, $I_{(2,2)} = A_4 \oplus B_4$. If $u = u_{(2,2)}$, then $g \cdot u \in A_4 \oplus B_4$, so that $g \cdot u = a_4 + b_4$, $a_4 \in A_4$ and $b_4 \in B_4$. Since $\dim B_4 = 2$, it is spanned over F by $h_4 \cdot u = x_2[x_3, x_4]x_1 \cdot u$ and $h'_4 u = x_2[x_4, x_1]x_3 \cdot u$. Therefore, there are $-\alpha$, $-\beta \in F$ such that $b_4 = -\alpha h_4 u - \beta h'_4 u$ and $gu = a_4 - \alpha h_4 u - \beta h'_4 u$. Hence

$$(g + \alpha h_4 + \beta h'_4)u = a_4 \in K_4 \cap I_{(2,2)},$$

which implies that $g + \alpha h_4 + \beta h'_4 \in K_4$. Q.E.D.

LEMMA 3.10. $[x_1, x_2][x_3, x_4]$ *has no component in $I_{(3,1)}$ (and in $I_{(4)}$).*

PROOF. Let $D_4 = V_4([x_1, x_2]x_3x_4)$, $D'_4 = V_4(x_1x_2[x_3, x_4])$ and $E_4 = V_4([x_1, x_2][x_3, x_4])$. Clearly, $E_4 \subset D_4 \cap D'_4$, so that $E_4 \cap I_{(3,1)} \subseteq (D_4 \cap I_{(3,1)}) \cap (D'_4 \cap I_{(3,1)})$. The character of D_4 (and of D'_4) can be computed by twice applying [3, theor. 3.3], and it is $[1^4] + 2[2, 1^2] + [2, 2] + [3, 1]$, so that $D_4 \cap I_{(3,1)}$ and $D'_4 \cap I_{(3,1)}$ are minimal. This implies that $D_4 \cap I_{(3,1)}$ is generated over V_4 by the idempotent that corresponds to

1	3	4
2		

and $D'_4 \cap I_{(3,1)}$ by

3	1	2
4		

Obviously, these two idempotents are orthogonal, which implies that

$$(D_4 \cap I_{(3,1)}) \cap (D'_4 \cap I_{(3,1)}) = (0).$$

Hence $E_4 \cap I_{(3,1)} = (0)$. Q.E.D.

We can now prove our main lemma:

LEMMA 3.11. *The following monomials in* $V_4^{(3)}$ *are* $\operatorname{dep}(U_4^{(3)} + K_4)$:

1) $x_3x_4x_1x_2$ *and* $x_3x_4x_2x_1$ *are* $\operatorname{dep}(U_4^{(3)} + K_4)$,
2) $x_3x_1x_4x_2$ *and* $x_3x_2x_4x_1$ *are* $\operatorname{dep}(U_4^{(3)} + K_4)$,
3) $x_3x_4x_1x_2$ *and* $x_3x_1x_4x_2$ *are* $\operatorname{dep}(U_4^{(3)} + K_4)$.

PROOF.

1) By Lemmas 3.9 and 3.10 there are $\alpha, \beta \in F$ such that

$$[x_1, x_2][x_3, x_4] + \alpha x_2[x_3, x_4]x_1 + \beta x_2[x_4, x_1]x_3 \in K_4$$

$$= x_1x_2x_3x_4 - x_1x_2x_4x_3 + v_2$$

where $v_2 \in V_4^{(2)}$. Hence $x_1x_2x_3x_4$, $x_1x_2x_4x_3$ are $\operatorname{dep}(V_4^{(2)} + K_4)$. The permutation $(1,3)(2,4)$ applied to the indices—and mapping $V_4^{(2)}$ isomorphically onto $U_4^{(3)} = V_4^{(4)}$—then implies (1).

2) Let

$$g = x_1x_2x_3x_4 - x_1x_4x_3x_2 + x_3x_4x_1x_2 - x_3x_2x_1x_4.$$

Conjugation by the transposition $(2,3)$ induces an automorphism in each I_λ. By a direct computation, $(2,3)g(2,3) = [x_1x_2][x_3, x_4]$ so we conclude that g has no components in $I_{(4)}$ and $I_{(3,1)}$. Therefore there are $\alpha, \beta \in F$ such that $g + \alpha x_3[x_2, x_4]x_1 + \beta x_3[x_4, x_1]x_2 \in K_4$, hence $x_1x_2x_3x_4$ and $x_1x_4x_3x_2$ are $\operatorname{dep}(V_4^{(3)} + K_4)$. Apply the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

to the last statement to conclude that $x_3x_1x_4x_2$ and $x_3x_2x_4x_1$ are $\operatorname{dep}(U_4^{(3)} + K_4)$.

3) Lemma 3.9 implies that there are $\alpha, \beta \in F$ such that

$$x_1[x_2, x_3]x_4 + \alpha x_2[x_3, x_4]x_1 + \beta x_2[x_4, x_1]x_3 \in K_4,$$

which shows that $x_1x_2x_3x_4$ and $x_1x_3x_2x_4$ are dep$(V_4^{(2)} + K_4]$. By applying the permutation $(1,3)(2,4)$ to the indices we obtain (3). Q.E.D.

NOTE. It can be shown that $x_3x_1x_2x_4$ and $x_3x_2x_1x_4$ are indep$(U_4^{(3)} + K_4)$. However, this is unnecessary for the later discussion.

COROLLARY 3.12. *Let*

$$A = \{x_3x_4x_1x_2,\ x_3x_4x_2x_1,\ x_3x_1x_4x_2,\ x_3x_2x_4x_1\}$$

and

$$B = \{x_3x_1x_2x_4,\ x_3x_2x_1x_4\}.$$

Then, for any $\mu_1 \in A$ and $\mu_2 \in B$, $\{\mu_1, \mu_2\}$ is a basis for $V_4^{(3)}$ modulo $U_4^{(3)} + K_4$.

PROOF. We know that $c_{3,4} = 2$ so there are at most two monomials in such a basis. Also, by the previous lemma, every two monomials in A are dep$(U_4^{(3)} + K_4)$. Let $\mu_1 \in A$, $\mu_2 \in B$ and assume they are dep$(U_4^{(3)} + K_4)$. Then $(1,2)\mu_1$ and $(1,2)\mu_2$ are also dep$(U_4^{(3)} + K_4)$. But $(1,2)\mu_1 \in A$, hence μ_1 and $(1,2)\mu_1$ are dep$(U_4^{(3)} + K_4)$, while $\{\mu_2, (1,2)\mu_2\} = B$. It now follows that every two monomials in $V_4^{(3)}$ are dep$(U_4^{(3)} + K_4)$ which would imply $c_{3,4} \leqq 1$, a contradiction. Q.E.D.

We can now turn to the

PROOF OF PROPOSITION 3.6, namely: $c_{n-1,n} \leqq 2$. We use induction on $n \geqq 3$ to show that the two monomials $\mu_1 = x_{n-1}x_1(x_2 \cdots x_{n-2}x_n)$ and $\mu_2 = x_{n-1}(x_2 \cdots x_{n-2}x_2)x_1$ span $V_n^{(n-1)}$ modulo $(U_n^{(n-1)} + K_n)$. If $n = 3$, there is nothing to prove, and the case $n = 4$ is implied by Corollary 3.12.

[NOTE. We shall use, several times, the "substitution argument", similar to that used in the proof of Proposition 3.5; namely, the one-to-one linear map that is induced by some substitution. The reader should check the corresponding properties 1, 2, 3 of such a map.]

Assume that

$$N_1 = x_{n-2}x_1(x_2 \cdots x_{n-3}x_{n-1})$$

and

$$N_2 = x_{n-2}(x_2 \cdots x_{n-3}x_{n-1})x_1$$

span $V_{n-1}^{(n-2)}$ mod$(U_{n-1}^{(n-2)} + K_{n-1})$, $s \leqq n$. For a given $2 \leqq i \leqq n$ consider the set of $(n-2)!$ monomials

$$M^{(i)} = \{M = x_{n-1} \cdots x_i x_1 \cdots\}.$$

As in the beginning of the proof of Proposition 3.5, the same substitution argument—together with the induction hypothesis—imply that

$$x_{n-1}x_2x_3\cdots x_{i-1}(x_ix_1)x_{i+1}\cdots x_{n-2}x_n \quad \text{and} \quad x_{n-2}x_3\cdots(x_ix_1)\cdots x_{n-2}x_nx_2$$

span $M^{(i)} \bmod (U_n^{(n-1)}+K_n)$. The following set of $2(n-1)$ monomials therefore span $V_n^{(n-1)} \bmod (U_n^{(n-1)}+K_n)$:

$$\begin{array}{ll} S_1=(x_{n-1}x_1)x_2(x_3\cdots x_{n-2}x_n) & T_1=(x_{n-1}x_1(x_3\cdots x_{n-2}x_n)x_2 \\ S_2=x_{n-1}(x_2x_1)(x_3\cdots x_{n-2}x_n) & T_2=x_{n-1}(x_3\cdots x_{n-2}x_n)(x_2x_1) \\ S_3=x_{n-1}x_2(x_3x_1)x_4\cdots x_{n-2}x_n & T_3=x_{n-1}(x_3x_1)x_4\cdots x_{n-2}x_nx_2 \\ \vdots & \vdots \\ S_{n-1}=x_{n-1}x_2\cdots x_{n-2}(x_nx_1) & T_{n-1}=x_{n-1}x_3\cdots x_{n-2}(x_nx_1)x_2 \end{array}$$

The case $n=5$. Here the 8 monomials are

$$\begin{array}{ll} S_1=x_4x_1x_2x_3x_5 & T_1=x_4x_1x_3x_5x_2 \\ S_2=x_4x_2x_1x_3x_5 & T_2=x_4x_3x_1x_5x_2 \\ S_3=x_4x_2x_3x_1x_5 & T_3=x_4x_3x_5x_1x_2 \\ S_4=x_4x_2x_3x_5x_1 & T_4=x_4x_3x_5x_2x_1 \end{array}$$

The monomials S_2, S_3, S_4 "contain" x_4x_2. The substitution (check the "substitution argument") $x_1\to x_1$, $x_2\to x_3$, $x_3\to x_4x_2$, $x_4\to x_5$ maps:

$$\bar S_2=x_3x_1x_2x_4\to S_2,$$

$$\bar S_3=x_3x_2x_1x_4\to S_3 \quad \text{and}$$

$$\bar S_4=x_3x_2x_4x_1\to S_4.$$

By Corollary 3.12, $\bar S_3$ and $\bar S_4$ span $\bar S_2 \bmod (U_4^{(3)}+K_4)$, so that S_3 and S_4 span $S_2 \bmod (U_5^{(4)}+K_5)$.

The remaining S_1, S_3, S_4 contain x_2x_3, and a similar substitution argument, based on the substitution $x_1\to x_1$, $x_2\to x_2x_3$, $x_3\to x_4$, $x_4\to x_5$, yields that S_3 is spanned, $\bmod (U_5^{(4)}+K_5)$, by S_1 and S_4.

The substitution $x_1\to x_1$, $x_2\to x_2$, $x_3\to x_4x_3$, $x_4\to x_5$ maps three monomials from A (see Corollary 3.12) to T_2, T_3, T_4. Hence T_4 spans the other two $\bmod (U_5^{(4)}+K_4)$. Write $M\sim N$ if M and N are $\mathrm{dep}(U_5^{(4)}+K_5)$. We are now constantly using Corollary 3.12 and substitution arguments:

$T_1 = x_4(x_1x_3)x_5x_2 \sim x_4x_5x_2(x_1x_3) = T$ (by the substitution argument $x_1 \to x_1x_3$, $x_2 \to x_2$, $x_3 \to x_4$, $x_4 \to x_5$), and

$T_4 = x_4x_3x_5(x_2x_1) \sim x_4x_5(x_2x_1)x_3 = T$ (by the substitution $x_1 \to x_2x_1 \to x_2 \to x_3$, $x_3 \to x_4$, $x_4 \to x_5$).

Finally, $S_4 = x_4(x_2x_3)x_5x_1 \sim x_4x_5(x_2x_3)x_1 = S$ (by $x_1 \to x_1$, $x_2 \to x_2x_3$, $x_3 \to x_4$, $x_4 \to x_5$) and

$S \sim x_4(x_5x_2)x_1x_3 = T$ ($x_1 \to x_1$, $x_2 \to x_3$, $x_3 \to x_5$, $x_4 \to x_5x_2$). The conclusion is that S_4 spans T_1 and $T_4 \bmod (U_5^{(4)} + K_5)$; hence S_1 and S_4 span $V_5^{(4)} \bmod (U_5^{(4)} + K_5)$, as was to be shown.

The case $6 \leqq n$. Again, consider the $2n-2$ monomials $S_1, \cdots, S_{n-1}$, $T_1, \cdots, T_{n-1}$.

Induction and the substitution $x_1 \to x_1$, $x_2 \to x_2x_3$, $x_i \to x_{i+1}$ if $3 \leqq i \leqq n-1$ imply that S_3 is spanned $\bmod (U_n^{(n-1)} + K_4)$ by S_1 and S_{n-1}.

Induction and the substitution $x_1 \to x_1$, $x_j \to x_{j+1}$ if $2 \leqq j \leqq n-2$, $x_{n-1} \to x_nx_2$ imply that T_3 is spanned by T_1 and $T_2 \bmod (U_n^{(n-1)} + K_n)$.

We can thus erase S_3 and T_3 and the remaining $2n-4$ monomials span $V_n^{(n-1)} \bmod (U_n^{(n-1)} + K_n)$. Moreover, each of these contain x_3x_4. Since $4 \lneqq n-1$, we can use the substitution $x_1 \to x_1$, $x_2 \to x_2$, $x_3 \to x_3x_4$, $x_i \to x_{i+1}$ if $4 \leqq i \leqq n-1$ which, together with the induction hypothesis imply that S_1 and S_{n-1} span the other monomials, and therefore span $V_n^{(n-1)} \bmod (U_n^{(n-1)} + K_n)$. The proof of Proposition 3.6 is now completed. Q.E.D.

THEOREM 3.13. *Let* $\{c_n\}$ *be the codimensions of* $K = T_0(s_3)$; *then* $c_n \leqq n(n+1)/2 - 1$ *for* $n \geqq 2$.

PROOF. By induction on n. If $n = 2, 3, 4$, equality holds. By Propositions 3.5 and 3.6, $c_n \leqq 1 + \cdots + 1 + 2 + c_{n,n} \leqq 1 + \cdots + 1 + 2 + c_{n-1} = n + c_{n-1} \leqq n + (n-1)n/2 - 1 = n(n+1)/2 - 1$. Q.E.D.

§4. The structure of $K = T_0(s_3)$.

We combine now the results of §§2, 3 together with an unpublished result by R. Rasala, [5], to obtain a close estimate of the structure of $K = T_0(s_3)$.

LEMMA 4.1. *Let* $n \geqq 4$, *then*

a) $K_n \supseteq I_{(1^n)}$,

b) $K_n \supseteq I_{(2,1^{n-2})}$.

PROOF. Part (a) is trivial since $I_{(1^n)} = F \cdot s_n[x]$, where $s_n = s_n[x_1, \cdots, x_n]$ is the standard polynomial of degree n.

Part (b) is proved by induction on $n \geqq 4$. If $n = 4$, this is shown in [4]. Let $n + 1 \geqq 5$ and assume $K_n \supseteq I_{(1^n)} \oplus I_{(2,1^{n-1})}$. By [3, theor. 3.3], $V_{n+1}K_n x_{n+1} \cap I_{(2,1^{n-1})}$ has length at least $(n-1)+1 = n$. Since n is the length of $I_{(2,1^{n-1})}$, it follows that $K_{n+1} \supseteq V_{n+1}K_n x_{n+1} \supseteq I_{(2,1^{n-1})}$. Q.E.D.

COROLLARY 4.2. *Let $n \geqq 4$, $\mu \in \mathrm{Par}(n)$ and suppose $I_\mu \cap K_n \subsetneqq I_\mu \cap Q_n$, then* $(2, 1^{n-2}) \underset{\neq}{<} \mu \underset{\neq}{<} (n-1, 1)$.

PROOF. Follows directly from Proposition 2.5, Lemma 4.1 and the fact that $\lambda = (n)$ does not occur in K_n nor in Q_n.

DEFINITION. For a given n, consider the dimensions of the minimal left ideals in $V_n = F[S_n]$ and write them in an ascending order: $a_n^{(1)} < a_n^{(2)} < \cdots$, thus obtaining the infinite sequences $\{a_n^{(1)}\}_{n=1}^{\infty}$, $\{a_n^{(2)}\}_{n=1}^{\infty}, \cdots$.

NOTE. Obviously, $a_n^{(1)} = 1$ for all n. It was shown in [2] that, except for $n = 4$, $a_n^{(2)} = n - 1$. Moreover, if $n \geqq 7$, the only partitions that yield dimension $n - 1$ are $(n-1, 1)$ and $(2, 1^{n-2})$. Recently, the following result, among others, was obtained by R. Rasala [5].

THEOREM 4.3 (Rasala). *Let $n \geqq 9$, then $a_n^{(3)} = n(n-3)/2$ and the only partitions that yield this dimension are $(n-2, 2)$ and $(2^2, 1^{n-4})$.*

We can now prove our main result, namely:

THEOREM 4.4. *Let χ_n be the n-th co-character of K and let $n \geqq 9$; then*

$$\chi_n = [n] + 2[n-1, 1] + \alpha[n-2, 2] + \beta[2^2, 1^{n-4}], \quad \alpha + \beta \leqq 1.$$

PROOF. We have already found three components in the co-character χ_n: one in $I_{(n)}$ and exactly two in $I_{(n-1,1)}$. If $K_n = Q_n$, we are done: $\alpha = \beta = 0$. Assume $K_n \neq Q_n$. Hence there is a $\mu \in \mathrm{Par}(n)$ such that $I_\mu \cap K_n \subsetneqq I_\mu \cap Q_n$, and by Corollary 4.2, $(2, 1^{n-2}) \underset{\neq}{<} \mu \underset{\neq}{<} (n-1, 1)$. Theorem 4.3 then implies that $\dim(Q_n \cap I_\mu) - \dim(K_n \cap I_\mu) \geqq n(n-3)/2$, so that $\dim Q_n - \dim K_n \geqq n(n-3)/2$. Since $K_n \subset Q_n$ and $n! - \dim Q_n = 2n - 1$ it follows that

$$n! - \dim K_n \geqq 2n - 1 + n(n-3)/2 = n(n+1)/2 - 1.$$

In other words, $c_n = \operatorname{codim} K_n \geqq n(n+1)/2 - 1$. By Theorem 3.13, $c_n = n(n+1)/2 - 1$.

Moreover, by applying Theorem 4.3 again, we obtain the statement about the co-character χ_n. Q.E.D.

COROLLARY 4.5. *Let $\{l_n\}$ be the co-lengths, $\{c_n\}$ the codimensions of K and let $n \geqq 9$. Then $l_n = 3$ or $l_n = 4$, $c_n = 2n - 1$ or $c_n = n(n+1)/2 - 1$.*

PROOF. Is obviously included in that of Theorem 4.4: If $K_n = Q_n$, then $l_n = 3$ and $c_n = 2n - 1$. If $K_n \subsetneqq Q_n$, then $l_n = 4$ and $c_n = n(n+1)/2 - 1$.

We end this paper with two conjectures.

CONJECTURE 1. For $n \geqq 4$

$$\chi_n = \chi_n(K) = [n] + 2[n-1,1] + [n-2,2]$$

(which implies that $c_n(K) = n(n+1)/2 - 1$ and $l_n = 4$).

NOTE. It can be shown that $c_n(K) = n(n+1)/2 - 1$ for $n \geqq 4$ if and only if a generalized form of the cancellation theorem (Theorem 1.1) holds. We therefore make

CONJECTURE 2. Let $d \geqq 2$ be any integer and $n \geqq d$. If

$$f(x_1, \cdots, x_n) \cdot x_{n+1} \in T_0(s_d[x_1, \cdots, x_d])$$

then

$$f(x_1, \cdots, x_n) \in T_0(s_d[x_1, \cdots, x_d]).$$

The same holds for $x_{n+1} f(x_1, \cdots, x_n)$.

Added in proof. Recently we obtained the following results: Call

$$d_h[x;y] = d_h[x_1, \cdots, x_h;\, y_1, \cdots, y_{h-1}] = \sum_{\sigma \in S_h} (-1)^{\sigma} x_{\sigma_1} y_1 x_{\sigma_2} y_2 \cdots y_{h-1} x_{\sigma_h}$$

the "Capelli" polynomial of (x) degree h.

If $\lambda = (a_1, \cdots, a_h) \in \mathrm{Par}(n)$ and $a_1 \geqq \cdots \geqq a_h \neq 0$, then $h = h(\lambda)$ is the "height" of λ (and $D(\lambda)$).

THEOREM 1. *Assume the T-ideal Q contains $d_h[x;y]$. Then for every n, $Q_n \supseteq I_\lambda$ for all $\lambda \in \mathrm{Par}(n)$ such that $h(\lambda) \geqq h$.*

THEOREM 2. *Indeed: $d_3[x;y] \in T_0(s_3[x]) = K$ (and similar results were obtained for $T_0(s_i[x])$, $2 \leqq i \leqq 7$).*

As a result, $[2^2, 1^{n-4}]$ is eliminated from $\chi_n(K)$ for $n \geqq 5$, since $h(2^2, 1^{n-4}) = n - 2$.

Details will appear elsewhere.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
CHICAGO, ILL. 60637 USA

AND

THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT, ISRAEL