COCYCLES AND THE STRUCTURE OF ERGODIC GROUP ACTIONS

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ABSTRACT

We show, for a large class of groups, the existence of cocycles taking values in these groups and which define ergodic skew products. We apply this to prove a generalization of Ambrose's representation theorem for ergodic actions of these groups.

1. The concept of a flow built under a function was introduced into ergodic theory by von Neumann in [9]. The significance of this concept was demonstrated by Ambrose [1] when he showed that every ergodic flow with an invariant probability measure could be represented as a flow built under a function. In a variety of situations, this allows one to reduce questions about flows to questions about automorphisms. Ambrose's theorem, as well as the very concept of a flow built under a function, was put in a new light by Mackey when he introduced the idea of looking at ergodic theory as a theory of "virtual subgroups" in order to bring out certain analogies between ergodic theory and group theory [8]. In particular, Mackey defined the notion of the range of a cocycle, or homomorphism, of an ergodic action into a locally compact group, which includes as a special case flows built under functions. In this context, Ambrose's theorem implies that every ergodic flow is the range of a cocycle of an action of the integers, and Mackey then raised the question: For which locally compact groups is it true that every ergodic action is the range of a cocycle of an automorphism? This question was considered by Forrest in [5], where he describes a class of groups for which this is true, and which includes $Z^k \times \mathbf{R}^n$. In this paper, we approach the question with different, perhaps simpler, techniques, and show that Mackey's question has a positive answer for a class of groups which includes (among others; see section 3 for details) $Z^k \times \mathbf{R}^n$, recurrent groups, and connected nilpotent lie groups. Section 2 below reduces the general

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problem to constructing cocycles into groups whose range is the whole group, and this problem is then examined in section 3.

2. We begin by recalling Mackey's construction of the range [8]. Let G be a locally compact second countable group and S a standard Borel space. Suppose there is a right Borel action of G on S and a probability measure μ on S quasi-invariant and ergodic under the G-action. If H is another locally compact (second countable) group a Borel function $\alpha: S \times G \to H$ is called a cocycle if for all $g, h \in G$, $\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$ for almost all s. One can then define an action of $G \times H$ on $S \times H$ given (almost everywhere; see [10, section 3]) by $(s, h)(g, k) = (sg, k^{-1}h\alpha(s, g))$. This action preserves the measure class of $\mu \times \mu_H$, where μ_H is Haar measure on H, and this measure is ergodic under the $G \times H$ action. It is also possible that $S \times H$ is actually ergodic under the action of $G \times \{e\}$. In this case, we say that α has range H. In any event, we can let $B_0 \subset B(S \times H)$ be the Boolean σ-algebra of Borel sets modulo null sets that are invariant under $G \times \{e\}$. Then B_0 will be a Boolean H-space and hence isomorphic to $B(X)$ for some (essentially unique) H-space X [7]. The H-space X is called the range of α and generalizes the construction of flow built under a function [8, section 6].

Our aim in this section is to prove the following theorem.

THEOREM 2.1. *i) If there is a cocycle $\alpha: S \times G \to H$ with range H, then for any ergodic H-space X, there is an ergodic G-space Y and cocycle $\beta: Y \times G \to H$ with range X.*

ii) Suppose further that S can be chosen to have a finite invariant measure. Then if X has a finite (σ-finite) invariant measure, we can choose Y so that it will also have a finite (σ-finite) invariant measure.

We will need the following lemma, which we shall also use in section 3.

LEMMA 2.2. *Suppose $\alpha: S \times G \to H$ is a cocycle with range H. Let X be an ergodic H-space. Define an action of G on $S \times X$ by $(s, x)g = (sg, x\alpha(s, g))$. Then $S \times X$ is an ergodic G-space with the product measure. (We will denote this G-space by $S \times_\alpha X$.)*

PROOF. (We remark first that $(s, x)g$ doesn't quite define an action but a near action [10, def. 3.1]. However, there is an action which agrees with this a.e. for each $g \in G$ [10, prop. 3.2].)

Suppose that there is a G-invariant Borel set $A \subset S \times X$. For $s \in S$, let $A_s = \{x \in X \mid (s, x) \in A\}$. Let $U: H \to U(L^2(X))$ be the natural induced unitary representation of H on $L^2(X)$. For each Borel set $E \subset X$, let P_E be the projection operator in $L^2(X)$ defined by multiplication by the characteristic function of E, and denote P_{A_s} by P_s. Then $s \to P_s$ is a Borel map of S into $L(L^2(X))$ where the latter has the weak Borel structure. G-invariance of A implies that for each $g \in G$, $A_s \cdot \alpha(s,g) = A_{sg}$ modulo null sets in X for almost all s. Hence for each g, $U(\alpha(s,g))^{-1}P_sU(s,g) = P_{sg}$ for almost all s. Define $\theta: S \times H \to L(L^2(X))$ by $\theta(s,h) = U(h)P_sU(h)^{-1}$. Then θ is Borel. Furthermore, for each $g \in G$, $h \in H$,

$$\begin{aligned}\theta(sg, h\alpha(s,g)) &= U(h)U(\alpha(s,g))P_{sg}U(\alpha(s,g))^{-1}U(h)^{-1}\\ &= U(h)P_sU(h)^{-1}\\ &= \theta(s,h) \quad \text{for almost all } s.\end{aligned}$$

Thus, θ is an essentially G-invariant Borel map and since $L(L^2(X))$ is countably separated, θ must be essentially constant by the ergodicity of G on $S \times H$. Thus, for some h and almost all (s,t),

$$U(h)P_sU(h)^{-1} = U(h)P_tU(h)^{-1}, \quad \text{so} \quad P_s = P_t.$$

Hence we can write $P_s = P$ for almost all s, and we have for each g, $U(\alpha(s,g))^{-1}PU(\alpha(s,g)) = P$ for almost all s. Let $H_0 = \{h \in H \mid U(h)^{-1}PU(h) = P\}$. Then $H_0 \subset H$ is a closed subgroup and by changing α on a null set, we see that α is equivalent to a cocycle into H_0. Since $S \times_\alpha H$ is ergodic, $H_0 = H$. Since H acts ergodically on X, $P = I$ or $P = 0$, which implies that A is either null or conull. This proves the lemma.

We now proceed to the proof of Theorem 2.1.

PROOF (OF THEOREM 2.1). Let $Y = S \times_\alpha X$, which by the preceding lemma is ergodic. Define β to be the "restriction" of α to $S \times X$ (cf. [10, example 2.7]), i.e. $\beta(s,x,g) = \alpha(s,g)$. Thus, $\beta: Y \times G \to H$ is a cocycle. We claim that the range of β is X.

Consider first the action of $H \times H$ on $X \times H$ defined by $(x,h)\cdot(m,k) = (x\cdot m, k^{-1}hm)$. Let $q: X \times H \to X$ be $q(x,h) = xh^{-1}$. Then q is invariant under $H \times \{e\}$ and for each $z \in X$, $H \times \{e\}$ is transitive on $q^{-1}(z)$. Hence, $q: X \times H \to X$ provides an ergodic decomposition of the $H \times \{e\}$ action on $X \times H$. Decompose the product measure on $X \times H$ over the fibers of q. This yields a Borel field of measures on $X \times H$, $z \to \mu_z$, $z \in X$, such that μ_z is supported on $q^{-1}(z)$ and $(X \times H, \mu_z)$ is ergodic under $H \times \{e\}$. Furthermore, the Boolean subalgebra of $H \times \{e\}$ invariant sets in $B(X \times H)$ is just $q^*(B(X))$, where $q^*: B(X) \to B(X \times H)$ is the induced map, and the ergodic $\{e\} \times H$ action on $q^*(B(X))$ is isomorphic to the given action of H on $B(X)$.

Now form the $G \times H$ space $Y \times_\beta H = S \times X \times H$. The $G \times H$ action is defined (almost everywhere) by

$$(*) \qquad (s, x, h)(g, k) = (sg, x\alpha(s, g), k^{-1}h\alpha(s, g)).$$

Suppose $E \subset S \times X \times H$ is a Borel set essentially invariant under the Boolean $G \times \{e\}$ action. Let $\phi: S \times X \times H \to X \times H$ be projection and $p = q \circ \phi$. Then the product measure on $S \times X \times H$ can be decomposed over the fibers of p as $\int_X^{\oplus} (\mu \times \mu_z) dz$. For each $z \in X$, $(*)$ also defines a near action of $G \times \{e\}$ on $(S \times X \times H, \mu \times \mu_z)$ and the Boolean action it defines is ergodic by Lemma 2.2, since $(X \times H, \mu_z)$ is an ergodic H-space. For $z \in X$, let $E_z = E \cap p^{-1}\{z\}$. Since E is essentially invariant, for each g we will have $E_z \cdot g = E_z$ modulo μ_z null sets for almost all z. Then for $G_0 \subset G$ a countable dense subgroup, we will have on a conull set of z, $E_z \cdot g = E_z \bmod 0$ for all $g \in G_0$. Thus $E_z \cdot g = E_z$ in $B(S \times X \times H, \mu \times \mu_z)$ for all $g \in G_0$, which implies by the ergodicity of $\mu \times \mu_z$ that E_z is either μ_z-null or μ_z-conull for almost all z. This implies that $E \in p^*(B(X)) \subset B(S \times X \times H)$. Conversely, since p is essentially G-invariant, every element of $p^*(B(X))$ is G-invariant. Thus, for the $G \times H$ space $Y \times_\beta H = S \times X \times H$, the Boolean subalgebra of $G \times \{e\}$ invariant elements is precisely $p^*(B(X)) = \phi^*(q^*(B(X)))$. Since $\phi^*: B(X \times H) \to B(S \times X \times H)$ is a Boolean H-map, the range of β can be identified with the H-space defined by the Boolean H-space $q^*(B(X))$, which is just X. Finally, it follows directly from the construction that if S and X have finite (or σ-finite) invariant measures, so does Y. This completes the proof.

Remarks. i) If the H-space X is the range of a cocycle of a Lebesgue Z-space (i.e. a Z-space with an invariant probability measure) then it is the range of a cocycle of any Lebesgue Z-space. This follows from the Dye–Belinskaya theorem. Namely, suppose S and T are Lebesgue Z-spaces. Then (perhaps by passing to invariant conull sets) there is an orbit preserving measure preserving map $\phi: S \to T$ [2]. Thus, there is a function $\theta: S \times Z \to Z$ such that $\phi(s \cdot n) = \phi(s) \cdot \theta(s, n)$. It is straightforward to see that θ is a cocycle and this imples that if $\alpha: T \times Z \to H$ is a strict cocycle, so is $\beta: S \times Z \to H$ defined by $\beta(s, n) = \alpha(\phi(s), \theta(s, n))$. Now define $\psi: S \times_\beta H \to T \times_\alpha H$ by $\psi(s, h) = (\phi(s), h)$. It is easy to check that ψ is an orbit preserving isomorphism of the $Z \times \{e\}$ actions and hence ψ^* is an isomorphism of the $Z \times \{e\}$-invariant Boolean algebras. Since ψ is clearly an H-map, the range of β will equal the range of α.

ii) An argument similar to the proof of Theorem 2.1, using Ambrose's theorem as in [11, theor. 2], shows that if X is the range of a cocycle on a Lebesgue Z-space, it is also the range of a cocycle on an arbitrary Lebesgue $\mathbf{R}$-space.

3. We now turn to the problem of constructing cocycles whose range is a given locally compact group. We shall be interested primarily in Lebesgue Z-actions. In [11], it was shown that every compact group is the range of a cocycle of such an action. The same technique works for a larger class of groups, namely the recurrent groups, by which we mean those groups admitting a non-singular recurrent random walk. (Non-singular means that some power of the law of the random walk has an absolutely continuous component with respect to Haar measure.)

THEOREM 3.1. *Let H be a recurrent group. Then for any ergodic Lebesgue Z-space S there is a cocycle on S whose range is H.*

PROOF. The construction and proof of Theorem 3 of [11] apply for any locally compact group. Since the sample sequence space of a non-singular recurrent Markov process is ergodic (see [6], e.g.), the general result follows from the remarks at the end of section 2.

REMARKS. i) Among the groups known to be recurrent are countable Abelian groups with rank $\leqq 2$ [4] (in particular Z and Z^2), $\mathbf{R}, \mathbf{R}^2$, and the group of displacements of the plane [3].

ii) Theorems 2.1 and 3.1 provide a proof of a weak form of Ambrose's theorem. ("Weak" in the sense that Mackey's range construction is somewhat more general than the flow built under a function construction; see [8, section 6].)

We shall now describe some inductive procedures for constructing cocycles "onto" large groups given cocycles "onto" small groups. We will need the following lemma.

LEMMA 3.2. *Suppose S is an ergodic G-space, X an ergodic H-space, $\alpha: S \times G \to H$ has range H and $\beta: X \times H \to L$ has range L. Then there is a G-space Y and a cocycle $\gamma: Y \times G \to L$ with range L. If S and X have finite (or σ-finite) invariant measures, Y can be chosen with this property as well.*

PROOF. Let $Y = S \times_\alpha X$, i.e. the G-space defined by $(s, x)g = (sg, x\alpha(s, g))$. Then Y is ergodic by Lemma 2.2. Define $\gamma: Y \times G \to L$ by $\gamma(s, x, g) =$

$\beta(x, \alpha(s,g))$. Then G acts on $Y \times_\gamma L = S \times X \times L$ by $(s,x,1)g = (sg, x\alpha(s,g), 1\beta(x, \alpha(s,g)))$. However, we can also consider this action as $S \times_\alpha (X \times_\beta L)$, and since $X \times_\beta L$ is an ergodic H-space, and α has range H, the result follows by another application of Lemma 2.2. (The final statement of Lemma 3.2 is immediate from the construction.)

THEOREM 3.3. *Suppose S is a Lebesgue Z-space and $\alpha: S \times Z \to H$, $\beta: S \times Z \to L$ are cocycles with range H and L respectively. Then there is a Lebesgue Z-space Y and a cocycle $\gamma: Y \times Z \to H \times L$ with range $H \times L$.*

PROOF. $S \times S$ is an ergodic Z^2-space under the action $(s,t)\cdot(n,p) = (sn, tp)$. It is straightforward to check that the function $\theta: S \times S \times Z^2 \to H \times L$, $\theta(s,t,n,p) = (\alpha(s,n), \ \beta(t,p))$ is a cocycle and that the Z^2-space $(S \times S) \times_\theta (H \times L)$ is isomorphic to the product Z^2-space $(S \times_\alpha H) \times (S \times_\beta L)$, and hence is ergodic. Thus, $H \times L$ is the range of a cocycle of an ergodic Lebesgue Z^2-space and the theorem follows from Lemma 3.2 and (the remarks following) Theorem 3.1.

COROLLARY 3.4. *For any Lebesgue Z- or $\mathbf{R}$- space, there is a cocycle with range $Z^k \times \mathbf{R}^n \times K$, where K is compact.*

As a consequence of Corollary 3.4 and Theorem 2.1, we obtain the theorem of P. Forrest referred to in the introduction. We also note that Lemma 3.2 implies the following.

PROPOSITION 3.5. *Suppose $D \subset H$ is a closed subgroup such that H/D has finite invariant measure. If there is a cocycle on a Lebesgue Z-space with range H, then D has this property as well.*

PROOF. Let $s: H/D \to H$ be a section of the natural projection. Then it is straightforward to check that $\alpha: H/D \times H \to D$ defined by $\alpha(x,h) = s(x)h\, s(xh)^{-1}$ is a cocycle with range D. Lemma 3.2 now implies the result.

By modifying the argument of Theorem 3.3, we obtain the following result which will enable us to construct cocycles whose ranges are nilpotent groups.

THEOREM 3.6. *Suppose H is a locally compact group and $N \subset H$ is a closed subgroup contained in the center of H. If S is a Lebesgue Z-space and $\alpha: S \times Z \to N$ and $\beta: S \times Z \to H/N$ have range N and H/N respectively, then there is a Lebesgue Z-space Y and a cocycle $\gamma: Y \times Z \to H$ with range H.*

PROOF. Let $s: H/N \to H$ be a Borel section of the natural projection p. Let $\gamma: S \times Z \to H$ be a cocycle such that $p \cdot \gamma = \beta$. Define $\theta: S \times S \times Z^2 \to H$ by

$\theta(s,t,n,p)=\alpha(s,n)\gamma(t,p)$. Since N is in the center of H, it is easy to check that θ is a cocycle. Form the Z^2 space $(S\times S)\times_\theta H$. We can define a Borel isomorphism $H\to N\times H/N$ by $x\to(s(p(x))^{-1}x,[x])$. Since the range of α is N, it is easy to see that the space of $Z\times\{0\}$ ergodic pieces of $S\times S\times H$ under the restriction of the Z^2 action is precisely $S\times H/N$. But $\{0\}\times Z$ acts ergodically on this space since β has range H/N, and it follows that the Z^2 action on $S\times S\times_\theta H$ is ergodic. The theorem again follows from Lemma 3.2 and Theorem 3.1.

Corollary 3.7. *Every connected nilpotent lie group is the range of a cocycle on a Lebesgue Z or* **R**-*space.*

Proof. This follows from Theorem 3.6 and Corollary 3.4 via an induction on the dimension of the group.

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