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Disproof of a Conjecture in the Domination Theory

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Abstract. In [1] C. Barefoot, F. Harary and K. Jones conjectured that for cubic graphs with connectivity three the difference between the domination and independent domination numbers is at most one. We disprove this conjecture and give an exhaustive answer to the question: "What is the difference between the domination and independent domination numbers for cubic graphs with given connectivity?"

We basically follow the standard terminology of [2]. A set S of vertices in a graph G is a dominating set if every vertex v not in S is adjacent to at least one vertex in S. A set I of vertices is an independent set if no two vertices in I are adjacent. A set D of vertices is an independent dominating set if D is both a dominating set and independent set. The domination number, $\alpha(G)$, is the smallest number of vertices in a dominating set of G. Similarly, the independent domination number, $\alpha'(G)$, is the smallest number of vertices in an independent dominating set of G. Clearly $\alpha' - \alpha \geq 0$ for all graphs.

We solve the following problem that was addressed in [1, 3]:

Problem. *For any $c \in \{0, 1, 2, 3\}$ determine whether the difference $\alpha' - \alpha$ can be arbitrary large for cubic graphs with connectivity c. (We remind the reader that the connectivity of any cubic graph does not exceed 3).*

This problem is solved in the affirmative for $c = 0$ (trivially), $c = 1$ ([3]) and $c = 2$ ([1]). The case $c = 3$ is more complicated. It is known that there exist cubic graphs with connectivity three for which $\alpha' - \alpha = 1$ ($K_{3,3}$ and the prism $C_k \times K_2$, where $k = 5 \pmod{12}$ [3]).

Conjecture [1]. *There are no any cubic graphs with connectivity three for which $\alpha' - \alpha > 1$.*

In theorem 1 we give an exhaustive answer to the question: "What is the difference between the domination and independent domination numbers for cubic graphs with given connectivity?" In particular we disprove the conjecture above. The minimal counterexample $G(3, 2, 1)$ which we construct has 92 vertices.

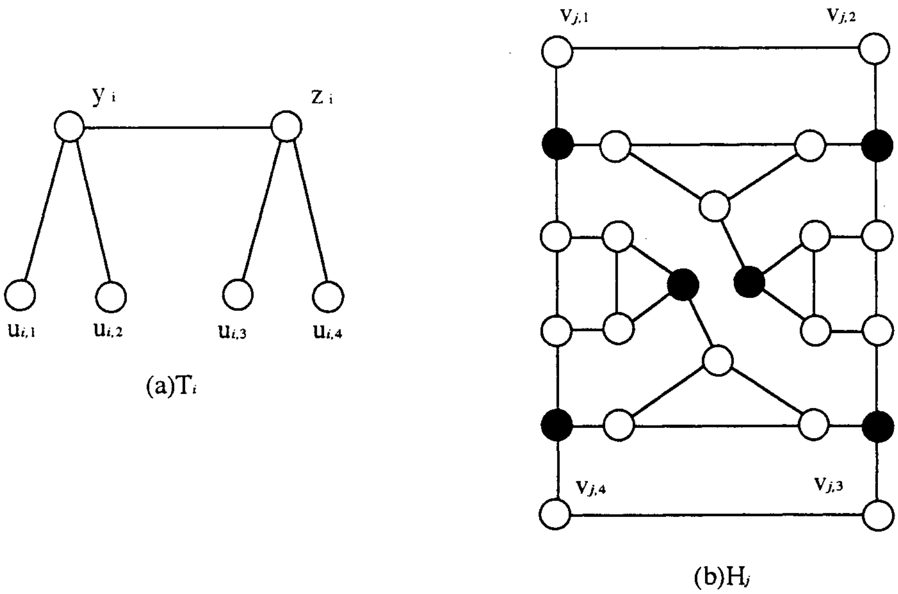

Fig. 1

Proposition 1. *Let G be a cubic graph for which there exists a partition of its vertex set*

$$VG = V_1 \cup \cdots \cup V_k \cup W_1 \cup \cdots \cup W_l \tag{1}$$

such that V_i induces T_i (Fig. 1a), *$i = 1, \ldots, k$, and $G(W_j)$ has a spanning subgraph H_j (Fig.* 1b), *$j = 1, \ldots, l$. Then $\alpha(G) \leq 2k + 6l$* and *$\alpha'(G) \leq 3k + 6l$.*

Proof. It is clear that $\{y_i, z_i\}$ is a dominating set of T_i. The set of 6 black vertices in Figure 1b is a dominating set of H_j. The union of these sets taken over all $i = 1, \ldots, k$ and $j = 1, \ldots, l$ gives a dominating set of G. Hence $\alpha(G) \leq 2k + 6l$.

If we replace $\{y_i, z_i\}$ by $\{u_{i,1}, u_{i,2}, z_i\}$ in this construction, then we obtain a set which contains an independent dominating set of G. So $\alpha'(G) \leq 3k + 6l$. □

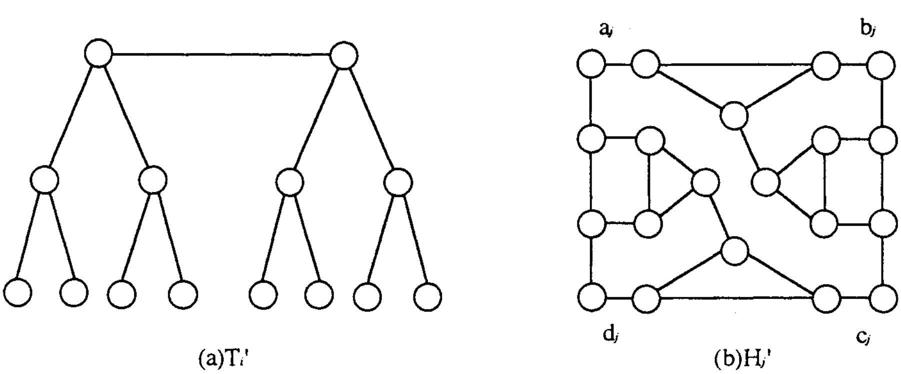

Fig. 2

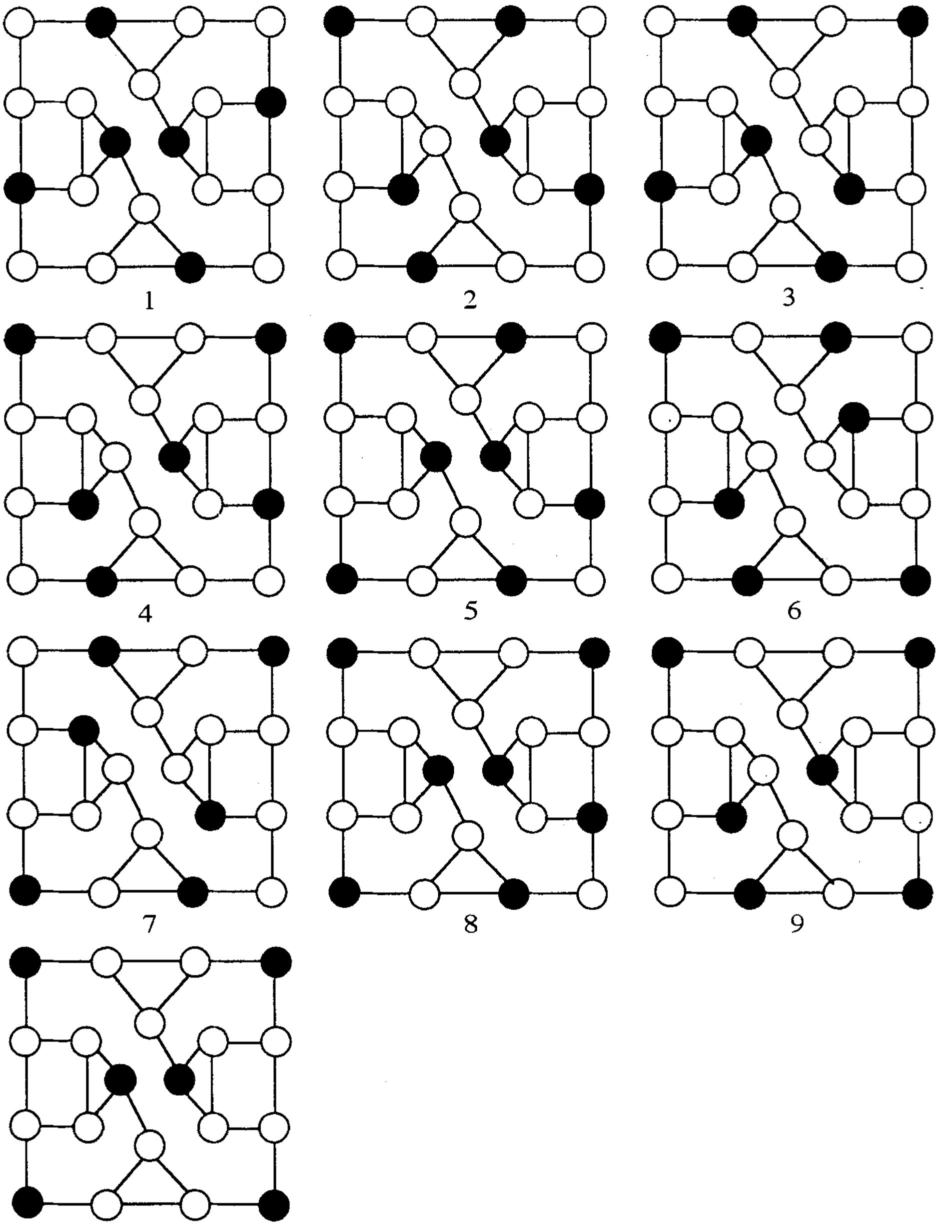

Fig. 3

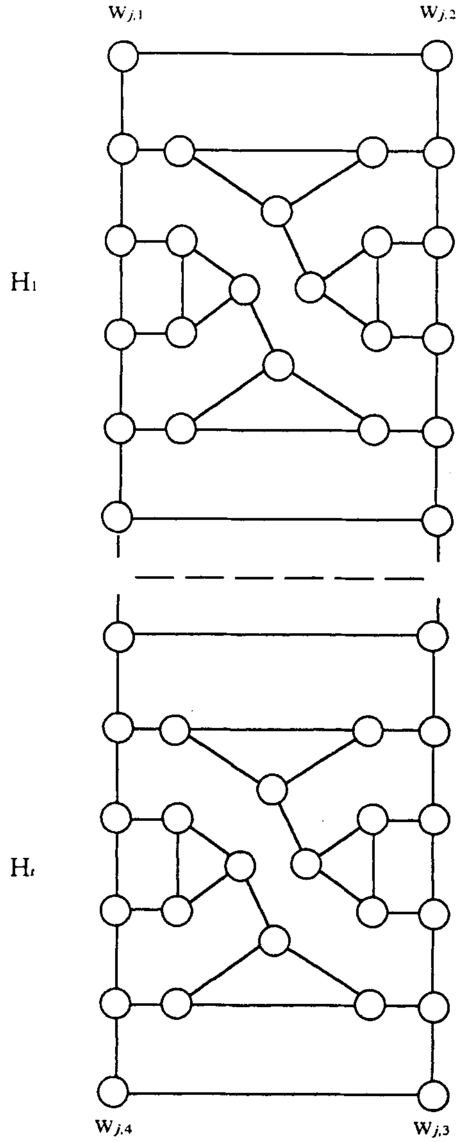

Fig. 4. The graph H_j'

Proposition 2. *Let G be a cubic graph for which there exists a partition of its vertex set*

$$VG = X_1 \cup \cdots \cup X_k \cup Y_1 \cup \cdots \cup Y_l \cup Z \tag{2}$$

such that $G(X_i)$ *has a spanning subgraph* T_i' *(Fig.* 2a), $i = 1, \ldots, k$, $G(Y_j)$ *is isomorphic to* H_j' *(Fig.* 2b), $j = 1, \ldots, l$, *and Z may be empty. Then* $\alpha(G) \geq 2k + 6l$ *and* $\alpha'(G) \geq 3k + 6l$.

Proof. Let S and D be a minimum dominating set and a minimum independent dominating set of G respectively. Obviously $|S \cap X_i| \geq 2$ and $|D \cap X_i| \geq 3$.

Now we show that $|S \cap Y_j| \geq 6$ and hence $|D \cap Y_j| \geq 6$. Put $A = \{a_j, b_j, c_j, d_j\}$ (Fig. 2b). There are 10 non-symmetric cases for $S \cap A$ (Fig. 3, where white vertices do not belong to S and black vertices do). Since A separates the vertices of $Y_j - A$ from $VG - Y_j$, we can easily find the cardinality of $S \cap (Y_j - A)$. A possible variant of the intersection $S \cap (Y_j - A)$ is shown in Fig. 3 for any case 1–10. It is easily checked that $|S \cap Y_j| \geq 6$ for any case 1–10.

Thus, $\alpha(G) = |S| = |S \cap X_i|k + |S \cap Y_j|l + |S \cap Z| \geq 2k + 6l$ and $\alpha'(G) = |D| = |D \cap X_i|k + |D \cap Y_j|l + |D \cap Z| \geq 3k + 6l$. □

Theorem 1. *For any $c \in \{0, 1, 2, 3\}$ and any integer $k \geq 0$ there exist infinitely many cubic graphs with connectivity c for which $\alpha' - \alpha = k$.*

Proof. At first we define a graph H^t, $t \geq 2$, as a union of t disjoint copies $H_1, \ldots, H_t$ of H_j (Fig. 1b) and edges of the form $v_{i,3}v_{i+1,2}$, $v_{i,4}v_{i+1,1}$ $(i = 1, \ldots, t-1)$. For $t = 1$ we put $H^1 = H_1$. The vertices of degree 2 in a copy H_j^t of H^t we denote by $w_{j,1}$, $w_{j,2}$, $w_{j,3}$, $w_{j,4}$ (Fig. 4).

Now we construct a cubic graph $G(c, k, t)$ where $c \in \{0, 1, 2, 3\}$, $k \geq 0$ and $t \geq 1$. For $k \geq 2$ we take k disjoint copies $T_1, \ldots, T_k$ of T_i (Fig. 1a), $2k$ disjoint copies $H_1^t, \ldots, H_{2k}^t$ of H_j^t and extra edge set E_c of the form

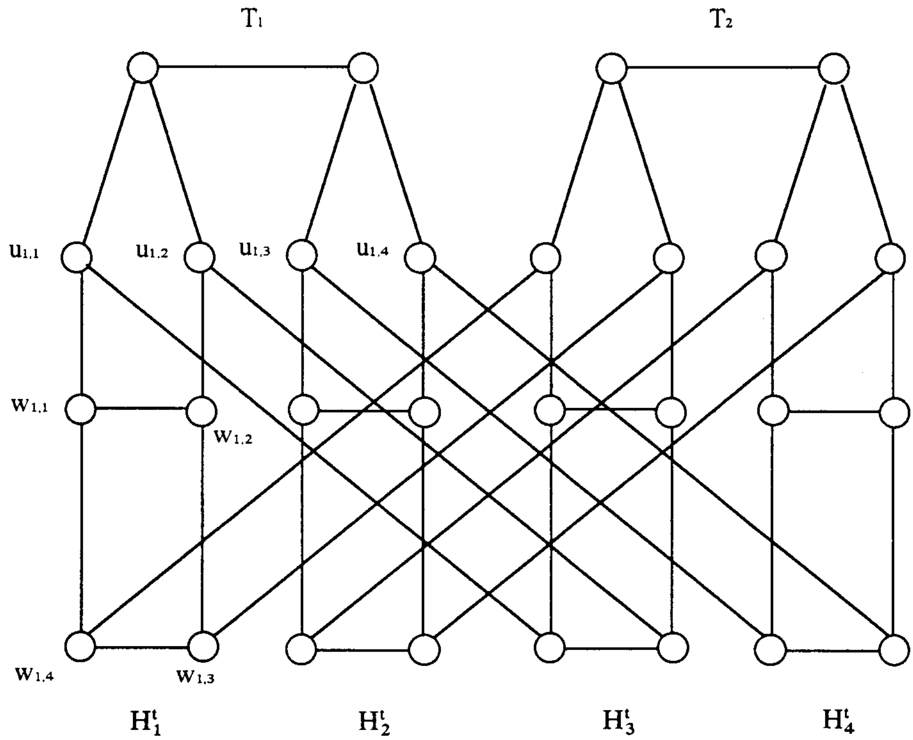

Fig. 5. The graph $G(3, 2, t)$

$1 \le i \le k, j = 1, 2$	E_0	E_2	E_3
$u_{i,1}$ is adjacent to	$w_{2i-1,j}$	$w_{2i-1,j}$	$w_{2i-1,1}$, $w_{2i+1,4}$
$u_{i,2}$ is adjacent to	$w_{2i-1,j+2}$	$w_{2i,j+2}$	$w_{2i-1,2}$, $w_{2i+1,3}$
$u_{i,3}$ is adjacent to	$w_{2i,j}$	$w_{2i,j}$	$w_{2i,1}$, $w_{2i+2,4}$
$u_{i,4}$ is adjacent to	$w_{2i,j+2}$	$w_{2i+1,j+2}$	$w_{2i,2}$, $w_{2i+2,3}$

Here $w_{2k+1,j} = w_{1,j}$ and $w_{2k+2,j} = w_{2j}$. In addition, E_1 equals to $(E_2 - \{u_{1,2}w_{2,j}, u_{k,4}w_{1,j}: j = 3, 4\}) \cup \{u_{1,2}w_{1,j}, u_{k,4}w_{2,j}: j = 3, 4\}$.

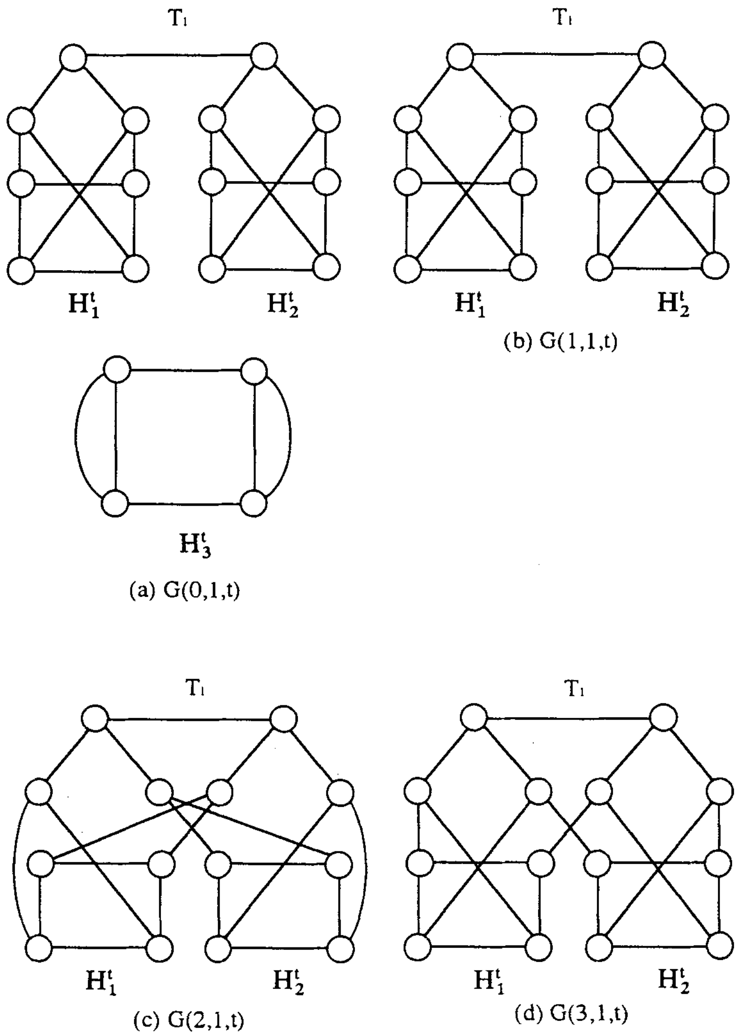

Fig. 6. Constructions for $k = 1$

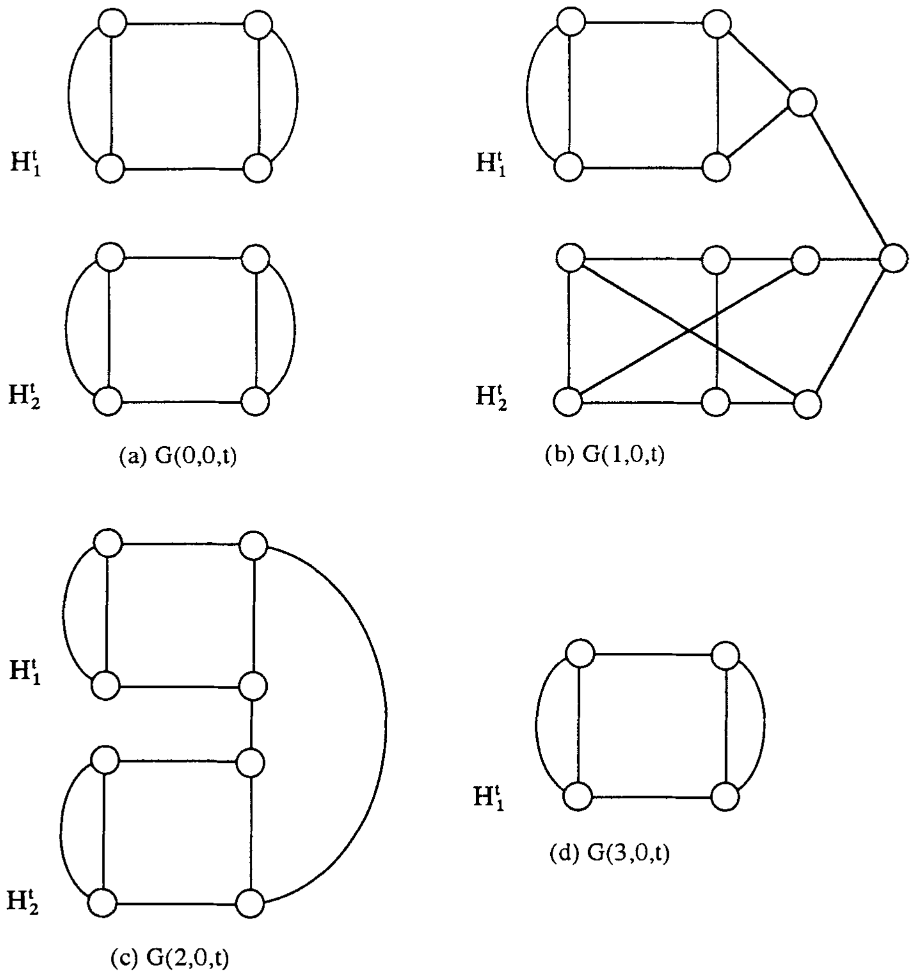

Fig. 7. Constructions for $k = 0$

For $k = 2$ and $c = 3$ the resulting graph $G(3, 2, t)$ is shown in Fig. 5 where every square denotes a copy of H^t. It can be seen that $G(c, k, t)$ has connectivity c. Further, $G(c, k, t)$ has a partition of the form (1) as well as a partition of the form (2). By Propositions 1 and 2 we obtain $\alpha' - \alpha = k$ for this graph.

It remains to construct $G(c, k, t)$ for $k = 0, 1$. Since such constructions are simple, we give illustrations only (Fig. 6 for $k = 1$ and Fig. 7 for $k = 0$). □

Conjecture. *For any $r \geq 4$, $c \in \{0, 1, \dots, r\}$ and $k \geq 0$ there exists an r-regular graph with connectivity c for which $\alpha' - \alpha = k$.*

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