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Independent Cycles with Limited Size in a Graph

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Abstract. Let k and s be two positive integers with $s \geq 3$. Let G be a graph of order $n \geq sk$. Write $n = qk + r, 0 \leq r \leq k - 1$. Suppose that G has minimum degree at least $(s - 1)k$. Then G contains k independent cycles $C_1, C_2, \ldots, C_k$ such that $s \leq l(C_i) \leq q$ for $1 \leq i \leq k - r$ and $s \leq l(C_i) \leq q + 1$ for $k - r < i \leq k$, where $l(C_i)$ denotes the length of C_i.

1. Introduction

Let G be a graph. A set of subgraphs of G is said to be independent in G if no two of them have any common vertex in G. The length of a cycle C is denoted by $l(C)$. Corrádi and Hajnal [2] investigated the maximum number of independent cycles in a graph. They proved the following: If G is a graph of order $n = qk + r$, where $q \geq 3, k \geq 1$ and $0 \leq r < k$, and G has minimum degree at least $2k$, then G contains k independent cycles $C_1, C_2, \ldots, C_k$ such that $l(C_i) \leq q$ for $1 \leq i \leq k - r$ and $l(C_i) \leq q + 1$ for $k - r < i \leq k$. In particular, when $n = 3k$ this result means that G contains k independent triangles. Hajnal and Szemerédi [3] proved that if G is a graph of order sk with $s \geq 3$ and $k \geq 1$ and G has minimum degree at least $(s - 1)k$ then G contains k independent complete subgraphs of order s. In this paper, we generalize Corrádi and Hajnal's result, proving the following theorem.

Theorem. *Let k and s be two positive integers with $s \geq 3$. Let G be a graph of order $n \geq sk$. Write $n = qk + r, 0 \leq r \leq k - 1$. Suppose that G has minimum degree at least $(s - 1)k$. Then G contains k independent cycles $C_1, C_2, \ldots, C_k$ such that*

$$s \leq l(C_i) \leq q \text{ for } 1 \leq i \leq k - r \quad \text{and} \quad s \leq l(C_i) \leq q + 1 \text{ for } k - r < i \leq k \quad (1)$$

We recall some terminology and notation. For a graph G, $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. For a vertex $u \in V(G)$ and a subset $U \subseteq V(G)$, we define $N(u, U)$ to be the set of all those vertices in U that are adjacent to u in G and let $d(u, U) = |N(u, U)|$. If H is a subgraph of G, define $N(u, H)$ and $d(u, H)$ by $N(u, V(H))$ and $d(u, V(H))$, respectively. Thus $d(u, G)$ is the degree of u in G. We also use $G[U]$ to denote the subgraph of G induced by U. Moreover $H + u$ is the subgraph of G obtained from H by adding to H the vertex u and all the edges of G between u and H. All graphs will be simple. Unexplained terminology and notation are adopted from [1].

2. Lemmas

Our proof of the theorem needs the following lemmas. In the following, p, q, s and t are fixed positive integers, G is a graph, $C = x_1x_2\dots x_tx_1$ is a cycle of G and $P = y_1y_2\dots y_p$ is a path of G independent of C. The subscripts of the x_i's will be reduced modulo t. A segment of C from x_i to x_j ($x_i \neq x_j$) is the path $x_ix_{i+1}\dots x_{j-1}x_j$, denoted by $C[x_i, x_j]$, of C. Note that $C[x_i, x_j]$ and $C[x_j, x_i]$ have no common vertices except x_i and x_j. A subpath of P with two endvertices y_i and y_j is denoted by $P[y_i, y_j]$ and called a segment of P.

Lemma 2.1. *Suppose that $t > s \geq 3$ and G has a vertex $y_0 \in V(G) - V(C)$ such that $d(y_0, C) \geq \frac{1}{2}s + 1$. Then $C + y_0$ contains a cycle C' such that $s \leq l(C') < t$.*

Proof. On the contrary, we suppose that the lemma fails. Let t have the smallest value with $t > s$ such that $C + y_0$ does not contain a cycle satisfying the requirement. Clearly, $d(y_0, C) < t$. If $t > s + 1$, let x_i be such that $x_iy_0 \notin E(G)$. Consider $H = G - x_i + x_{i-1}x_{i+1}$ and $C_1 = C - x_i + x_{i-1}x_{i+1}$. Then $d(y_0, C_1) \geq \frac{1}{2}s + 1$ holds in H. By the minimality of t, we see that in H, $C_1 + y_0$ contains a cycle C' with $s \leq l(C') < t - 1$. Then C' is not a cycle of $C + y_0$. Let $C'' = C' - x_{i-1}x_{i+1} + x_{i-1}x_i + x_ix_{i+1}$. Then C'' is a cycle of $C + y_0$ with $s \leq l(C'') < t$. So $t = s + 1$ holds. Since $d(y_0, C) \geq \frac{1}{2}s + 1$, there exists i such that x_iy_0, $x_{i+3}y_0 \in E(G)$. W.l.o.g., say x_1y_0, $x_4y_0 \in E(G)$. Then the cycle $x_1y_0x_4x_5\dots x_{s+1}x_1$ of $C + y_0$ has length s. □

Lemma 2.2. *Suppose that $t \geq s + 2$ and $s \geq 7$. If G contains a vertex $y_0 \in V(G) - V(C)$ such that $d(y_0, C) \geq \lceil \frac{1}{2}s \rceil$ then $C + y_0$ contains a cycle C' such that $s \leq l(C') < t$ unless $s = 12$ and $t = 14$. When $s = 12$ and $t = 14$, if G contains another vertex $y_1 \in V(G) - V(C)$ such that $d(y_1, C) \geq 6$ then $C + y_0 + y_1$ contains a cycle C' such that $12 \leq l(C') < 14$.*

Proof. On the contrary, we suppose that the lemma fails. In the natural way, we partition $N(y_0, C)$ into segments of C, say $I_1, I_2, \dots, I_k$, in order along C. Let J_i denote the segment of C between I_i and I_{i+1}, where the subscripts are reduced modulo k. Clearly $|V(I_i)| \leq 3$ for all i, $1 \leq i \leq k$, for otherwise obviously $C + y_0$ has a cycle of length $t - 1$. Similarly, we see that either $|V(J_i)| = 1$ or $|V(J_i)| \geq 4$ for all i, $1 \leq i \leq k$. Suppose that there is some J_i such that $|V(J_i)| = 1$. Let $V(J_i) = \{x_j\}$. Add the edge x_jy_0 to G. By Lemma 2.1, $C + y_0 + x_jy_0$ contains a cycle C_1 with $s \leq l(C_1) < t$. Then C_1 must contain x_jy_0. It is also clear that C_1 contains exactly one of the edges $x_{j-1}x_j$ and x_jx_{j+1}. W.l.o.g., say C_1 contains $x_{j-1}x_j$. Let $C' = C_1 - x_jy_0 + x_jx_{j+1} + x_{j+1}y_0$ if $l(C_1) = s$, or let $C' = C_1 - x_j + x_{j-1}y_0$ if $l(C_1) \geq s + 1$. Then C' is a cycle of $C + y_0$ with $s \leq l(C') < t$, a contradiction. So $|V(J_i)| \geq 4$ for all i, $1 \leq i \leq k$. We next claim that $|V(J_i)| \geq t - s + 2$ for all i, $1 \leq i \leq k$. Let the first and last vertices of J_i be x_j and x_h. Then $C'' = x_1x_2\dots x_{j-1}y_0x_{h+1}x_{h+2}\dots x_tx_1$ is a cycle of length less than t. Hence $l(C'') < s$ for otherwise we are done. Thus $|V(J_i)| \geq t - (s - 2)$. So the claim holds. Therefore we have

$$t = \sum_{i=1}^{k} |V(I_i)| + \sum_{i=1}^{k} |V(J_i)| \geq \left\lceil \frac{1}{2}s \right\rceil + k(t - s + 2) \tag{2}$$

$$= t + (k-1)(t - s + 2) - (s - 2) + \left\lceil \frac{1}{2}s \right\rceil. \tag{3}$$

It is easy to see that (2) does not hold if s is odd since $k \geq \lceil \frac{1}{6}(s + 1) \rceil \geq 2$. If s is even, write $s = 6m + r$ with $0 \leq r < 6$. Note that $k \geq m + \lceil \frac{1}{6}r \rceil$ and r is even. Then it is not difficult to see that (2) does not hold unless $r = 0$, $t = s + 2$, $k = 2$ and equality in (2) holds. Hence $|I_1| = |I_2| = 3$ and $|J_1| = |J_2| = 4$ hold and thus we have $s = 12$ and $t = 14$. Similarly, we define I_i' and J_i', $i = 1, 2, \ldots, k'$ with respect to $N(y_1, C)$ and apply the above argument to I_i' and J_i' $(1 \leq i \leq k')$. Then it is easy to check that when $s = 12$ and $t = 14$, $C + y_0 + y_1$ contains a cycle C' such that $12 \leq l(C') < 14$, a contradiction. This proves the lemma. □

Lemma 2.3. *Suppose that $t > s \geq 3$. Assume that $d(x_i, C) + d(x_{i+1}, C) \geq s + 2$ for some i, $1 \leq i \leq t$. Then $G[V(C)]$ contains a cycle C' with $s \leq l(C') < t$.*

Proof. On the contrary, we suppose that the lemma fails. Clearly, C has a chord in G. This implies $s \geq 5$. We may let t be the smallest integer with $t > s$ such that $G[V(C)]$ does not contain a cycle satisfying the requirement. W.l.o.g., we may assume that $d(x_1, C) + d(x_2, C) \geq s + 2$. If $t \geq s + 2$, then x_4 is not adjacent to x_1, nor to x_2. So by the minimality of t, $G[V(C) - \{x_4\}] + x_3x_5$ contains a cycle C_1 with $s \leq l(C_1) < t - 1$. Therefore C_1 must contain the edge x_3x_5. Let $C' = C_1 - x_3x_5 + x_3x_4 + x_4x_5$. Then C' is a cycle of $G[V(C)]$ satisfying the requirement. Hence we have $t = s + 1$. It is easy to see that for each $i \in \{1, 2, \ldots, s + 1\}$, if $x_1x_i \in E(G)$ then $x_2x_{i+2} \notin E(G)$ for otherwise $G[V(C) - \{x_{i+1}\}]$ contains a cycle of length s. Let I be the segment of C from x_4 to $s - 1$ and J from x_5 to $s + 1$. Then $d(x_2, J) \leq d(x_2, J - x_5) + 1 \leq |V(J - x_5)| - d(x_1, I) + 1 = |V(J)| - d(x_1, I)$. We also have $d(x_1, C) = d(x_1, I) + 2$ and $d(x_2, C) = d(x_2, J) + 2$. Thus $d(x_1, C) + d(x_2, C) \leq d(x_1, I) + 2 + |V(J)| - d(x_1, I) + 2 = s + 1$, a contradiction. This proves the lemma. □

Lemma 2.4. *Suppose that $t > s \geq 3$ and $G[V(C)]$ does not contain a cycle with length at least s but less than t. Let $a > s$ and $b \geq 0$ be two integers such that $t = a + b$. Then there exists a segment P of C with a vertices such that $\sum_{i=1}^{t} d(x_i, P) \leq \frac{1}{2}(s + 1)a + \frac{1}{2}(s - 3)b + 2$.*

Proof. We first show that $\sum_{x \in V(P)} d(x, P) \leq \frac{1}{2}(s + 1)a$ for any segment P on a vertices of C. For the sake of simplicity, let $P = x_1x_2 \ldots x_a$. By Lemma 2.3, we must have $d(x_i, P) + d(x_{i+1}, P) \leq d(x_i, C) + d(x_{i+1}, C) \leq s + 1$ for all i, $1 \leq i \leq t$. Therefore, if a is even, then $\sum_{i=1}^{a} d(x_i, P) \leq \frac{1}{2}(s + 1)a$ holds. Assume that a is odd. It is easy to see that if $G[V(P)] + x_1x_a$ contains a cycle C' with $s \leq l(C') < a$ then, by replacing x_1x_a by the segment of C from x_a to x_1, we would obtain a cycle C'' of $G[V(C)]$ with $s \leq l(C'') < t$, contradicting the assumption of $G[V(C)]$. Hence by Lemma 2.3, we must have $d(x_1, P) + d(x_a, P) \leq s + 1$. W.l.o.g., we assume that $d(x_a, P) \leq \frac{1}{2}(s + 1)$. Hence $\sum_{i=1}^{a} d(x_i, P) = \frac{1}{2}(s + 1) + \sum_{i=1}^{a-1} d(x_i, P) \leq \frac{1}{2}(s + 1) + \frac{1}{2}(s + 1)(a - 1) = \frac{1}{2}(s + 1)a$.

Now we assume that $a < t$. Again by Lemma 2.3, there exists a vertex, say x_b, such that $d(x_b, C) \leq \frac{1}{2}(s + 1)$. Let $L = x_1x_2 \ldots x_b$ and $P = C - V(L)$. We shall prove $\sum_{i=1}^{b} d(x_i, P) \leq \frac{1}{2}(s - 3)b + 2$. This is true when $b = 1$. When b is even, we have $\sum_{1 \leq i \leq (1/2)b} (d(x_{2i-1}, P) + d(x_{2i}, P)) \leq \sum_{1 \leq i \leq (1/2)b} (d(x_{2i-1}, C) + d(x_{2i}, C) - 4) + 2 \leq \frac{1}{2}(s - 3)b + 2$ as claimed. When b is odd, we have $\sum_{1 \leq i \leq (1/2)(b-1)} (d(x_{2i-1}, P) +$

$d(x_{2i}, P)) + d(x_b, P) \le \sum_{1 \le i \le (1/2)(b-1)} (d(x_{2i-1}, C) + d(x_{2i}, C) - 4) + 1 + d(x_b, P) \le \frac{1}{2}(s-3)(b-1) + 1 + \frac{1}{2}(s+1) - 1 = \frac{1}{2}(s-3)b + 2$ as claimed again. This proves the lemma. □

Lemma 2.5. *Suppose that $t \ge 4$ and $t + 1 \ge p \ge t$. Assume that $\sum_{i=1}^{t} d(y_i, C) \ge t^2 - t + 1$ if $p = t$ and $\sum_{i=1}^{t+1} d(y_i, C) \ge t^2$ if $p = t + 1$. Then $G[V(C \cup P)]$ contains two independent cycles of length t.*

Proof. First assume $p = t$. Since $t^2 \ge \sum_{i=1}^{t} d(y_i, C) \ge t^2 - t + 1$, we see that $d(y_1, C) + d(y_t, C) \ge t + 1$. This implies that there exist two consecutive vertices of C that are adjacent to y_1 and y_t, respectively. W.l.o.g., say $x_1 y_1, x_t y_t \in E(G)$. Thus $C \cup P - x_1 x_t + x_1 y_1 + x_t y_t$ is a hamiltonian cycle of $G[V(C \cup P)]$. Consider $t - 1$ pairwise disjoint pairs of edges $\{y_i x_{t-i}, y_{i+1} x_{t-i+1}\}$ $(1 \le i \le t-1)$. These edges are not necessarily in G. Each one of the $t - 1$ pairs divides the hamiltonian cycle into two independent cycles of length t. Since $G[V(C \cup P)]$ misses at most $t - 1$ of those possible edges between C and P, we may assume that exactly one of the two edges $y_i x_{t-i}$, $y_{i+1} x_{t-i+1}$ is not in G for all i, $1 \le i \le t - 1$. But then $y_t x_3 x_4 \ldots x_t x_1 y_t$ and $x_2 y_1 y_2 \ldots y_{t-1} x_2$ are two independent cycles of length t in $G[V(C \cup P)]$.

Now assume that $p = t + 1$. If $d(y_1, C) \le t - 1$ then we have $\sum_{i=2}^{t+1} d(y_i, C) \ge t^2 - t + 1$ and therefore we can use the above argument. So we may assume that $d(y_1, C) = t$. Similarly, we may assume that $d(y_{t+1}, C) = t$. It is easy to see that if $d(y_3, C) \ge 1$ or $d(y_{t-1}, C) \ge 1$ then we have two independent cycles of length t in $G[V(C \cup P)]$. So we may assume that $d(y_3, C) = 0 = d(y_{t-1}, C)$. If $t \ne 4$, then $\sum_{i=1}^{t+1} d(y_i, C) \le t^2 - t$, a contradiction. If $t = 4$, then $d(y_i, C) = 4$ for $i = 1, 2, 4, 5$, and so $G[V(C \cup P) - \{y_3\}]$ contains two independent cycles of length 4. This proves the lemma. □

For a subgraph H of G and a vertex $x \in V(G) - V(H)$, we define $\bar{d}(x, H)$ to be the number of vertices y of H that are not adjacent to x in G, i.e., $\bar{d}(x, H) = |V(H)| - d(x, H)$. The proofs of the following two lemmas share much in common, especially when we deduce that $t = s$ and (b) follows from (a).

Lemma 2.6. *Suppose that $p \ge q \ge s \ge 7$ and $t \ge s$. Set $\sigma = 0$ or 1 according to whether s is even or odd, respectively. Let Y be a subset of $V(P)$ with $|Y| = q$ and $I = \sum_{y \in Y} d(y, C)$. Suppose that $G[V(C \cup P)]$ does not contain a cycle of length at least s but less than t. Then the following two statements hold:*

(a) *If $I \ge \frac{1}{2}(s - \sigma)q + \frac{1}{2}t(s - 2 + \sigma) + 1$ then $G[V(C \cup P)]$ contains two independent cycles of length at least t.*
(b) *If $q > s$ and $I \ge \frac{1}{2}(s - \sigma)(q - 1) + \frac{1}{2}t(s - 2 + \sigma) + s$ then $G[V(C \cup P)]$ contains two independent cycles of length at least t but at most $p - 1$.*

Proof. Let $r = p + q + t$. On the contrary, we suppose that the lemma fails and assume that $r = p + q + t$ has the smallest value with p, q and t satisfying the conditions of the lemma such that $G[V(C \cup P)]$ does not contain two cycles satisfying the requirement. We first prove (a) and then (b) will follow. To prove (a), we distinguish two cases: $t > s$ or $t = s$.

Assume first that $t > s$. If $q \le t + 1$, then $\lceil I/q \rceil \ge s - 1$. Therefore, there exists $y \in Y$ such that $d(y, C) \ge s - 1$. By Lemma 2.1, $C + y$ contains a cycle of length at least s but less than t, contradicting the assumption of $G[V(C \cup P)]$. So $q \ge t + 2$. By the minimality of r, we see that $d(y, C) \ge \frac{1}{2}(s - \sigma) + 1$ for all $y \in Y$. Thus, by Lemmas 2.1 and 2.2, we see that $t = s + 1$ and s is odd.

Suppose that there are two consecutive vertices, say x_i and x_{i+1}, of C such that x_i and x_{i+1} have at most $\frac{1}{2}(s - 2 + \sigma)$ common neighbours in Y. We obtain a new graph G' and a new cycle C' from G and C by contracting the edge $x_i x_{i+1}$ to a new vertex z. Then $\sum_{x \in V(C')} d(x, Y) \ge I - \frac{1}{2}(s - 2 + \sigma) = \frac{1}{2}(s - \sigma)q + \frac{1}{2}(t - 1)(s - 2 + \sigma) + 1$ holds in G'. If $G'[V(C' \cup P)]$ contains a cycle C'' with $s \le l(C'') < t - 1$ then we obtain a cycle C''' in $G[V(C \cup P)]$ by undoing the contraction. Clearly $l(C'') \le l(C''') \le l(C'') + 1 < t$, a contradiction. Therefore, by the minimality of r, $G'[V(C' \cup P)]$ contains two independent cycles C_1 and C_2 of length at least $t - 1$. Again, by undoing the contraction, we obtain two independent cycles C_1' and C_2' in $G[V(C \cup P)]$ from C_1 and C_2 with length at least $t - 1$. These two cycles must have length at least t by the assumption on $G[V(C \cup P)]$. Hence $|N(x_i, Y) \cap N(x_{i+1}, Y)| \ge \frac{1}{2}(s + \sigma)$ for all i, $1 \le i \le t$.

Choose any two distinct vertices $z_1, z_2 \in N(x_1, Y) \cap N(x_2, Y)$. Since $d(z_1, C) + d(z_2, C) \ge s + 1$, there exists i such that $x_i z_1, x_{i+4} z_2 \in E(G)$. It is easy to check that $C + z_1 + z_2$ contains a cycle of length s, a contradiction. Hence $t = s$.

By Lemma 2.5, it $p = s$, (a) is true. So $p \ge s + 1$. We now show that $Y = V(P)$ and $|N(y_i, C) \cap N(y_{i+1}, C)| \ge \frac{1}{2}(s - \sigma) + 1$ for all i, $1 \le i \le p - 1$. Suppose that $|N(y_i, C) \cap N(y_{i+1}, C)| \le \frac{1}{2}(s - \sigma)$, or $\{y_i, y_{i+1}\} \nsubseteq Y$ for some i, $1 \le i \le p - 1$. We obtain a new graph G', a new path P' and a new subset Y' of $V(P')$ from G, P and Y, respectively by contracting the edge $y_i y_{i+1}$ to a new vertex w. Then in G', $\sum_{y \in Y'} d(y, C) \ge \frac{1}{2}(s - \sigma)q + \frac{1}{2}(s - 2 + \sigma)t + 1$ if $\{y_i, y_{i+1}\} \nsubseteq Y$ and $\sum_{y \in Y'} d(y, C) \ge \frac{1}{2}(s - \sigma)(q - 1) + \frac{1}{2}(s - 2 + \sigma)t + 1$ if $\{y_i, y_{i+1}\} \subseteq Y$. By the minimality of r, $G'[V(C \cup P')]$ contains two independent cycles of length at least s from which we readily obtain two independent cycles of length at least s in $G[V(C \cup P)]$ by undoing the contraction. Therefore $Y = V(P)$ and $|N(y_i, C) \cap N(y_{i+1}, C)| \ge \frac{1}{2}(s - \sigma) + 1$ for all i, $1 \le i \le p - 1$.

Let $\bar{I} = \sum_{i=1}^{s} \bar{d}(x_i, P)$. Then

$$\bar{I} = ps - I \le \tfrac{1}{2}(s + \sigma)(p + 2 - s) - 1 - \sigma \tag{4}$$

We shall derive a lower bound for $\bar{I}$ to obtain a contradiction with (4). Since $d(y_1, C) + d(y_p, C) \ge s + 1$, there exist two consecutive vertices, say x_1 and x_s, of C such that $x_1 y_1, x_s y_p \in E(G)$. For each i, $1 \le i \le s - 1$, let

$$B_i = y_{s-i} y_{s-i+1} \cdots y_{p-i+1} \tag{5}$$

If $d(x_i, B_i) + d(x_{i+1}, B_i) \ge |V(B_i)| + 2$, then there are two vertices y_j and y_k on B_i with $j < k$ such that $x_i y_j$, $x_{i+1} y_k \in E(G)$. Then $x_1 x_2 \dots x_i y_j y_{j-1} \dots y_1 x_1$ and $x_{i+1} x_{i+2} \dots x_s y_p y_{p-1} \dots y_k x_{i+1}$ are two independent cycles of length at least s, a contradiction. So we must have

$$d(x_i, B_i) + d(x_{i+1}, B_i) \le |V(B_i)| + 1 \textit{ for } i = 1, 2, \dots, s - 1 \tag{6}$$

and therefore

$$\bar{d}(x_i, B_i) + \bar{d}(x_{i+1}, B_i) \geq 2|V(B_i)| - |V(B_i)| - 1 = p + 1 - s \text{ for } i = 1, 2, \ldots, s-1 \quad (7)$$

Let $X = \{x_i | \bar{d}(x_i, P) \geq \frac{1}{2}(p+1-s), 1 \leq i \leq s\}$. By (7), we see that $|X| \geq \frac{1}{2}(s-1)$ and no two vertices in $V(C) - X$ are consecutive on the path $C - x_1 x_s$. We discuss the following two cases.

Case 1. $|X| \geq \frac{1}{2}(s+\sigma)$.

Then $p+1-s$ must be even for otherwise $\bar{d}(x, P) \geq \frac{1}{2}(p+2-s)$ for all $x \in X$ and so $\bar{I} \geq \frac{1}{2}(s+\sigma)(p+2-s)$, contradicting (4). Let $X_0 = \{x_i | \bar{d}(x_i, P) = 0, 1 \leq i \leq s\}$. If $\bar{d}(x_i, P) = 0$, i.e., $d(x_i, P) = p$ then, by (7), $\bar{d}(x_{j_i}, P) \geq p - s + 1 \geq \frac{1}{2}(p+1-s) + 1$ for some $x_{j_i} \in \{x_{i-1}, x_{i+1}\}$. Since $|X| \geq \frac{1}{2}(s+\sigma)$, we can choose distinct x_{j_i} for all $x_i \in X_0$. Thus $\bar{I} \geq \frac{1}{2}|X|(p+1-s) + s - |X| \geq \frac{1}{2}(s+\sigma)(p+2-s) - \sigma$, contradicting (4).

Case 2. $|X| < \frac{1}{2}(s+\sigma)$.

Then s must be odd, $|X| = \frac{1}{2}(s-1)$ and $X = \{x_2, x_4, x_6, \ldots, x_{s-1}\}$. It is easy to see, similar to obtaining (7), that $\bar{d}(x_i, B_i - y_{p-i+1}) + \bar{d}(x_{i+2}, B_i - y_{p-i+1}) \geq p - s$ for all i, $1 \leq i \leq s-2$. If $p+1-s$ is even, then $p-s$ is odd. Therefore either $\bar{d}(x_1, P) \geq \frac{1}{2}(p+1-s)$ or $\bar{d}(x_3, P) \geq \frac{1}{2}(p+1-s)$ and so $|X| \geq \frac{1}{2}(s+1)$, a contradiction. If $p+1-s$ is odd then $\bar{d}(x_{2i}, P) \geq \frac{1}{2}(p+2-s)$ for $i = 1, 2, \ldots, \frac{1}{2}(s-1)$ and therefore $\bar{I} \geq \frac{1}{2}(s-1)(p+2-s) + 2(p-s) \geq \frac{1}{2}(s+1)(p+2-s) - 1$, contradicting (4). This proves (a).

We now turn to the proof of (b) which easily follows from (a). If $t > s$, we can easily show, as before, that $q \geq t+2$ by the minimality of r. Also by the minimality of r, we have $y_1, y_p \in Y$. If $t = s$, we may assume, by Lemma 2.5, that $p \geq s+2$, and again by the minimality of r, we can easily show, as before, that $Y = V(P)$.

If $d(y_p, C) \leq s-1$ then $\sum_{y \in Y - \{y_p\}} d(y, C) \geq I - s + 1 \geq \frac{1}{2}(s-2-\sigma)(q-1) + \frac{1}{2}t(s+\sigma) + 1$. By (a), $G[V(C \cup P) - \{y_p\}]$ contains two independent cycles of length at least t. Obviously these two cycles have length at most $p-1$. So we may assume that $d(y_p, C) \geq s$, and similarly, $d(y_1, C) \geq s$. Therefore $t = s$ by Lemma 2.1. Then $d(y_i, C) = 0 = d(y_j, C)$ for i, j, $s-1 \leq i \leq p-2$ and $3 \leq j \leq p-s+2$ for otherwise we obtain two independent cycles in $G[V(C \cup P)]$ of length at least s but at most $p-1$. So $I < s^2$. But by the condition of (b), we have $I \geq s^2$. This proves (b) and therefore the lemma. □

Lemma 2.7. *Suppose that $6 \geq s \geq 4$, $p \geq q \geq s$ and $t \geq s$. Set $\sigma = 0$ or 1 according to whether s is even or odd, respectively. Let Y be a subset of $V(P)$ with $|Y| = q$ and $I = \sum_{y \in Y} d(y, C)$. Suppose that $G[V(C \cup P)]$ does not contain a cycle of length at least s but less than t. Then the following two statements hold:*

(a) *If $I \geq \frac{1}{2}(s-2-\sigma)q + \frac{1}{2}t(s+\sigma) + 1$ then $G[V(C \cup P)]$ contains two independent cycles of length at least t.*
(b) *If $q > s$ and $I \geq \frac{1}{2}(s-2-\sigma)(q-1) + \frac{1}{2}t(s+\sigma) + s$ then $G[V(C \cup P)]$ contains two independent cycles of length at least t but at most $p-1$.*

Proof. As we did in the proof of Lemma 2.6, it is easy to see that (b) follows from (a). So we shall give the proof of (a). Let $r = p + q + t$. On the contrary, suppose that (a) fails and assume that r has the smallest value with p, q and t satisfying the conditions of the lemma (a) such that $G[V(C \cup P)]$ does not contain two independent cycles of length at least t.

To a contradiction, suppose that $t > s$. As we did in the proof of Lemma 2.6, we can easily deduce that $q \geq t + 2$, $|N(x_i, Y) \cap N(x_{i+1}, Y)| \geq \frac{1}{2}(s + \sigma) + 1$ for all i, $1 \leq i \leq t$ and $d(y, C) \geq \frac{1}{2}(s - \sigma)$ for all $y \in Y$. Thus $G[V(C \cup P)]$ has a cycle of length 4. Therefore $s = 5$ or 6. Let $Y_i = N(x_i, Y) \cap N(x_{i+1}, Y)$ for $i = 1, 2, \ldots, t$. Then $|Y_i| \geq 4$ for all i, $1 \leq i \leq t$. If $s = 5$, $Y_1 \cap Y_2 = \emptyset$; otherwise $G[V(C \cup P)]$ contains a cycle of length 5. Hence $d(x_2, Y) \geq 8$. Let z_i $(1 \leq i \leq 8)$ be eight vertices in $N(x_2, Y)$ in order along P. Then $x_2P[z_1, z_4]x_2$ is a cycle of length at least 5 and $G[V(C \cup P[z_5, z_8]) - \{x_2\}]$ contains a cycle of length at least 5 since $d(z_5, C - x_2) \geq 1$ and $d(z_8, C - x_2) \geq 1$. If $s = 6$, Then $|Y_1 \cap Y_2| \leq 1$, $|Y_2 \cap Y_3| \leq 1$ and $Y_1 \cap Y_3 = \emptyset$ for otherwise we have a cycle of length 6 in $G[V(C \cup P)]$. So $|N(x_2, Y) \cup N(x_3, Y)| \geq 10$. The rest of the argument is similar to the above.

Now we prove (a) for $t = s$. By Lemma 2.5, when $p = s$, (a) is true. So assume that $p > s$. Again, as we did in the proof of Lemma 2.6, we see that $Y = V(P)$ and $|N(y_i, C) \cap N(y_{i+1}, C)| \geq \frac{1}{2}(s - \sigma)$ for all i, $1 \leq i \leq p - 1$.

Let x_i ad x_j be two distinct vertices of C such that $x_iy_1, x_jy_p \in E(G)$. It is easy to see that there are two independent segments P_1 and P_2 of C and two vertices z_1, z_2 of $N(y_{s-1}, C) \cap N(y_s, C)$ such that $x_i, z_1 \in V(P_1)$ and $x_j, z_2 \in V(P_2)$. If $p \geq 2(s - 1)$ then $G[V(C \cup P)]$ contains two independent cycles of length at least s. This idea is also used (by choosing x_i, x_j, z_1 and z_2 properly) in the following three cases while $p \leq 2s - 3$.

Case 1. $p \geq s + 2$.

Then $s = 5$ or 6. First assume that $s = 5$. Then $p = 7$, $I \geq 23$ and $N(y_i, C) \cap N(y_{i+1}, C) \geq 2$ $(1 \leq i \leq 6)$. It is easy to see that $N(y_4, C) \cap N(y_5, C)$ must contain x_j for otherwise we readily get two independent cycles of length at least 5. Similarly, $x_i \in N(y_3, C) \cap N(y_4, C)$. Then it is easy to see that if y_5 or y_7 is adjacent to a vertex $x_k \in V(C) - \{x_i, x_j\}$ then $G[V(C \cup P)]$ contains two independent cycles of length at least 5 with one being $x_iy_1y_2y_3y_4x_i$. So we may assume that $N(y_5, C) = \{x_i, x_j\} = N(y_7, C)$. Similarly, $N(y_1, C) = \{x_i, x_j\} = N(y_3, C)$. Since $I \geq 23$, we see that $d(y_2, C) = d(y_4, C) = d(y_6, C) = 5$ and so $G[V(C \cup P)]$ contains two independent cycles of length 5.

Now let $s = 6$. Then $8 \leq p \leq 9$, $I \geq 2p + 19$ and $N(y_i, C) \cap N(y_{i+1}, C) \geq 3$ $(1 \leq i \leq p - 1)$. Let x_a, x_b, x_c be three distinct vertices in $N(y_5, C) \cap N(y_6, C)$ in order along C. It is easy to see that if there is a vertex $u \in V(C) - \{x_a, x_b, x_c\}$ such that $uy_p \in E(G)$ then there is a vertex $v \in N(y_1, C)$ such that the graph $C \cup P \cup \{uy_p, vy_1\}$ together with the six edges between $\{y_5, y_6\}$ and $\{x_a, x_b, x_c\}$ contains two independent cycles of length at least 6. Thus $N(y_p, C) = \{x_a, x_b, x_c\}$. It is easy to see that if the three vertices x_a, x_b, x_c are not consecutive on C, then there exists $v \in N(y_1, C)$ such that the graph $C \cup P + vy_1$ together with the nine edges between $\{y_5, y_6, y_p\}$ and $\{x_a, x_b, x_c\}$ contains two independent cycles of length at least 6. So we may assume that $\{x_a, x_b, x_c\} = \{x_1, x_2, x_3\} = N(y_p, C)$. Thus we see that $x_2y_1 \notin E(G)$ for the same reason. Similarly, we may assume that y_1 is adjacent to three vertices in $N(y_3, C) \cap N(y_4, C)$ that are consecutive on C. We now see that $G[V(C \cup P)]$ contains two independent cycles of length 6.

Case 2. $p = s + 1$ and no two consecutive vertices of C are adjacent to y_1 and y_{s+1}, respectively.

In this case, we may assume w.l.o.g. that $N(y_1, C) = \{x_1, x_3\} = N(y_{s+1}, C)$ if $s = 4$ or 5 and $N(y_1, C) = \{x_1, x_3, x_5\} = N(y_7, C)$ if $s = 6$. If $s = 4$, then $I \geq 14$ and both $N(y_2, C)$ and $N(y_4, C)$ contain the two vertices x_1 and x_3. To avoid the occurrence of two independent cycles of length at least 4, x_3 must be in $N(y_3, C) \cap N(y_4, C)$. Similarly, x_1 must be in $N(y_3, C) \cap N(y_2, C)$ and hence $d(y_2, C) = 2 = d(y_4, C)$ must hold. Therefore $I \leq 4 \cdot 2 + 4 = 12$, a contradiction.

Similarly, if $s = 5$, we have $I \geq 22$ and $\{x_1, x_3\} \subseteq N(y_2, C) \cap N(y_5, C)$. To avoid the occurrence of two independent cycles of length at least 5, one of x_1 and x_3, say x_1, must be in $N(y_3, C) \cap N(y_4, C)$. For the same reason, we see that neither of the two edges y_2x_5 and y_5x_5 is in G, nor is one of y_3x_5 and y_4x_4. Hence $I \leq 2 \cdot 2 + 3 \cdot 4 + 5 = 21$, a contradiction.

If $s = 6$, then $I \geq 33$ and $\{x_1, x_3, x_5\} \subseteq N(y_2, C) \cap N(y_6, C)$. We have $30 \geq \sum_{i=2}^{6} d(y_i, C) \geq 33 - 6 = 27$, which easily implies that $G[V(C \cup P)]$ contains two independent cycles of length at least 6.

Case 3. $p = s + 1$ and there exist two consecutive vertices of C that are adjacent to y_1 and y_{s+1}, respectively.

Say y_1x_1, $y_{s+1}x_s \in E(G)$. As in Lemma 2.5, we see that each of the $2(s-1)$ pairwise disjoint pairs of edges (not necessarily in G) $\{y_ix_{s-i}, y_{i+1}x_{s-i+1}\}$ and $\{y_{i+1}x_{s-i}, y_{i+2}x_{s-i+1}\}$ $(1 \leq i \leq s-1)$ contains an edge which is not in G. Hence $I \leq 14$ if $s = 4$, $I \leq 22$ if $s = 5$ and $I \leq 32$ if $s = 6$. Therefore we must have $s = 4$ or 5. When $s = 4$, $x_1x_2y_1y_2x_1$ and $x_3x_4y_4y_5x_3$ are two independent cycles of $G[V(C \cup P)]$. When $s = 5$, $x_1x_2y_1y_2y_3x_1$ and $x_4x_5y_4y_5y_6x_4$ are two independent cycles of $G[V(C \cup P)]$. This proves (a) and therefore (b) follows. □

3. Proof of the Theorem

Let k, n, s be integers with $k \geq 1$, $s \geq 3$ and $n \geq sk$. Let G be a graph of order n with minimum degree at least $(s-1)k$. Write $n = qk + r$, $0 \leq r < k$. We shall prove that G contains k independent cycles satisfying (1). Corrádi and Hajnal's result [2] shows the theorem is true for $s = 3$. So we may assume that $s \geq 4$ in the following. It is well known that if a graph H has minimum degree $\delta \geq 2$ then H contains a cycle of length at least $\delta + 1$. We first claim:

Claim 1. *G contains a cycle C with $s \leq l(C) \leq q$.*

On the contrary, suppose that every cycle of G with length at least s has length at least $q + 1$. Since G has a cycle of length at least s, we may choose m cycles C_1, $C_2, \ldots, C_m$ of G such that C_1 is a smallest cycle of length at least s in G and C_i is a smallest cycle of length at least s in $G - \bigcup_{j=1}^{i-1} V(C_j)$ for $i = 2, 3, \ldots, m$ but $G -$

$\bigcup_{j=1}^{m} V(C_j)$ does not have a cycle of length at least s. By the assumption, we have $l(C_i) \geq q + 1$ for $i = 1, 2, \ldots, m$. This implies that $m < k$. Suppose that $V(G) \neq \bigcup_{i=1}^{m} V(C_i)$. Then $G - \bigcup_{i=1}^{m} V(C_i)$ has a vertex y_0 such that $d(y_0, G - \bigcup_{i=1}^{m} V(C_i)) \leq s - 2$. Then $d(y_0, \bigcup_{i=1}^{m} C_i) \geq (s-1)k - (s-2) = (s-1)(k-1) + 1$ and so there exists i_0 such that $d(y_0, C_{i_0}) \geq s$. If $V(G) = \bigcup_{i=1}^{m} V(C_i)$ then, by the choice of C_m and Lemma 2.3, we see that C_m contains a vertex y_0 such that $d(y_0, C_m) \leq \lfloor \frac{1}{2}(s+1) \rfloor$. Then $d(y_0, \bigcup_{i=1}^{m-1} C_i) \geq (s-1)k - \lfloor \frac{1}{2}(s+1) \rfloor \geq (s-1)(k-1) + 1$ and so there exists i_0 such that $d(y_0, C_{i_0}) \geq s$. By Lemma 2.1, $C + y_0$ contains a cycle C' with $s \leq l(C') < l(C_{i_0})$, contradicting the choice of C_{i_0}. This proves the claim.

Let k_0 be the greatest integer such that G contains k_0 independent cycles $C_1, C_2, \ldots, C_{k_0}$ such that

$$s \leq l(C_i) \leq q \textit{ for } 1 \leq i \leq k - r \quad \textit{and} \quad s \leq l(C_i) \leq q + 1 \textit{ for } k - r + 1 \leq i \leq k_0 \tag{8}$$

Subject to (8), we may choose C_i's such that

$$\sum_{i=1}^{k_0} l(C_i) \textit{ is minimum.} \tag{9}$$

By Claim 1, $k_0 \geq 1$. For the proof of the theorem, we may assume that $k_0 < k$. We shall prove that this is a contradiction.

Claim 2. *G contains k_0 independent cycles C_i satisfying* (8) *and* (9) *such that $G - \bigcup_{i=1}^{k_0} V(C_i)$ contains a cycle of length at least s.*

Suppose that this claim fails. Then we choose, subject to (8) and (9), k_0 independent cycles C_i such that $G - \bigcup_{i=1}^{k_0} V(C_i)$ contains a longest path. Let $H = \bigcup_{i=1}^{k_0} C_i$, $D = G - \bigcup_{i=1}^{k_0} V(C_i)$ and $P = x_1 x_2 \ldots x_p$ be a longest path of D. Then $d(x_1, D) = d(x_1, P) \leq s - 2$ and $d(x_p, D) = d(x_p, P) \leq s - 2$ hold.

We now show that D is connected. If not, let D_0 denote a component of D which does not contain P. Then, since D_0 does not contain a cycle of length at least s, D_0 contains a vertex x_0 such that $d(x_0, D_0) \leq s - 2$. Hence we have $d(x_0, H) + d(x_1, H) \geq 2(s-1)k - 2(s-2) = 2(s-1)(k-1) + 2$. Therefore there exists i_0 such that $d(x_0, C_{i_0}) + d(x_1, C_{i_0}) \geq 2(s-1) + 1$. So either $d(x_0, C_{i_0}) \geq s$ or $d(x_1, C_{i_0}) \geq s$. By Lemma 2.1 and (9), we have $l(C_{i_0}) = s$. Then C_{i_0} contains three consecutive vertices, say y_1, y_2, y_3, that are adjacent to both x_0 and x_1. Thus $C'_{i_0} = C_{i_0} - y_2 + x_0 y_1 + x_0 y_3$ is a cycle of length s and $P + y_2 x_1$ is a path longer than P, contradicting the choice of P. This proves that D is connected. It follows from this argument that $D - V(P)$ does not contain a vertex adjacent to x_2 or x_{p-1}; for if such a vertex exists, say x_0 again, then $d(x_0, D) = d(x_0, P) \leq s - 2$ and a contradiction follows from this argument. This, in turn, implies that $p \geq 4$ since $|V(D)| \geq q \geq s \geq 4$.

We now consider $R = d(x_1, H) + d(x_2, H) + d(x_{p-1}, H) + d(x_p, H)$. Then $R \geq 4(s-1)k - (s-2) - (s-1) - (s-1) - (s-2) = 4(s-1)(k-1) + 2$. Thus there exists i_0 such that $d(x_1, C_{i_0}) + d(x_2, C_{i_0}) + d(x_{p-1}, C_{i_0}) + d(x_p, C_{i_0}) \geq 4s - 3$. This, together with Lemma 2.1 and (9), implies that $l(C_{i_0}) = s$. Let $C_{i_0} = y_1 y_2 \ldots y_s y_1$.

Since $d(x_1, C_{i_0}) + d(x_2, C_{i_0}) + d(x_{p-1}, C_{i_0}) + d(x_p, C_{i_0}) \geq 4s - 3$, either $d(x_1, C_{i_0}) \geq s - 1$ or $d(x_p, C_{i_0}) \geq s - 1$. W.l.o.g., say $d(x_1, C_{i_0}) = s - \tau$, where $\tau = 0$ or 1.

Then

$$d(x_2, C_{i_0}) + d(x_p, C_{i_0}) \geq 4s - 3 - (s - \tau) - s = 2s - 3 + \tau$$

This implies that $|N(x_2, C_{i_0}) \cap N(x_p, C_{i_0})| \geq s - 3 + \tau$. Therefore if $d(x_1, C_{i_0}) = s$, we obtain two independent cycles C' and C'' of lengths s and p, respectively. With C_{i_0} replaced by C', we obtain $D' = G - \bigcup_{i \neq i_0} V(C_i) - V(C')$, which contains C''. Since D' must be connected, we see that either C'' contains all vertices of D', contradicting the assumption that the claim is false, or D' contains a path longer than P, contradicting the choice of D. For the same reason, if $d(x_1, C_{i_0}) = s - 1$, say $x_1 y_2 \notin E(G)$, then $N(x_2, C_{i_0}) \cap N(x_p, C_{i_0}) \subseteq \{y_1, y_3\}$. Therefore $s = 4$. It is easy now to see that $G[V(C_{i_0} \cup P)]$ contains two independent cycles of lengths of 4 and p, respectively. This contradiction completes the proof of the claim.

By Claim 2, we may choose k_0 independent cycles $C_1, C_2, \ldots, C_{k_0}$ of G satisfying (8) and (9) such that

$$G - \bigcup_{i=1}^{k_0} V(C_i) \textit{ contains a smallest cycle of length at least } s. \tag{10}$$

Let $L = \bigcup_{i=1}^{k_0} C_i$, $F = G - V(L)$, $C = x_1 x_2 \ldots x_t x_1$ be a smallest cycle of length at least s in F and $F_0 = F - V(C)$. Then $t \geq q + 1$. By the maximality of k_0, when $t = q + 1$ we have $k_0 < k - r$. By Lemma 2.4, we may assume that $P = x_1 x_2 \ldots x_{q+1}$ satisfies

$$\sum_{i=1}^{q+1} d(x_i, C) \leq \frac{1}{2}(q+1)(s+1) + \frac{1}{2}(s-3)l_0 + 2, \textit{ where } l_0 = t - q - 1 \tag{11}$$

We define three numbers as follows:

$$I_1 = \sum_{i=1}^{q+1} d(x_i, L); \qquad I_2 = \sum_{i=1}^{q+1} d(x_i, C); \qquad I_3 = \sum_{i=1}^{q+1} d(x_i, F_0) \tag{12}$$

Clearly,

$$I_1 + I_2 + I_3 = \sum_{i=1}^{q+1} d(x_i, G) \geq (q+1)(s-1)k \tag{13}$$

We shall estimate the lower bound for I_1 and then apply Lemmas 2.6 and 2.7. To do so, we first estimate the upper bounds for I_2 and I_3. Define $\sigma = 0$ or 1 according to whether s is even or odd, respectively. Let $f_0 = |V(F_0)|$ and $p_0 = \sum_{i=1}^{k_0} l(C_i)$. Then $l_0 + f_0 = qk + r - p_0 - q - 1$. We distinguish two cases: $s \geq 7$ or $4 \leq s \leq 6$.

Assume first that $s \geq 7$. If $t = s + 1$ then $q = s$, $k_0 < k - r$ and $l(C_i) = s$ for all i, $1 \leq i \leq k_0$. By Lemma 2.1 and the minimality of C, we have $d(x, P) \leq d(x, C) \leq \frac{1}{2}(s + \sigma)$ for all $x \in V(F_0)$. Together with (11), we obtain $I_2 + I_3 \leq \frac{1}{2}(q+1)(s+1) + \frac{1}{2}(s+\sigma)(f_0 + l_0) + 2$. From this and (13), we obtain $I_1 \geq (\frac{1}{2}(s-\sigma)s - 1)k_0 + \frac{1}{2}(s+\sigma)p_0 + 1$. This implies that there exists i_0 such that $\sum_{i=1}^{s+1} d(x_i, C_{i_0}) \geq \frac{1}{2}(s-\sigma)s - 1 + \frac{1}{2}(s+\sigma)s + 1 = s^2$. Then by Lemma 2.5, $G[V(C_{i_0} \cup P)]$ contains two independent cycles of length s, contradicting the maximality of k_0. If $t \geq s + 2$ then, by Lemmas 2.1 and 2.2, $\sum_{y \in V(F_0)} d(x, P) \leq \frac{1}{2}(s - 2 + \sigma)f_0 + 1$. Together with (11) and

(13), we obtain

$$\begin{aligned} I_1 &\geq (q+1)(s-1)k - \tfrac{1}{2}(q+1)(s+1) - \tfrac{1}{2}(s-3)l_0 - 2 - \tfrac{1}{2}(s-2+\sigma)f_0 - 1 \\ &\geq (q+1)(s-1)k - \tfrac{1}{2}(q+1)(s+1) - \tfrac{1}{2}(s-2+\sigma)(qk+r-p_0-q-1) - 3 \\ &= (\tfrac{1}{2}(s-\sigma)q+s-1)k + \tfrac{1}{2}(s-2+\sigma)p_0 - \tfrac{1}{2}(3-\sigma)(q+1) - \tfrac{1}{2}(s-2+\sigma)r - 3 \end{aligned} \tag{14}$$

From (14), we deduce that when $k_0 < k - r$, $I_1 \geq (\frac{1}{2}(s-\sigma)q + s - 1)k_0 + \frac{1}{2}(s - 2 + \sigma)p_0 + 1$. This implies that there exists i_0 such that $\sum_{i=1}^{q+1} d(x_i, C_{i_0}) \geq \frac{1}{2}(s-\sigma)q + \frac{1}{2}(s-2+\sigma)l(C_{i_0}) + s$. If $k_0 \geq k - r$, then $I_1 \geq \frac{1}{2}(s-\sigma)(q+1)(k-1) + \frac{1}{2}(s - 2 + \sigma)p_0 + 1$ by maximizing r to $k-1$. This implies that there exists i_0 such that $\sum_{i=1}^{q+1} d(x_i, C_{i_0}) \geq \frac{1}{2}(s-\sigma)(q+1) + \frac{1}{2}(s-2+\sigma)l(C_{i_0}) + 1$.

For the case that $4 \leq s \leq 6$, by Lemma 2.1, we have $d(y,P) \leq d(y,C) \leq \frac{1}{2}(s+\sigma)$ for all $y \in V(F_0)$. Therefore we obtain

$$\begin{aligned} I_1 &\geq (q+1)(s-1)k - \tfrac{1}{2}(q+1)(s+1) - \tfrac{1}{2}(s-3)l_0 - 2 - \tfrac{1}{2}f_0(s+\sigma) \\ &\geq (q+1)(s-1)k - \tfrac{1}{2}(q+1)(s+1) - \tfrac{1}{2}(s+\sigma)(qk+r-p_0-q-1) - 2 \\ &= (\tfrac{1}{2}(s-2-\sigma)q+s-1)k + \tfrac{1}{2}(s+\sigma)p_0 - \tfrac{1}{2}(s+\sigma)r - \tfrac{1}{2}(1-\sigma)(q+1) - 2 \end{aligned} \tag{15}$$

From (15), we deduce that if $k_0 < k - r$, then $I_1 \geq (\frac{1}{2}(s-2-\sigma)q + s - 1)k_0 + \frac{1}{2}(s+\sigma)p_0 + 1$ and therefore there exists i_0 such that $\sum_{i=1}^{q+1} d(x_i, C_{i_0}) \geq \frac{1}{2}(s-2-\sigma)q + \frac{1}{2}(s+\sigma)l(C_{i_0}) + s$. If $k_0 \geq k-r$, then $I_1 \geq \frac{1}{2}(s-2-\sigma)(q+1)(k-1) + \frac{1}{2}(s+\sigma)p_0 + 1$ and therefore there exists i_0 such that $\sum_{i=1}^{q+1} d(x_i, C_{i_0}) \geq \frac{1}{2}(s-2-\sigma)(q+1) + \frac{1}{2}(s+\sigma)l(C_{i_0}) + 1$.

In both cases, we have $l(C_{i_0}) < q + 1$ for otherwise there exists x_i such that $d(x_i, C_{i_0}) \geq s - 1$, contradicting (9) by Lemma 2.1. Then by Lemma 2.6 or Lemma 2.7, $G[V(C_{i_0} \cup P)]$ contains two independent cycles C' and C'' such that $l(C_{i_0}) \leq l(C'), l(C'') \leq q$ if $k_0 < k - r$, or $l(C_{i_0}) \leq l(C') \leq q$ and $l(C_{i_0}) \leq l(C'') \leq q + 1$ since $l(C_{i_0}) + q + 1 < 2(q+1)$, contradicting the maximality of k_0. This proves the theorem.

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