LCA Methodology

A Special View on the Nature of the Allocation Problem

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Abstract

One of the remaining important problems of life cycle inventory analysis is the allocation problem. A proper solution of this problem calls for a proper understanding of the nature of the problem itself. This paper argues that the established definition of the allocation problem as the fact that one unit process produces more than one function, is not appropriate. That definition points to an important reason of the occurrence of the problem, but the situation of internal (closed-loop) recycling already indicates that there may be product systems which contain multifunction processes, but which nevertheless need not exhibit an allocation problem. The paper proceeds by examining a number of simple hypothetical cases, and proposes a precise and operational definition of the allocation problem. This enables a systematic categorization of approaches for dealing with the allocation problem.

Keywords: Allocation problem, algorithmetic procedures; LCA methodology, allocation problem, algorithmetic procedures; matrix inversion, allocation problem, LCA methodology

1 Introduction

Many books and papers which discuss the principles of life cycle inventory analysis describe the allocation problem as the fact that one process provides more than one valuable function (see, for instance, CONSOLI et al., 1993; HUPPES & SCHNEIDER, 1994; LINDFORS et al., 1995; ANONYMOUS, 1998 and AZAPAGIC, 1996). This then immediately offers a starting point to discuss solutions to the allocation problem: the multi-function process must be split into a number of virtual processes by means of so-called allocation factors, or a process must be subtracted from the product system so that the coproduced function avoids that some other system produces it.

This way of defining the allocation problem has proven to be an efficient way of communicating: the allocation problem is now perceived by many scientists and practitioners in more or less one meaning. Also, solutions to the allocation problem have been brought to a much higher level than before. The earlier books wrote about a simple allocation on a mass basis (e.g. FAVA et al., 1991; FECKER, 1992), while the current developments show a sophisticated hierarchy of solutions, sometimes with criteria for accepting or rejecting a certain solution (e.g. HUPPES & FRISCHKNECHT, 1995; LINDFORS et al. 1995; ANONYMOUS, 1998).

Nonetheless, there remain some unclarities in these discussions. One instance of this is provided by the situation of internal recycling (alternatively called closed-loop recycling): here we definitely have a multi-function process, providing the functions waste treatment and production of secondary material, but there is not necessarily an allocation problem.

As an example, consider the simple system of the two processes "electricity production" and "fuel production", with the electricity production process producing a waste that can be used as an input for the fuel production process. The fuel production process thus effectively recycles waste into fuel. The complete specifications are:

- electricity production: output of 10 kWh electricity, output of 2 kg waste, input of 1 l fuel;
- fuel production: output of 100 l fuel, input of 500 kWh electricity, input of 200 kg waste.

As a functional unit we choose 1000 kWh electricity. It is easily checked that a multiplication of the electricity production process by 200 and a multiplication of the fuel production process by 2 leads to a consistent balance: there is a gross production of 2000 kWh electricity, half of which is used as an input for fuel production. The 200 l fuel that is produced is exactly consumed in the production of electricity, and exactly the 400 kg waste that is produced is fed into the production of fuel. In other words, we have included a recycling process without having performed an allocation step.

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From the above example, we can easily illustrate an important conclusion of the mathematical portion of this paper. A central task of the inventory analysis is the scaling of the data of the unit processes such that there is a system-wide balance per material and per service. If there are I of these balances and J unit processes, we need to find J scaling factors by means of I equations. Finding the scaling factors is straightforward if $I = J$, that is, if all processes produce a single function. When some processes produce more than one function, we have more equations than unknowns ($I > J$). This overdetermined system of equations will in general not have a unique solution, but it may have one in special cases, among which is the case of internal (closed-loop) recycling and the numerical example described above. The identification of situations in which there is a unique solution, and the way an allocation step affects I and/or J is the main topic of this paper. The paper does not discuss how the allocation problem should be dealt with, although some main directions are indicated in the concluding section.

This paper starts with the algorithmetic procedures of the inventory analysis for a descriptive LCA. Next, a number of very simple cases will be analyzed, with the purpose of coming to an exact definition of where and how computational problems occur that are related to the allocation problem. The paper closes with a summary of the findings from these cases, an exact definition of the allocation problem, and a number of implications for designing strategies to deal with the allocation problem.

2 The Algorithmetic Procedures of the Inventory Analysis

This paper presumes that the reader has a basic knowledge of the principles of life cycle inventory analysis. Nevertheless it starts with a concise overview of selected elements of inventory analysis. The reason for doing so is that it concentrates on its algorithmetic structure. Although the reader might feel that the algorithm for finding an inventory table is a dull exercise not to be discussed in methodological papers, we will see that this is a crucial aspect of appreciating the nature of the allocation problem. Moreover, in doing so we are able to discover connections to other branches of science, in particular to economic activity analysis (see e.g. KOOPMANS 1951b), a field of science that goes back to the early 1950s and that has acquired a much more solid foundation than life cycle inventory analysis has at present.

We assume that the economic unit processes which build the product system have been recorded in some format so that there is a clear separation between inputs from other processes ("economic inputs"), outputs to other processes ("economic outputs"), inputs from the environment ("environmental inputs"), and outputs to the environment ("environmental outputs"). (ISO labels economic flows as "intermediate product flows" and environmental flows as "elementary flows".) In order to construct an algorithmetic procedure we must first assign mathematical symbols to these flows. For that purpose we adopt the following rules:

- the ith economic flow of process j is given the symbol a_{ij};
- the kth environmental flow of process j is given the symbol b_{kj};
- input flows are represented by negative values of the variables a and b;
- output flows are represented by positive values of the variables a and b;
- flows that are not involved in a certain process have values 0 for the variables a and b;
- the number of economic flows is denoted by I, the number of processes by J, and the number of environmental commodities by K; these numbers are not necessarily equal to one of the others, neither is there a fixed ordering of these numbers in the sense that, say, $K < J < I$.

Figure 1 illustrates these conventions.

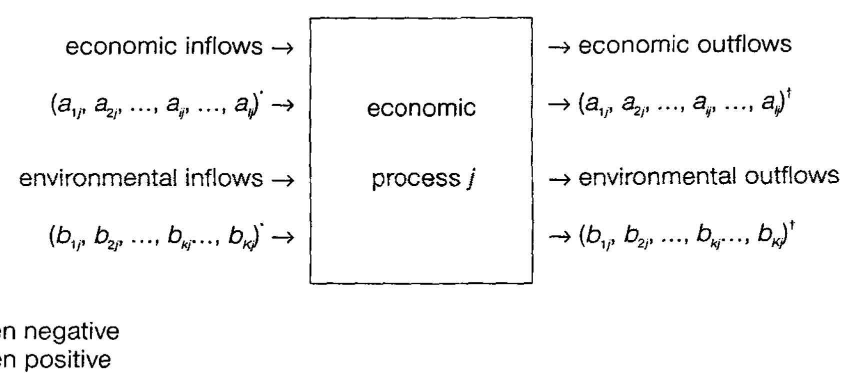

Fig. 1: Representation of an economic unit process

The above definitions comply with previous papers on the algorithmetic structure of life cycle inventory analysis, in particular with MÖLLER (1992), HEIJUNGS (1994), FRISCHKNECHT & KOLM (1995), and HEIJUNGS (1997), although irrelevant deviations of the chosen symbols may occur. Restricted to economic flows, the same structure appears in the much older literature of economic activity analysis (see for instance VON NEUMANN, 1945/1946; KOOPMANS, 1951a; GEORGESCU-ROEGEN, 1951; CHENERY & CLARK, 1959; DEBREU, 1959; ROSENBLUTH, 1968; SAXTON & AYRES, 1971 and GINSBURGH & WAELBROECK, 1981).

It should be observed that every process can in principle be multi-input and multi-output, and that representation of multi-function processes is not in any sense problematic. There is therefore no allocation problem at this stage of data collection, translation into an algorithm, and representation in a product system.

Usually a certain functional unit has been defined in the goal definition of the Life Cycle Assessment. Let us assume that the functional unit is 1000 kWh electricity (In ISO terminology: the functional unit is electricity and the reference flow is 1000 kWh). The last step of the inventory analysis is to scale the processes that build the product system in an appropriate way such that the system as a whole produces exactly the functional unit, and that all other economic flows have been traced back to environmental flows. In other words, there is one economic output: 1000 kWh electricity, and there are no other economic outputs or inputs. The translation of this statement into a mathematical expression which can be subject to an algorithm is the following: find multiplication factors for every process such that the net economic flows is restricted to an output of 1000 kWh electricity, while all other net economic flows are zero. Denoting the multiplication factor for process j by t_j, we seek to determine the t_js such that the aggregated amount of economic flow i over all scaled processes equals either the magnitude of the functional unit for the flow that corresponds to the nature of the functional unit, or that it equals 0 for all other economic flows. If we denote the net amount of economic flow i to or from the system as α_i (which is again negative for a flow entering the system and positive for a flow leaving the system), we thus have that one of these α_i is equal to +1000 kWh electricity, and the other α_is are equal to 0. The mathematical procedure to find the scaling factors t_j for every process j can now be stated as:

$$\forall\ i = 1, ..., I: \sum_{j=1}^{J} a_{ij} t_j = \alpha_i \quad (1)$$

Readers that are familiar with matrix algebra will immediately observe that the quantities in this equation can be summarized into a matrix equation:

$$\mathbf{A} \cdot \mathbf{t} = \boldsymbol{\alpha} \quad (2)$$

where the columns of the matrix A represent data with respect to one unit process, the rows of the matrix A and the rows of the vector $\boldsymbol{\alpha}$ represent data with respect to economic flows of one single type, and the rows of the vector t represent the scaling factors for the different processes. The matrices are thus defined as follows:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1J} \\ a_{21} & a_{22} & \cdots & a_{2J} & \cdots & a_{2J} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{iJ} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{I1} & a_{I2} & \cdots & a_{Ij} & \cdots & a_{IJ} \end{pmatrix};\ \boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_i \\ \cdots \\ \alpha_I \end{pmatrix};\ \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ \cdots \\ t_j \\ \cdots \\ t_J \end{pmatrix} \quad (3)$$

The matrix A is in the economic literature often called the technology matrix (VAN RIJCKEGHEM, 1967; TEN RAA, 1995). The scaling factors t_j are variously called intensities (VON NEUMANN, 1945/1946), activity levels (CHENERY & CLARK, 1995; SAXTON & AYRES, 1975) or operating times (HEIJUNGS, 1997). The set of scaling factors is sometimes called a program (CHENERY & CLARK, 1959).

Finding the scaling factors for the processes is of course not the aim of the inventory analysis: it is an intermediate step in the computation of the total environmental flows. To be more precise, the scaling factor of a process scales all data of that process, not only the data on economic flows. The scaled environmental flows are therefore given by the product of the recorded flow b_{kj} and the scaling factor for that process t_j. Aggregation of this amount over all processes yields the aggregated amount of environmental flow k. Denoting the aggregated amount of environmental flow k by β_k, we have symbolically

$$\forall\ k = 1, ..., K: \beta_k = \sum_{j=1}^{J} b_{kj} t_j \quad (4)$$

Again we may easily identify this with matrix operations by putting the elements b_{kj} into a matrix B and the elements β_k into a vector $\boldsymbol{\beta}$:

$$\boldsymbol{\beta} = \mathbf{B} \cdot \mathbf{t} \quad (5)$$

Here, the matrix B and the vector $\boldsymbol{\beta}$ are defined as

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1J} \\ b_{21} & b_{22} & \cdots & b_{2J} & \cdots & b_{2J} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kJ} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{K1} & b_{K2} & \cdots & b_{Kj} & \cdots & b_{KJ} \end{pmatrix};\ \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_k \\ \cdots \\ \beta_K \end{pmatrix} \quad (6)$$

The vector β which is calculated here contains K environmental flows, often called the environmental interventions (ISO does not reserve a seperate term for this, and employs "elementary flows"). It can therefore be said to represent the inventory table associated with the functional unit defined by the only non-zero element of the vector α. Matrix B has, as far as we know, not been denoted otherwise than the intervention matrix (HEIJUNGS, 1997).

Once we know the scaling factors (the vector t) the calculation of the inventory table (the vector β) is straightforward. The problem for the moment is the computation of the scaling factors from the process data (the technology matrix A) and the functional unit (the vector α). Equation (2) is an equation which states the relationship between **A**, **t**, and α, but does not express how **t** is to be found for a given A and α. Elementary matrix algebra states that **t** can be found by inversion of the matrix A and postmultiplication with the vector α:

$$\mathbf{t} = \mathbf{A}^{-1} \cdot \alpha \tag{7}$$

provided A is invertible.

It is exactly this condition of invertibility which determines if there is a computational problem that is very closely related to the allocation problem (cf. HEIJUNGS, 1994). If A is invertible, we can simply calculate **t** according to Equation (7) and proceed to calculate the inventory table according to Equation (5). Or in one step, substituting Equation (7) into Equation (5), we obtain the fundamental equation of life cycle inventory analysis:

$$\beta = \mathbf{B} \cdot \mathbf{A}^{-1} \cdot \alpha \tag{8}$$

If however the technology matrix A is not invertible we must either give up the idea of calculating an inventory table or we must go into a trick to find some kind of solution.

3 Some Illustrative Case Studies

This section is devoted to a discussion of a number of cases in which there are certain problems of finding the inverse of the technology matrix and thereby finding the inventory table. It should be emphasized that the cases are completely fictitious, the figures are chosen such that transparent manipulation is possible, instead of representing real process data. The first case will serve to illustrate the general formalism that is described above. The other four cases will throughout be based on small modifications to the original process data of the first one. For easy reference, the modifications will be marked by shading.

3.1 Case i: a simple illustration without complications

Suppose that we have two unit processes: fuel production and electricity production (→ *Fig.* 2), and that hypothetical process data are given (→ *Table 1*). The mathematical symbols to be assigned to these data are now as follows: a_{11} = 10, a_{21} = -1, b_{11} = 1, etc. Although we have not yet defined a functional unit, we may draw a system boundary around these two processes and thereby construct a product system. The output that is to be delivered can then be imposed by requiring a certain functional unit to be produced. The technology matrix is given by the first two rows of Table 1:

$$\mathbf{A} = \begin{pmatrix} 10 & -500 \\ -1 & 100 \end{pmatrix} \tag{9}$$

The intervention matrix is given by the last three rows:

$$\mathbf{B} = \begin{pmatrix} 1 & 10 \\ 0.1 & 2 \\ 0 & -50 \end{pmatrix} \tag{10}$$

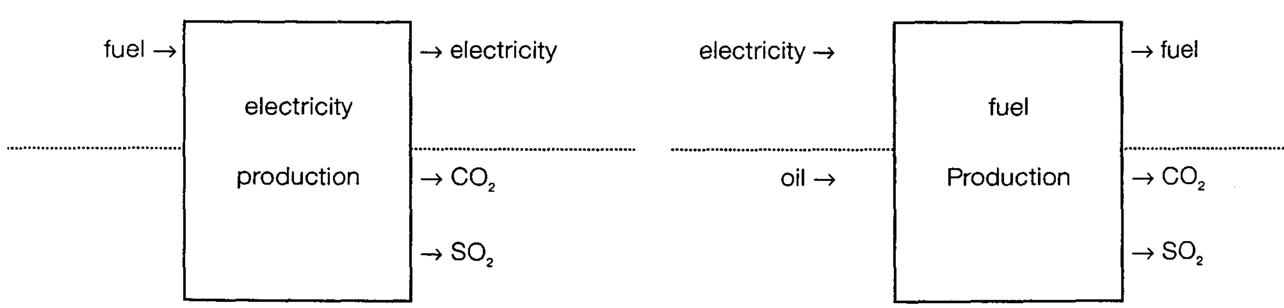

Fig. 2: The two hypothetical unit processes that play a central role in the examples

Table 1: Hypothetical process data for the simple example of two processes and two economic flows (Case i)

economic or environmental input or output	process 1 (electricity production)	process 2 (fuel production)
economic flow 1 (electricity in kWh)	10	-500
economic flow 2 (fuel in l)	-1	100
environmental flow 1 (CO_2 in kg)	1	10
environmental flow 2 (SO_2 in kg)	0.1	2
environmental flow 3 (crude oil in l)	0	-50

Assuming like above the functional unit to be 1000 kWh electricity, we can build a vector of external economic flows α:

$$\alpha = \begin{pmatrix} 1000 \\ 0 \end{pmatrix} \tag{11}$$

Calculation of the scaling factors is now simply given by

$$t = \begin{pmatrix} 10 & -500 \\ -1 & 100 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1000 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.2 & 1 \\ 0.002 & 0.02 \end{pmatrix} \cdot \begin{pmatrix} 1000 \\ 0 \end{pmatrix} = \begin{pmatrix} 200 \\ 2 \end{pmatrix} \tag{12}$$

Hence the scaling factor for process 1 is 200 and for process 2 it is 2. The inventory table is easily found as

$$\beta = \begin{pmatrix} 1 & 10 \\ 0.1 & 2 \\ 0 & -50 \end{pmatrix} \cdot \begin{pmatrix} 200 \\ 2 \end{pmatrix} = \begin{pmatrix} 220 \\ 24 \\ -100 \end{pmatrix} \tag{13}$$

In other words, the environmental interventions for the cradle-to-grave production of 1000 kWh electricity are an emission of 220 kg CO_2, an emission of 24 kg SO_2, and an extraction of 100 l crude oil.

We may interpret these matrix equations in terms of a system of two equations with two variables. The variables are the scaling factors for the two processes t_1 and t_2, and the equations express the balancing requirement (hence the obsolete name ecobalance) for each economic flow:

$$\begin{cases} 10t_1 + -500t_2 = 1000 \\ -1t_1 + 100t_2 = 0 \end{cases} \tag{14}$$

The matrix formalism enables to easily find the values for the variables t_1 and t_2.

From this example two things are immediately clear:

- The formalism has no problems in dealing with loops, such as that electricity production needs fuel and fuel production needs electricity (cf. FAVA et al., 1991 and HEIJUNGS et al., 1992).
- The same inverse of the technology matrix can be used to calculate any desired functional unit. As finding the inverse of a matrix is the most time-consuming step of the calculations one might consider to provide with every database the inverse of the technology matrix (cf. FRISCHKNECHT et al., 1994).

The next example will introduce a very small complication related to a process which is not involved in the product system providing the functional unit.

3.2 Case ii: a process that is not involved in the product system

Suppose that our database contains an additional process: waste incineration. The process data are given by **Table 2**. The technology matrix is now given by the first three rows of **Table 2**, and it contains an additional column for the newly introduced process; furthermore the intervention matrix also contains an extra column, and the vector of external economic flows an extra row:

$$\mathbf{A} = \begin{pmatrix} 10 & -500 & -5 \\ -1 & 100 & 0 \\ 0 & 0 & -1000 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} 1 & 10 & 900 \\ 0.1 & 2 & 10 \\ 0 & -50 & 0 \end{pmatrix}; \alpha = \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

Calculation of the scaling factors is still simple:

$$t = \begin{pmatrix} 10 & -500 & -5 \\ -1 & 100 & 0 \\ 0 & 0 & -1000 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 200 \\ 2 \\ 0 \end{pmatrix} \tag{16}$$

We see that the scaling factors for processes 1 and 2 remain unaffected, and that the scaling factor for process 3 is 0, which means that this process is not involved in the product system that provides the functional unit. The inventory table is therefore the same as before:

$$\beta = \begin{pmatrix} 1 & 10 & 900 \\ 0.1 & 2 & 10 \\ 0 & -50 & 0 \end{pmatrix} \cdot \begin{pmatrix} 200 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 220 \\ 24 \\ -100 \end{pmatrix} \tag{17}$$

Table 2: Hypothetical process data for the simple example of three processes and three economic flows, one process and one flow being not involved in providing the functional unit (Case ii)

economic or environmental input or output	process 1 (electricity production)	process 2 (fuel production)	process 3 (waste incineration)
economic flow 1 (electricity in kWh)	10	-500	-5
economic flow 2 (fuel in l)	-1	100	0
economic flow 3 (waste in kg)	0	0	-1000
environmental flow 1 (CO_2 in kg)	1	10	900
environmental flow 2 (SO_2 in kg)	0.1	2	10
environmental flow 3 (crude oil in l)	0	-50	0

From this example we learn one additional thing:

- The system may contain processes which are not involved in a certain product system. In other words, it is not necessary to delete or "hide" processes from the database in order to calculate the inventory table. With respect to memory requirements of computer equipment it may be advisable to do so, but in the light of the previous conclusion concerning the universal applicability of the inverse of the full technology matrix, it could still be advised to include all processes.

The same should obviously hold true for multi-function processes that are not involved in the product system. That situation, however, introduces some important complications, and therefore deserves a separate discussion.

3.3 Case iii: a two-function process that is not involved in the product system

Suppose that the process of waste incineration of Case ii fulfils two functions: processing of organic waste and processing of chemical waste (→ *Table 3* for the process data). The technology matrix and the vector of economic flows now have four rows:

$$A = \begin{pmatrix} 10 & -500 & -5 \\ -1 & 100 & 0 \\ 0 & 0 & -1000 \\ 0 & 0 & -200 \end{pmatrix}; \quad \alpha = \begin{pmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (18)$$

This matrix can not be inverted because it is not square: there are more rows than columns. We may interpret this as the problem of an overdetermined system: there are three variables (the scaling factors for the three processes) and there are four equations (one for each economic flow):

$$\begin{cases} 10t_1 + -500t_2 + -5t_3 = 1000 \\ -1t_1 + 100t_2 + 0t_3 = 0 \\ 0t_1 + 0t_2 + -1000t_3 = 0 \\ 0t_1 + 0t_2 + -200t_3 = 0 \end{cases} \qquad (19)$$

In general, one cannot solve an overdetermined system of equations, because there is a conflict between two or more equations.

An intuitive argument, however, says that the situation is not different from that of Case ii: process 3 is not active and columns 3 and rows 3 and 4 might have been discarded without altering the characteristics of the system as involved

Table 3: Hypothetical process data for the simple example of three processes and three economic flows, one multi-function process and two flows being not involved in the product system that provides the functional unit (Case iii)

economic or environmental input or output	process 1 (electricity production)	process 2 (fuel production)	process 3 (waste incineration)
economic flow 1 (electricity in kWh)	10	-500	-5
economic flow 2 (fuel in l)	-1	100	0
economic flow 3 (organic waste in kg)	0	0	-1000
economic flow 4 (chemical waste in kg)	0	0	-200
environmental flow 1 (CO_2 in kg)	1	10	1000
environmental flow 2 (SO_2 in kg)	0.1	2	30
environmental flow 3 (crude oil in l)	0	-50	0

for the functional unit. In fact the third and fourth equation are redundant: it follows directly from

$$\begin{cases} 0t_1 + 0t_2 + -1000t_3 = 0 \\ 0t_1 + 0t_2 + -200t_3 = 0 \end{cases} \quad (20)$$

that $t_3 = 0$. An important question is now how this intuitive aspect can be formalized so that software for life cycle inventory analysis is able to deal with this situation.

For that purpose we need to go briefly into the theory of the pseudo-inverse of matrices (or Moore-Penrose generalized inverse; see GOLUB & VAN LOAN, 1993 and STEWART & SUN, 1990 for a theoretical account, and PRESS et al., 1992) for a "numerical recipe" for calculating a pseudo-inverse. Furthermore, many mathematical packages (like Mathematica, Matlab and Linpack) contain a standard routine for it. The pseudo-inverse of a matrix A is equal to the inverse for a normal non-singular square matrix, and is defined as well for rectangular matrices and for square matrices for which the normal inverse does not exist. The pseudo-inverse can be found by means of singular value decomposition, and has the property that it is an approximation to what a normal inverse should be. While a normal inverse A^{-1} has the property that it gives the unit matrix I when multiplied with the matrix itself ($A \cdot A^{-1} = I$), the pseudo-inverse $A^{\dagger}$ is the matrix which gives, when multiplied with the matrix itself, a result that is closest to the unit matrix. This should be understood in terms of the systems of equations $\alpha = A \cdot t$ in which α and A are known and the unknown t can be found by $t = A^{-1} \cdot A \cdot t = A^{-1} \cdot \alpha$. For a normal inverse A^{-1} we have thus

$$\| A \cdot A^{-1} \cdot \alpha - \alpha \| = 0 \quad (21)$$

while for a pseudo-inverse $A^{\dagger}$ we have

$$\| A \cdot A^{\dagger} \cdot \alpha - \alpha \| = \text{minimal} \quad (22)$$

The vector $A \cdot A^{\dagger} \cdot \alpha - \alpha$ is called the minimum residual (GOLUB & VAN LOAN, 1993). Its length $\| A \cdot A^{\dagger} \cdot \alpha - \alpha \|$ indicates the distance in terms of the square root of the sum of the squares of the different elements of the expression inside.

In the example above we can compute the pseudo-inverse of the technology matrix and postmultiply it with α to obtain the scaling factors:

$$t = \begin{pmatrix} 10 & -500 & -5 \\ -1 & 100 & 0 \\ 0 & 0 & -1000 \\ 0 & 0 & -200 \end{pmatrix}^{\dagger} \cdot \begin{pmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 200 \\ 2 \\ 0 \end{pmatrix} \quad (23)$$

Clearly we find that process 3 is not involved for the functional unit. An important check is now whether the pseudo-inverse has yielded the closest but yet inexact solution or whether it has found an exact solution. This can be investigated by computing the distance between what is actually delivered externally ($A \cdot t = A \cdot A^{\dagger} \cdot \alpha$) and what was supposed to be delivered externally (α):

$$\| A \cdot A^{\dagger} \cdot \alpha - \alpha \| = \left\| \begin{pmatrix} 10 & -500 & -5 \\ -1 & 100 & 0 \\ 0 & 0 & -1000 \\ 0 & 0 & -200 \end{pmatrix} \cdot \begin{pmatrix} 200 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\| = 0 \quad (24)$$

In this case the solution found with the pseudo-inverse is therefore an exact solution to the problem that was to be solved. The inventory table can be found in exactly the same way as was discussed in Case ii.

One further remark here is that the system could be solved for this particular functional unit. The approach with the pseudo-inverse will for some choices for the functional unit not lead to an exact answer. For example, for the functional unit "treatment of 1000 kg chemical waste", we have

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1000 \end{pmatrix} \quad (25)$$

but the solution found with the pseudo-inverse appears to produce an external economic flow that is given by

$$A \cdot A^{\dagger} \cdot \alpha = \begin{pmatrix} 0 \\ 0 \\ -192 \\ -38.5 \end{pmatrix} \quad (26)$$

with a minimum residual distance

$$\| A \cdot A^{\dagger} \cdot \alpha - \alpha \| = 196 \quad (27)$$

as the apparently closest solution. In the above calculations, we have neglected to specify the dimensions of the elements. A situation with non-zero minimum residuals will most often exhibit bizarre unphysical units; see also Case iv.

One new aspect has been learned:

- Systems with multi-function processes can, when these processes are not involved in the product system that provides the specified functional unit, be solved in an exact way without entering an allocation procedure.

The next case discusses the less trivial situation that a multi-function process is involved in the product system.

Table 4: Hypothetical process data for the simple example of two processes and three economic flows, one economic process being a multi-function process (Case iv)

economic or environmental input or output	process 1 (electricity production)	process 2 (fuel production)
economic flow 1 (electricity in kWh)	10	-500
economic flow 2 (fuel in l)	-1	100
economic flow 3 (steam in MJ)	1	0
environmental flow 1 (CO_2 in kg)	1	10
environmental flow 2 (SO_2 in kg)	0.1	2
environmental flow 3 (crude oil in l)	0	-50

3.4 Case iv: a two-function process that is involved in the product system

Suppose that the process of electricity production coproduces a certain amount of steam. The process data are in **Table 4**. The technology matrix is now

$$\mathbf{A} = \begin{pmatrix} 10 & -500 \\ -1 & 100 \\ 1 & 0 \end{pmatrix} \tag{28}$$

Like in the previous case, the matrix can not be inverted because it is not square. We will explore the procedure with the pseudo-inverse. The technology matrix A can be pseudo-inverted and premultiplied by the vector of external flow $\boldsymbol{\alpha}$ to provide the vector of scaling factors t:

$$\mathbf{t} = \begin{pmatrix} 98 \\ 0 \end{pmatrix} \tag{29}$$

Calculation of the external flows which are effectively produce with these scaling factors gives

$$\mathbf{A}\cdot\mathbf{t} = \begin{pmatrix} 980 \\ -98 \\ 98 \end{pmatrix} \tag{30}$$

The distance between imposed and obtained external flow is therefore

$$\|\mathbf{A}\cdot\mathbf{A}^{\dagger}\cdot\alpha - \alpha\| = \left\| \begin{pmatrix} 980 \\ -98 \\ 98 \end{pmatrix} - \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix} \right\| = 140 \tag{31}$$

The solution that is produced with the pseudo-inverse is therefore not an exact solution. Moreover, it violates the rules of dimension analysis, according to which different units may never be added. The residual that has been minimized has the bizarre dimension $\sqrt{\mathrm{kWh}^2 + \mathrm{l}^2 + \mathrm{MJ}^2}$. If we express the third flow in GJ instead of MJ, thereby replacing the 1 for a_{31} into 0.001, we obtain quite different operating times:

$$\mathbf{t} = \begin{pmatrix} 200 \\ 2 \end{pmatrix} \tag{32}$$

producing an external flow of

$$\alpha \approx \begin{pmatrix} 1000 \\ -0.0002 \\ 0.2 \end{pmatrix} \tag{33}$$

with a subsequently reduced minimum residual, but a still unphysical dimension: $\sqrt{\mathrm{kWh}^2 + \mathrm{l}^2 + \mathrm{GJ}^2}$. One might argue that kWh and GJ are energy units, and that l oil could be converted into energy units as well, so that everything inside the square root is squared energy indeed. But in a more general situation we will have something like $\sqrt{\mathrm{MJ}^2 + \mathrm{kg}^2 + \mathrm{m}^2 + (\mathrm{m}^2)^2 + \mathrm{hr}^2 + ..}$ with MJ of energy, kg of steel, m of wire, m^2 of coating, and hr of labour mixed up. The point is, that minimization of the sum of squares of dimension-bearing quantities always results in a dimension-dependent minimum.

We conclude from the above:

- The procedure with the pseudo-inverse does not in all situations produce an acceptable solution.

Only special choices of a multiple functional unit (for instance, 1000 kWh electricity and 200 MJ steam) resolves the inconsistency between the equations.

The last case discusses the situation that a multi-function process is involved in a rather special way.

Table 5: Hypothetical process data for the simple example of two processes and three economic flows, one process producing an internally used flow of recycled waste (Case v)

economic or environmental input or output	process 1 (electricity production)	process 2 (fuel production)
economic flow 1 (electricity in kWh)	10	-500
economic flow 2 (fuel in l)	-1	100
economic flow 3 (waste in kg)	2	-200
environmental flow 1 (CO_2 in kg)	1	10
environmental flow 2 (SO_2 in kg)	0.1	2
environmental flow 3 (crude oil in l)	0	-50

3.5 Case v: a two-function process of which both functions are involved in the "correct" proportion

Suppose that the electricity production process produces a certain amount of waste with an energetic content, and that the production process of fuel can use this waste. We would say that the fuel production process recycles waste into fuel. Because the waste is internally used, this situation is often referred to as internal or closed-loop recycling (→ *Table 5* for the process data). The technology matrix and the external flows are given by

$$\mathbf{A} = \begin{pmatrix} 10 & -500 \\ -1 & 100 \\ 2 & -200 \end{pmatrix}; \ \alpha = \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix} \quad (34)$$

Again, we have a non-invertible matrix, and again intuition tells us that this system must have an exact solution. After all, the second and third equation appear to be redundant, because the two equations are linear combinations of one another:

$$\begin{cases} 10t_1 + -500t_2 = 1000 \\ -1t_1 + 100t_2 = 0 \\ 2t_1 + -200t_2 = 0 \end{cases} \quad (35)$$

Indeed, if we employ the trick with the pseudo-inverse we are able to find a solution

$$\mathbf{t} = \begin{pmatrix} 10 & -500 \\ -1 & 100 \\ 2 & -200 \end{pmatrix}^{\dagger} \cdot \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 200 \\ 2 \end{pmatrix} \quad (36)$$

This solution can be shown to be an exact solution in the sense that the distance between calculated external flow and imposed external flow is 0. The inventory table can be calculated from this as usual.

This example was based on closed-loop recycling, but the conclusion can be extended to systems with multi-function process in general. The essential element is that the functions that are delivered in some proportion by the multi-function process must be used in the system in exactly the same proportion. The situation thus also may apply to coproduction or combined waste treatment, not only to multi-function processes with a recycling character. Like in Case iii, not all choices for the functional unit will give an exact result. Indeed, Case iii can be seen as a special instance of Case v: the two functions that are delivered by the two-function process are involved in the "correct" proportion: 0 and 0.

In conclusion:

- Systems with a multi-function process can, when the functions of this process are involved in the "correct" proportion, be solved in an exact way without entering an allocation procedure. Only one function is leaving the system, because the other functions of these processes in the system fully cover the need of the functional unit analyzed and therefore are entirely used up within the system.

If a multi-function process is involved in a more general way, we retrieve Case iv where an allocation procedure seems to be necessary for finding a solution.

This concludes our overview of hypothetical cases. It's time to review the findings.

4 Towards a Definition of the Allocation Problem

From the hypothetical cases we are able to infer a number of results:

- The matrix formalism provides a powerful algorithm for computing an inventory table, also for product systems with loops
- The inclusion of multi-function processes leads under a number of different and frequently occurring circumstances not to an allocation problem: when these proc-

esses turn out not to be involved in the product system, and when the useful flows of these processes turn out to be involved in the "correct" proportion
- The matrix formalism must be extended with a routine to calculate the pseudo-inverse of the technology matrix for those cases for which the simple matrix inversion does not work. It may happen that the solution found with the pseudo-inverse turns out to be an exact solution of the inventory problem, especially if the multi-function processes do not take part in the product system at hand, or if their useful flows are involved in exactly the "correct" proportion, such as is the case with internal or closed-loop recycling.

We thus have achieved a fairly complete picture of the algorithmetic structure of the inventory analysis, and have seen a number of quite different situations in which a simple solution to the calculation problem of the inventory analysis fails. For many of those situations (such as multi-function processes that are not involved and multi-function processes of which the functions are involved in the "correct" proportion), we have found a systematic procedure.

In the way described above, we have defined the allocation problem not in the usual way related to the occurrence of multi-function processes, but to the fact that the technology matrix A can not be inverted, added to the fact that application of the pseudo-inverse does not yield an exact solution. So now, finally, we are able to define the allocation problem:

The allocation problem refers to the fact that the scaling factors for the processes in the product system can not be determined so as to satisfy the functional unit. This condition can be formalized as $\| A \cdot A^{\dagger} \cdot \alpha - \alpha \| \neq 0$, *where* A *is the technology matrix that contains the process data,* α *is the vector of external flows that specifies the functional unit,* $A^{\dagger}$ *denotes either the normal inverse or the pseudo-inverse of the technology matrix, and* $\| \dots \|$ *represents the square root of the sum of the squares of the elements (the Pythagorean length).*

It must be emphasized that the allocation problem is hence not defined at a process level, but at a system level. In other words, one can not tell whether a particular process will create an allocation problem.

Furthermore, the allocation problem is to some extent defined without regard to the functional unit α: the allocation problem is in principle present or not only in relation to the processes of which data is known. One might argue that the set of processes included in the product system depends on the functional unit chosen. For practical reasons that is indeed the case. In a more fundamental respect, however, the set of processes included for any functional unit contains all processes that exist (HOFSTETTER, 1996). Therefore the flow chart of processes that build the product system is – in qualitative terms – theoretically identical for every functional unit. With the development of large highly connected databases, like the one of FRISCHKNECHT et al. (1994), this theoretical argument has already started to become a real fact.

In the above, it was stated that the allocation problem is only to some extent defined without regard to the functional unit. To some extent it is, however: Case iii, for instance, did not result in an allocation problem for the functional unit of 1000 kWh electricity, but did result in one for the functional unit of treating 1000 kg chemical waste.

The situation is as follows:

- If the normal inverse of the technology matrix exists, there is never an allocation problem (Cases i and ii)
- If the normal inverse of the technology matrix does not exist because the matrix has more rows than columns, the pseudo-inverse may in a substantial number of cases provide an exact solution (Cases iii and v). There is not for all choices of the functional unit an exact solution, which indicates an allocation problem
- If the procedure with the pseudo-inverse does not produce an exact solution, there is an allocation problem (Case iv)

This last point implies that it is the database itself, the collection of process data that is used for finding the inventory table for any functional unit, that can create the allocation problem. The allocation problem is there or is not there, independent of the case study at hand. However, the solution of the allocation problem may be (highly) dependent on the goal and scope of the case study. A first attempt to link goals and allocation procedures is provided by FRISCHKNECHT (1997).

The condition of being invertible in the normal sense can be translated into two requirements which both have to be satisfied:

- the matrix A must be square, *i.e.* the number of rows I must be equal to the number of columns J, *i.e.* the number of processes must be equal to the number of economic flows;
- the matrix A must be non-singular, *i.e.* the columns of A must be independent vectors, *i.e.* no process may be constructed by a linear combination of the other processes.

This second requirement means in practice that one must take care not to load the same process twice into the database. If one does, there will be two columns that are exactly identical. The question of independence is in general difficult to comprehend. In any case one should be cautious not to put an aggregated inventory table (say, of an electromotor) together with its constituent processes into the same database, because the aggregated inventory table is by definition a linear combination of the constituent processes.

One runs into the approach with the pseudo-inverse whenever the number of processes is smaller than the number of economic flows. This can only mean that there are processes that produce more than one economic flow (of course produce means here production of valuable commodities or elimination of waste). The argument is now that the occurrence of a multi-function process is a necessary condition for the existence of the allocation problem. It is not a sufficient condition, however: Case iv and v describe a system with a multi-function process without an allocation problem. Practitioners of life cycle inventory analysis and designers of databases should realize that they put too much effort into their work when they jump into an allocation procedure for every multi-function process which they see. It seems natural to first wait if the technology matrix is nevertheless invertible or if the procedure with the pseudo-inverse gives an exact solution, and only start an allocation procedure for Case iv. In the situation that the normal inverse does not exist (Cases iii, iv and v) it could be argued to anyhow start an allocation procedure in order to strive for a universally applicable database. Making the same appeal to universal applicability, one might argue to start the allocation procedure for any multi-function process. It can be shown that in those cases where the pseudo-inverse leads to an exact solution, one would have found the same answer when applying an allocation procedure with an arbitrary allocation parameter. Practically that means that one can save oneself a lot of trouble by only entering the allocation procedure when strictly required, and not, for instance, in the case of closed-loop recycling.

5 Towards a Solution of the Allocation Problem

This paper is concerned with the nature of the allocation problem, not with strategies to solve the problem. However, understanding the true nature of the problem adds suggestions to the discussion how to solve it. It turns out that the matrix representation offers once more a suitable framework for analysis and discussion. Most proposals for dealing with the allocation problem are easily described with the matrix formalism. A small number of remarks will be made with that respect.

The allocation problem has its origins in an impossibility to solve a matrix equation. We may easily identify three different strategies for solution:

- The first strategy is to modify the question. In Case iv we saw such a modification: if the vector of external demand were to be changed from

$$\alpha = \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix} \quad \text{into} \quad \alpha = \begin{pmatrix} 1000 \\ 0 \\ 200 \end{pmatrix},$$

  hence changing the functional unit from 1000 kWh electricity to a double functional unit of 1000 kWh electricity and 200 MJ steam, there was no allocation problem any longer. If, however, we are interested in the environmental interventions of the different coproducts, we may extend this strategy with an allocation procedure to partition the full set of environmental interventions to each of the net economic flows. This strategy is used in the systems expansion approach to achieve an equality of benefits (see Fleischer & Schmidt, 1996). It may be referred to as "allocation at the system level".
- The second strategy is to reduce the number of equations. This amounts to reducing the number of rows of the technology matrix and the vector of external flows. In Case iv, we could merge the rows that correspond to "electricity" and "steam" into one row bearing the name "energy". Doing so will take away the redundancies in the set of equations. This strategy is often employed when a certain material is coproduced and almost the same material is used by another process. Or if electricity and steam are coproduced in, say, The Netherlands and the electricity is used by another process in, say, Switzerland in the same product system that uses the Dutch steam. The suggested interpretation is that "open-loop recycling is treated as closed-loop recycling" (see Anonymous, 1998).
- The third strategy is to increase the number of variables. This amounts to introducing additional processes in the system. In Case iv, we might introduce an alternative process to produce steam. The inventory equation would become solvable, yielding a negative scaling factor for the newly introduced process of steam production if the functional unit is 1000 kWh electricity. This negative factor is often interpreted as "avoided production", and the whole procedure may be referred to as the "substitution method" (see Ekvall, 1992). An alternative form of the same strategy is to split the multi-function process that produces electricity and steam into two independent single-function processes. This would also result in a solvable inventory equation for a functional unit of 1000 kWh electricity, yielding a positive scaling factor for the electricity production process and a zero scaling factor for the steam production process. This procedure may be referred to as "allocation at the process level" (see Huppes, 1994).

An interesting feature of the above categorization of strategies is that it differs from the description and typology that is currently proposed by, amongst others, ISO (Anonymous, 1998). Our conjecture is that our proposal provides a categorization that is more systematic. It shows that the concepts of avoided emissions is only a special case of allocation of environmental interventions (both increasing the number of variables) and leads to a particular allocation of environmental interventions and requirements by referring to the environmental performance of competing processes (cf. Frischknecht, 1998a).

The three strategies must clearly be interpreted as frameworks which may contain different elaborations. For instance, the third strategy contains at least two families (the substitution method and allocation at the process level), and each of these can be implemented in different ways. For the first family there is the choice of the most adequate process to be subtracted, for the second family, the most striking difference is perhaps the choice of the allocation factor on, *e.g.*, a physical basis, an economic basis, marginal relationships, 50-50 partitioning, *et cetera*. Discussion of the different members to be distinguished within each family of strategies and of the ways to actually achieve the allocation procedure is, however, beyond the scope of this paper (see FRISCHKNECHT, 1998a, 1998b). Suffice it to point out for the moment that strategies for dealing with the allocation problem go back to the economic literature on activity analysis (see, for instance, VAN RIJCKEGHEM, 1967; TEN RAA et al., 1984 and KONIJN, 1994) and that a satisfactory answer has not yet been formulated (see ROSENBLUTH, 1968 and TEN RAA, 1988). It has also been studied in the context of environmental economics (see OENNING, 1997). We think that the debate on allocation in LCA can be enriched by taking this tradition into account.

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