LCA Methodology

Linear Programming as a Tool in Life Cycle Assessment

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Abstract

Linear Programming (LP) is a powerful mathematical technique that can be used as a tool in Life Cycle Assessment (LCA). In the Inventory and Impact Assessment phases, in addition to calculating the environmental impacts and burdens, it can be used for solving the problem of allocation in multiple-output systems. In the Improvement Assessment phase, it provides a systematic approach to identifying possibilities for system improvements by optimising the system on different environmental objective functions, defined as burdens or impacts. Ultimately, if the environmental impacts are aggregated to a single environmental impact function in the Valuation phase, LP optimisation can identify the overall environmental optimum of the system. However, the aggregation of impacts is not necessary: the system can be optimised on different environmental burdens or impacts simultaneously by using Multiobjective LP. As a result, a range of environmental optima is found offering a number of alternative options for system improvements and enabling the choice of the Best Practicable Environmental Option (BPEO). If, in addition, economic and social criteria are introduced in the model, LP can be used to identify the best compromise solution in a system with conflicting objectives. This approach is illustrated by a real case study of the borate products system.

Keywords: Allocation, multiobjective linear programming; Best Practical Environmental Option (BPEO); Boron; BPEO; dual values, LCA; LCA; Life Cycle Assessment, LCA; marginal analysis, LCA; operations research, LCA; Pareto Analysis; system optimisation, LCA

1 Introduction

One of the main applications of Life Cycle Assessment (LCA) is comparison of possible modifications to an existing product or process with the aim of improving its environmental performance. In most cases there are many possibilities for improvements and the choice of the best alternative is not obvious. Therefore, to identify the optimum options for system improvements, it is necessary to use a suitable optimisation technique. This paper illustrates the potential of applying one such technique – Linear Programming – to optimise the environmental performance of a product system as a part of the Improvement Assessment phase of LCA. In addition, it demonstrates the value of Linear Programming (LP) in solving the problem of allocation in multiple-function systems as a part of both the Inventory and Impact assessment phases, and shows how the approach can be used to identify the Best Practicable Environmental Option for a process or product system. The use of the approach is illustrated by a specific example of a multi-product system which produces different borate products from boron ores.

2 Linear Programming and Life Cycle Assessment

For present purposes, a model means a set of mathematical relationships which describe the operation of the unit processes forming a product system. LCA is based on linear homogeneous and unconstrained models of human economic activities and of their effect on the environment; i.e. environmental burdens and their impacts are assumed to be directly proportional to the number of functional units produced (e.g. HEIJUNGS, 1992; HUPPES and SCHNEIDER, 1994). The commonest and simplest form of linear model, usually known as an input/output model, takes the form of a set of homogeneous linear equations relating the outputs from and inputs to an economic system. In LCA, this type of model is extended to relate the environmental burdens associated with a product system to its economic outputs (e.g. HEIJUNGS, 1997). However, the simple input/output approach is not able to account for the internal structure of the system and it cannot help to improve the efficiency of industrial operations. Simple input/output modelling is also not applicable to the analysis of a product system whose operation is constrained, for example, by the capacity of existing unit processes, or by the availability of material or energy inputs. Since in reality, almost all systems are subject to a number of constraints, a more powerful approach to system analysis and optimisation in LCA is necessary.

The approach proposed here uses Linear Programming (LP) to model the behaviour of a linear system subject to constraints. The same method can be used for non-linear systems that can be approximated as linear. The general mathematical relationships are given in the text, with a more detailed account of LP in the Appendix. Other examples of the applications of LP to LCA, in addition to the specific example shown here, are given by AZAPAGIC (1996) and AZAPAGIC and CLIFT (1994, 1995a, 1995b, 1998a, 1998b, 1998c). The concepts behind the use of LP in LCA are shown schematically in **Figure 1**.

The main characteristic of this kind of modelling is that it is based on physical and technical relationships between the inputs and outputs and environmental interventions of the system. Therefore, a LP model describes the underlying physical causation in the system and thus lends itself naturally to solving allocation in multiple-function systems according to the procedure recommended by ISO 14041 (1997). Moreover, because LP modelling describes complex interactions between different parts of the system, it can describe changes in the operating state of the system and associated environmental interventions, resulting from changes in material or process properties. This approach therefore reveals how environmental burdens and impacts – and their allocation between different functions – change as the operation of the system is changed. These features are particularly useful in the Inventory and Impact Assessment phases, as discussed further in the paper.

The LP approach is also valuable in the Improvement Assessment phase for quantifying the trade-offs between different environmental burdens and impacts. In effect, the LP model identifies a range of optimum solutions for improved environmental performance which can be achieved through various modifications to the product system, including changes in economic performance. In this way, by combining both environmental and socio-economic criteria, this approach enables identification of the Best Practicable Environmental Option (BPEO) not entailing excessive costs.

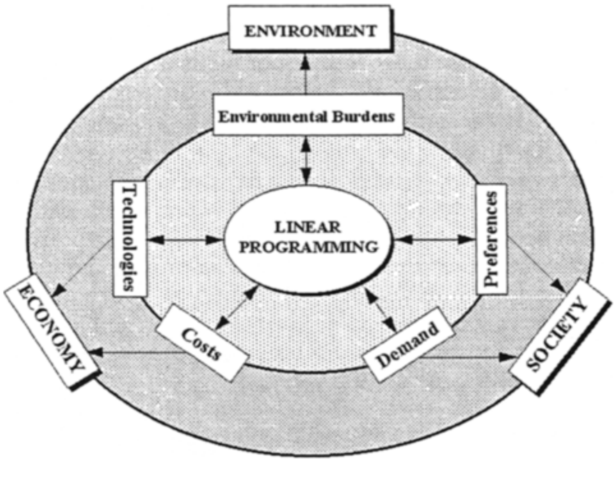

Fig. 1: Linear Programming as a tool in Life Cycle Assessment

2.1 Modelling in the Linear Programming Format

A conventional Linear Programming model of an economic system has the form:

$$\text{Maximise (or Minimise)} \quad F = \sum_{i=1}^{I} f_i x_i \tag{1}$$

$$\text{subject to} \quad \sum_{i=1}^{I} a_{c,i}\, x_i \le e_c \qquad c = 1,2,\ldots,C \tag{2}$$

$$\text{and} \quad x_i \ge 0 \qquad i = 1,2,\ldots,I \tag{3}$$

where eqn. (1) represents an objective function, usually a measure of economic performance (e.g. profit or cost) and eqns. (2)-(3) are linear constraints in the system, describing material and energy balance relationships, productive capacity, raw material availabilities, quality requirements, market demand and so on. The constraints can be defined as equalities or inequalities. The LP example in the Appendix shows how mass and energy balances lead to equality constraints, whereas raw material availability and other constraints can lead to inequalities. The variables x_i represent quantitative measures of material and energy flows including inputs, flows within the economic system, and outputs. In LP these variables are termed "activities", or more precisely, "activity levels". The coefficients $a_{i,i}$ are the factors of proportionality between activity (process) levels and inputs and outputs from the system. The right hand side coefficients, e_c, represent the limitations on the constraints.

The optimum point of a LP model is defined by the "active" constraints, i.e. by the constraints that are satisfied as equalities at the solution of the LP model. The other, non-active constraints do not influence the solution of the system and usually have "slack" or unused resources associated with it (→ *Appendix, p. 314*). At the solution, each active constraint has a dual or marginal value, which shows the change in the objective function with the change in the right hand side coefficient of that constraint. The discussion in the following section shows that the dual values represent the allocation coefficients which describe the physical causality in the system. Therefore, LP is useful in the Inventory and Impact Assessment phases for solving the allocation problem that arises in multiple-function systems.

The types of questions that conventional LP helps answer are, for instance, related to finding the optimum operating point in the system that maximises the profit (or minimises the costs) and uses the optimum amount of resources, subject to the constraints. In the context of LCA, the general LP model has the same form; however, the constraints (2) now encompass all activities from extraction of primary materials from earth through processing to final disposal. In addition, the functional output or outputs are also treated as activities. Furthermore, the objective functions are now defined by the environmental burdens if the analysis is at the

Inventory level, rather than an economic objective, as represented by Azapagic (1996):

$$\text{Minimise} \qquad B_j = \sum_{i=1} bc_{j,i} \; x_i \qquad (4)$$

where $bc_{j,i}$ is burden j from process or activity x_i. The objective functions can also be defined as the environmental impacts:

$$\text{Minimise} \qquad E_k = \sum_{j=1}^{J} ec_{k,j} B_j \qquad (5)$$

where $ec_{k,j}$ represents the relative contribution of burden B_j to impact E_k, as defined by the "problem oriented" approach to Impact Assessment (Heijungs et al., 1992). The LP example in the Appendix also show how environmental objectives can be formulated.

Depending on the goal of the study, the system can be optimised on one or a number of environmental and economic objective functions, to identify the optimum solutions for system improvements. This application of LP in Improvement Assessment is explained below.

2.2 Linear Programming in Inventory and Impact Assessment

The aim of the Inventory and Impact Assessment phases is to quantify total environmental burdens and impacts and not to optimise performance of the system. Therefore, the optimisation is not performed at this level of analysis; the LP model is solved to calculate the burdens and impacts for possible operating states of the system, which are normally defined by the optimum economic performance. For the type of systems that most LCA studies consider, i.e. unconstrained and homogeneous, this step is equivalent to the conventional LCA procedure of calculating the burdens and impacts. However, unlike the simple input/output analysis in the conventional approaches, formulation of the system as a LP model also provides a rigorous basis for allocation of the burdens and impacts in multiple-function systems by physical causality. This approach has already been proposed by Azapagic and Clift (1994, 1995a, 1995b, 1998b) and Azapagic (1996) and is explained in more detail below.

2.2.1 Allocation in multiple-function systems

The problem of allocation in multiple-function systems is to find a procedure which would assign to each of the functions of the system only those environmental burdens and impacts for which it is responsible. There are three types of multiple-function systems where allocation can be relevant: multiple-input systems (waste treatment processes), multiple-output systems (co-production), and multiple-use systems (open-loop recycling). Because the method for solving the allocation problem may depend on the goal of the study, different approaches to allocation are possible: allocation may be avoided by enlarging or disaggregating the system, or it can be solved by applying an appropriate causation principle (ISO, 1997). Avoiding allocation by system enlargement is an appealing way to deal with this problem; however, in many cases the systems considered will become more complicated or uncertain because of the additional data needed. Avoiding the allocation problem by disaggregation may be possible, but only if detailed data about the system are available and if the system can be broken down to eliminate all processes common to the functional outputs.

In all other cases some kind of allocation will still be necessary. The recommended approach (ISO, 1997) is to base allocation on the physical[1] causal relationships between burdens and functional outputs, which in turn requires a model of the system behaviour. Therefore, allocation on an arbitrary basis, such as mass or molar flow, must be avoided, unless the modelling of causalities shows such a basis to be appropriate. However, if it is not possible to apply the causation principle, for example if the functional outputs cannot be varied independently (see below), then economic relationships should be used as the only other relevant basis for allocation (Huppes and Frischknecht, 1995; ISO, 1997; Clift et al., 1998).

When the causal relationships are represented by a model which describes the real behaviour of the product system, the model can be used to allocate burdens between different functions by exploring how the burdens change when the quantity of one function is changed with the quantities of all the other functions kept constant (e.g. Azapagic and Clift, 1998b). An example of this kind of problem in the chemical industry is naphtha cracking, where the outputs can be varied independently by changing cracking conditions (Azapagic and Clift, 1995b).

The type of changes considered can be marginal, incremental or average which, in turn, depends on the goal and scope of the study and the questions to be answered by LCA (Clift et al., 1998; Azapagic and Clift, 1998b). Marginal changes are relevant when the performance of a specific system is analysed to determine the effect of infinitesimal changes. Because the changes are infinitesimal, this amounts to analysing the present state of the system, which most LCA studies are initially concerned with. This kind of analysis is relevant if the goal of the study is comparison of the products from a multiple-function system or comparison of one of the products from this system with a product from another system. Marginal changes can always be described by a linear model, even if the model is not homogenous (Clift and

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[1] The term "physical relationships" has a broader meaning in this context and includes physical, chemical, biological and technical relationships.

AZAPAGIC, 1995; CLIFT et al., 1998). Therefore, LP modelling can always be used to describe marginal changes. As discussed in the following section, the marginal values calculated at the solution of a LP model represent the marginal allocation coefficients which reflect the physical causal relationships in the system.

However, if the goal of the study is to consider incremental or average changes[2] to a new state of the system, for instance to describe product or process development, then a linear model may not be applicable (CLIFT and AZAPAGIC, 1995; CLIFT et al., 1998). Where the system behaviour can be linearised about the alternative operating states being considered, it may still be possible to use a LP model. This is normally the case for incremental changes, but is less likely to be applicable for average changes.

However, in some systems, the ratio between two or more functional units and their parameters in the system may be fixed; examples of this arise in the chemical industry, where the ratio of sodium-hydroxide (NaOH) to chlorine (Cl_2) produced by electrolysing brine is constant. In this case, allocation by physical causation cannot be used and socio-economic relationships, as the only other relevant choice, should be used instead. The argument for this is that economic relationships reflect socio-economic demand which causes the multiple-functional units to exist at all (CLIFT et al., 1998). These relationships are usually translated into some economic measure, such as net value at the point in the system where the product streams divide (HUPPES and FRISCHKNECHT, 1995).

2.2.2 Linear programming and allocation

The solution of the LP model defines for each constraint, as given by eqn. (2), a dual or marginal value which shows the effect on the objective function of a marginal change in the right-hand side coefficient, e_c, of the constraint with all the other constraints unchanged. This analysis is applicable only where the coefficients $e_1,\ldots e_C$ can in principle be subject to independent marginal changes.

The value of the objective function, as defined by eqn. (1), at the solution of the model can then also be written as a function of the right-hand side coefficients:

$$F = f[e_1, e_2, \ldots, e_C] \tag{6}$$

In the case of marginal changes in these coefficients, the corresponding change in the objective function is equal to:

$$dF = \sum_{c=1}^{C} \left(\frac{\partial F}{\partial e_c}\right)_{e_1,e_2,\ldots,e_{c-1},e_{c+1},\ldots,e_C} de_c \tag{7}$$

or in a simplified notation:

$$dF = \sum_{c=1}^{C} \lambda_c \, de_c \tag{8}$$

The partial derivative:

$$\lambda_c = \left(\frac{\partial F}{\partial e_c}\right)_{e_1,e_2,\ldots,e_{c-1},e_{c+1},\ldots,e_C} \tag{9}$$

represents the dual or marginal value which is calculated automatically at the solution of a LP model. For a marginal change, the derivative (9) will remain constant, i.e. the state of the system will not change, so that after integration, eqn. (8) becomes:

$$F = \sum_{c=1}^{C} \lambda_c e_c \tag{10}$$

so that, in these terms, the model is both linear and homogeneous. For the objective function defined as environmental burden B_j, then by analogy with eqn. (10), the total value of that burden is related to the dual values by:

$$B_j = \sum_{c=1}^{C} \lambda_{j,c} e_c \lambda_{j,c} = \left(\frac{\partial B_j}{\partial e_c}\right)_{e_1,\ldots,e_{c-1},e_{c+1},\ldots,e_C} \tag{11}$$

where $l_{j,c}$ is the marginal or dual value of the *c*th constraint for burden j, i.e. it represents the change in the total burden B_j with the change in coefficient e_c:

$$\lambda_{j,c} = \left(\frac{\partial B_j}{\partial e_c}\right)_{e_1,\ldots,e_{c-1},e_{c+1},\ldots,e_C} \tag{12}$$

Because the environmental impacts are linear homogenous functions of the burdens, they can be formulated in the same way. In general, eqn. (11) can be written as:

$$E_k = \sum_{c=1}^{C} \mu_{k,c} e_c \tag{13}$$

where:

$$\mu_{k,c} = \left(\frac{\partial E_k}{\partial e_c}\right)_{e_1,\ldots,e_{c-1},e_{c+1},\ldots,e_C} \tag{14}$$

The LP model can be formulated so that the quantities of functional outputs are represented by coefficients e_c, amongst other coefficients describing constraints of the system. If at the solution of the LP model only those constraints that are related to the functional outputs are active, then the dual

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[2] Incremental changes are small but finite changes which change the operating state of the system. Average changes are related to more substantial changes, for instance eliminating a functional output completely.

values represent the marginal allocation coefficients, which relate changes in a burden or impact to the infinitesimal changes in one output while all other functional outputs are held constant; i.e. $l_{j,c}$ and $m_{k,c}$ represent the allocated burden or impact, respectively. In LCA terms, such a system is unconstrained so that the burdens and impacts are allocated fully between the outputs (Azapagic, 1996; Clift et al., 1998; Azapagic and Clift, 1998b). Naturally, as noted above, the dual or marginal analysis is only applicable when all the coefficients e_c can be varied independently. Thus, as expected, the marginal allocation approach cannot be used for outputs which are produced in fixed proportions. However, whenever constraints which describe capacity or market demand are active at the solution, the burdens and impacts will be allocated in part to these constraints so that they are not fully allocated to the functional outputs (→ *Appendix*). This is consistent with the physical causation principle, as the changes in the system behaviour in this case are not determined by the functional outputs only, but by other properties of the system as well.

An illustration of the marginal allocation approach for a multiple-function system producing five boron products is presented in Section 3.

2.3 Linear Programming in Improvement Assessment

The main objective of the Improvement Assessment phase is to identify opportunities for improving the environmental performance of the system. To achieve this, the system has to be optimised on the environmental objective functions, B_j or E_k. There are different ways to approach the optimisation problem, depending on the goal of the study. For instance, the environmental impacts could be aggregated into a single impact function in the Valuation phase and the system optimised on this objective to give the overall environmental optimum of the system. However, we argue that, until the methodology is developed further, Valuation should be avoided and the system optimised on the set of environmental burdens or impacts instead (Azapagic, 1996). This is possible by using Multiobjective LP (MOLP) which enables simultaneous optimisation on a number of objective functions, resulting in a range of environmental optima of the system. These optima define the multidimensional non-inferior or Pareto surface, which is optimal in the sense that none of the objective functions can be improved without worsening one or more of the other objective functions. Therefore, trade-offs between objective functions are necessary in order to select the best compromise solution. For example, if two objective functions, defined as environmental impacts, are optimised simultaneously, the resulting Pareto optimum does not necessarily mean that these impacts are at their minima obtained when the system is optimised on each of them separately. The Pareto optimum, however, does mean that the set of best possible options has been identified for a system in which both impacts should be reduced. The value of MOLP in LCA, therefore, lies in offering a range of alternative solutions; they are all optimal in the Pareto sense, but the choice of the best one will depend on preferences and constraints imposed on decision-makers.

However, decisions are rarely made on the basis of environmental LCA only and other considerations, particularly economic and social, are usually involved. If, in addition to the environmental objectives, multiobjective optimisation also includes economic and social objectives, acceptable solutions which represent the compromise between conflicting objectives can be identified. This approach could be of particular importance for the process industries, which face the problem of having to keep production costs down while at the same time complying with environmental legislation and responding adequately to increased public awareness of environmental problems. For example, it shows how far from its economic optimum the system must be operated to achieve certain improvements in environmental performance.

The application of MOLP to the Improvement Assessment phase is now illustrated by a real case study of a multiple-function (co-product) system producing five borate products.

3 Case Study

The co-product system considered here produces the following products: 5 and 10 mol borate, boric acid (BA), anhydrous borax (ABA) and anhydrous boric acid (ABA). A simplified flow diagram of the process is shown in Figure 2. Further details of the system are given elsewhere (Azapagic, 1996; Azapagic and Clift, 1998d). Boron minerals – borax and kernite – are extracted in the mine, crushed and transported to the plant. 5 and 10 mol borates are produced by dissolving borax and kernite in water; Na-borates are then separated from insolubles, crystallised and dried to produce powder products. Boric acid is produced in a separate plant, by reacting kernite ore with sulphuric acid. The rest of the process is similar to the 5 and 10 mol production. Anhydrous borax and anhydrous boric acid are made in high-temperature furnaces from 5 mol borate and BA, respectively. All products are then either packed or shipped in bulk. Electric energy and the steam for the system are provided by the on-site natural gas cogeneration facility, which meets all of the electricity and most of the steam demand. Any additional steam is provided by a steam plant which is also fired by natural gas. The overburden from the mine and the gangue from the process are stocked in piles; the waste water from the refinery is discharged into self-contained ponds. All activities, from extraction of raw materials to the production of the boron products and materials used, are included in the system. However, the use and disposal phases of the products are not considered in this study ("cradle-to-gate" approach).

The goal of the study is to evaluate the environmental performance of the system and to identify the opportunities for environmental improvements. Since the study is intended for internal use, the functional unit is taken to be total yearly production of the boron products, i.e. 1062000 t/yr.

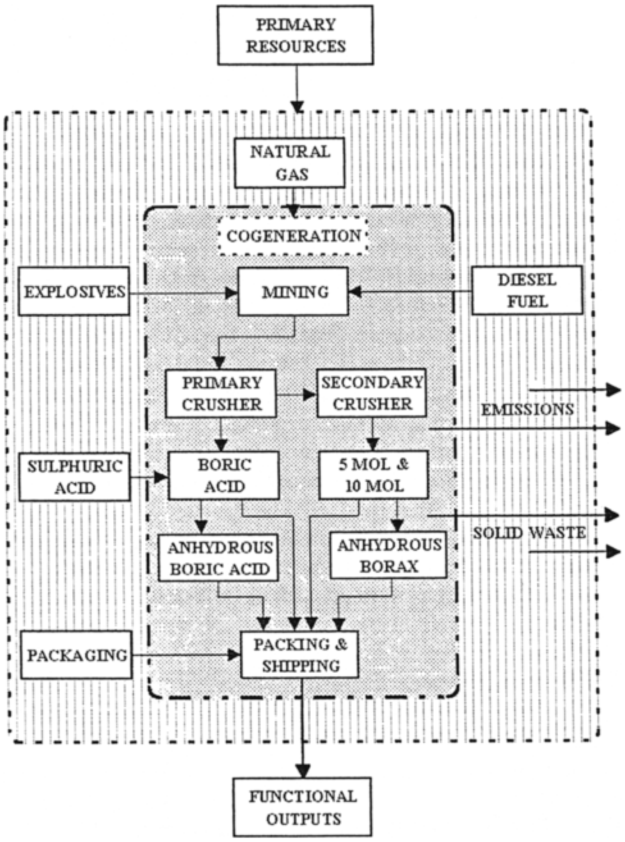

Fig. 2: LCA flow diagram of the boron products system

3.1 Allocation in the boron products system

The LP model of the boron system is described by the constraints defined by material balances, output of products, supply of primary and raw materials, capacity and heat requirements as given by:

Mass balance:
$$\sum_{i=1}^{I} a_{c,i} x_i = 0,\ i=1,2,\ldots,I \qquad (15)$$

Products output:
$$P_q \leq D_q,\ q=1,2,\ldots,Q \qquad (16)$$

Primary and raw material supply:
$$R_m \leq S_m,\ m=1,2,\ldots,M \qquad (17)$$

Productive capacity:
$$x_i \leq C_i,\ i=1,2,\ldots,I \qquad (18)$$

Heat requirements:
$$H_z \leq Q_z,\ z=1,2,\ldots,Z \qquad (19)$$

where production P_q is limited by the product demand D_q, primary and raw materials consumption R_m is determined by the supply S_m, activities or processes x_i are subject to the capacity limit C_i and the heat requirement H_z is constrained by the heat availability Q_z. Since the discussion in this chapter is related to the functional unit defined as the operation of the system for one year, the product demand D_q is taken to be equal to the total output of each product for one year. The objective functions of the system are environmental burdens and impacts, as defined by eqns. (4) and (5). A large scale LP software, XPRESSMP (Dash Associates, 1993), has been used to solve the system model and to calculate marginal values.

The functional outputs from the system can be changed independently and therefore allocation based on physical causation is appropriate. Since this study is concerned with a specific system and marginal changes around its operation, the burdens and impacts are allocated through marginal values of the LP model, found at the solution of the model. As noted above, for the purposes of allocation, the model is not optimised at this stage; it is solved to calculate the total burdens and impacts and the corresponding marginal values at the operating point of interest. The solution is found at the intersection of active constraints, which are in this case defined by the quantities of the product outputs (see eqn. (16)). Therefore, these constraints have non-zero marginal values, which are equivalent to the marginal allocation coefficients. Other constraints are non-active and their marginal values are zero; therefore, the burdens are fully allocated to the products. If the operating state of the system changed so that other constraints determined the operation of the system, then they would also be active at the solution and their marginal values would be non-zero; hence the burdens would be allocated to these constraints as well as to the products.

As an example, the results of the marginal allocation of the total CO_2 emissions of 295860 t/yr among different products are shown in **Table 1** and **Figure 3**. For instance, the marginal value, i.e. the emission of CO_2 allocated to 10 mol on the marginal basis, is equal to 0.18 t/t; i.e. if the output of this product increases by 1 t, the total CO_2 emissions increase by 0.18 t. The CO_2 emissions allocated to other products range from 0.24 t/t for 5 mol, through 0.41 t/t for BA to 0.98 t/t and 1.94 t/t for AB and ABA, respectively.

Table 1: CO_2 emissions allocated to the boron products

Product	Functional Output* (t/yr)	Marginal allocation (t/t)	Total allocated CO_2 (t/yr)
10 mol	81000	0.18	14580
5 mol	810000	0.24	194400
BA	150000	0.41	61500
AB	16000	0.98	15680
ABA	5000	1.94	9700
TOTAL	1062000		295860

*Production figures shown in this table are not based on actual production or sales figures in any particular year, but represent a possible production scenario.

Let us now compare the results of the marginal allocation approach with the arbitrary allocation methods most commonly used in co-product systems: mass and market value bases (→ *Fig. 3*). To illustrate the importance of system disaggregation, allocation on the mass basis for both aggregated and disaggregated systems is considered. System disaggregation was possible in this particular case study because the detailed data for the system were available. If allocation on the mass basis is done without system disaggregation, the total burdens are allocated among different products in proportion to the mass of their outputs. In that case all products in the boron system would have the same allocation factor of 0.28 t/t (total CO_2 emissions of 295860 t/yr divided by the total production of 1062000 t/yr). For the output of 10 mol of 81000 t/yr, for instance, the CO_2 emissions allocated to this product would be equal to 22680 t/yr. The marginal approach allocates 14580 t CO_2/yr to the same product, which represents a total difference of 36%. For ABA the difference is much higher and is equal to 86%. The reason for this is that allocation by mass without system disaggregation implicitly assumes that all processes for production of co-products are the same, which is obviously not true. These results illustrate the point that allocation done in this arbitrary way does not reflect physical causality and, therefore, cannot allocate the burdens realistically in complex industrial multiple-function systems. A similar conclusion emerges if allocation is based on the B_2O_3 content in different products. Although it seems that the B_2O_3 allocation coefficients increase as the degree of processing of different products increases to give a higher boron content (→ *Fig. 3*), this increase is actually related linearly to the content of B_2O_3 in the products and not to the burdens associated with their processing.

However, the situation is quite different if allocation on the mass basis is done after the system has been disaggregated to take into account differences in the processes for producing different products. In that case allocation on marginal and mass bases gives the same results (→ *Fig. 3*). This means that in case of the CO_2 emissions, allocation based on a physical quantity, i.e. mass, is appropriate. Thus, it can be correct to allocate the burdens on the basis of a physical quantity, provided that this basis emerges from analysis of causation in the system.

Allocation on the basis of market value[3] does not give correct results in this case. Although the burdens allocated on the marginal and market value bases are quite similar for 10Mol, 5Mol and BA, the difference is much larger for AB and ABA (51% and 38%, respectively). This implies that the external costs of the environmental burdens are not proportional to the current economic values of these products. Therefore, allocation by financial value can give misleading results and should not be used in systems where physical causality exists.

Allocation by physical causality and, in particular, the use of LP for marginal allocation are discussed in more detail in Azapagic and Clift (1998b, 1998c).

3.2 Improvements in the boron system

In addition to evaluating the environmental performance, the other goal of performing a LCA study of the boron products system was to identify a range of possibilities for minimising total environmental impacts from the system, while maximising production subject to product demand and keeping the production costs at a minimum. The information obtained would then serve as a basis for effecting improvements in the system.

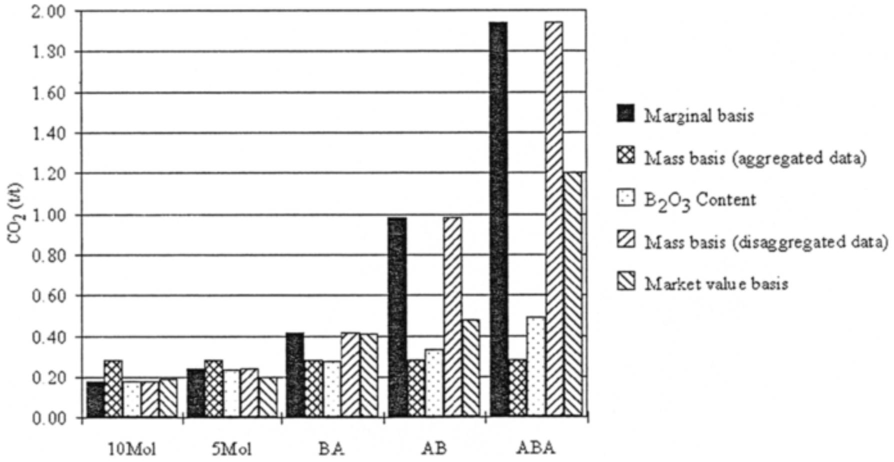

Fig. 3: Comparison of different allocation methods for CO_2 emissions

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[3] The market value is here taken to be gross selling price of the boron products, respectively: 10Mol: $239.5; 5Mol: $255.4; BA: $527.4; AB: $612.5; ABA: $1535.2

The objective functions of the LP model, therefore, include environmental impacts, costs, and total production. To enable the approach to be illustrated graphically, the system is here optimised on three objectives only: Costs (F), Global Warming Potential (GWP) and Production (P) as defined by:

$$\text{Minimise} \qquad F = \sum_{i=1}^{I} f_i x_i \tag{20}$$

$$\text{Minimise} \qquad GWP = \sum_{j=1}^{J} ec_j B_j \tag{21}$$

$$\text{Maximise} \qquad P = \sum_{q=1}^{Q} P_q \tag{22}$$

subject to the constraints defined by eqns. (15)-(19). Obviously, eqns. (20) and (21) are equivalent to eqns. (1) and (5) and they are expressed in £/yr and t/yr, respectively. Equation (22) is defined by the total output of the five boron products in t/yr (→ *Table 1*). Therefore, the model is similar to that used for allocation of environmental burdens, except that the product output constraints (16) are now different: instead of being defined by current operations, they are determined by the market demand projected on the basis of the trends in the previous years. In addition, several alternatives to the way the system is operated at present have been considered to identify the Best Practicable Environmental Options (BPEO). These alternatives include transport of the ore in the mine by a conveyor instead of trucks, and steam generation in the Cogeneration instead of the Steam plant. Depending on the position on the non-inferior surface, different alternatives are chosen in the optimisation and they represent the BPEO for that particular operating state. As the position on the surface changes, so does the BPEO. This is illustrated in **Figure 4**.

As a result of multiobjective optimisation on functions (20)-(22), the non-inferior surface ABCD with a range of optimum solutions is generated. The results shown in **Figure 4** are normalised to the optimum values of each objective, i.e. F*, GWP* and P*, obtained in single-objective optimisation, when the other two functions are ignored. Point A in Figure 4 represents the minimum of the cost objective function; however the production is at the minimum and GWP is 31% above its optimum value (G*). For instance, the BPEO at this point includes transport in the mine by the trucks and steam produced in the Steam plant. If a further decrease in the GWP objective by one tonne is required, that would result in a cost increase of £95 for constant product output; i.e. the marginal cost for reducing global warming potential from the system in this case is £95. Similarly, if the production were to increase by one tonne, the resulting increase in the costs would be £34 if GWP were kept constant.

By moving from point A along the non-inferior curve for constant GWP, both costs and production increase, to reach their maximum feasible values at point B. Here, the Cost function is 4% above its optimum value. If Production is increased by one tonne, £300 of the Costs objective have to be given up. Similarly, one tonne change in the GWP is associated with a cost change of £100; i.e. the marginal cost of reducing GWP changes as a result of the change in operating conditions. At this solution, the BPEO is defined by steam generation mainly in the Steam plant and the preferred transportation means in the mine are the conveyors.

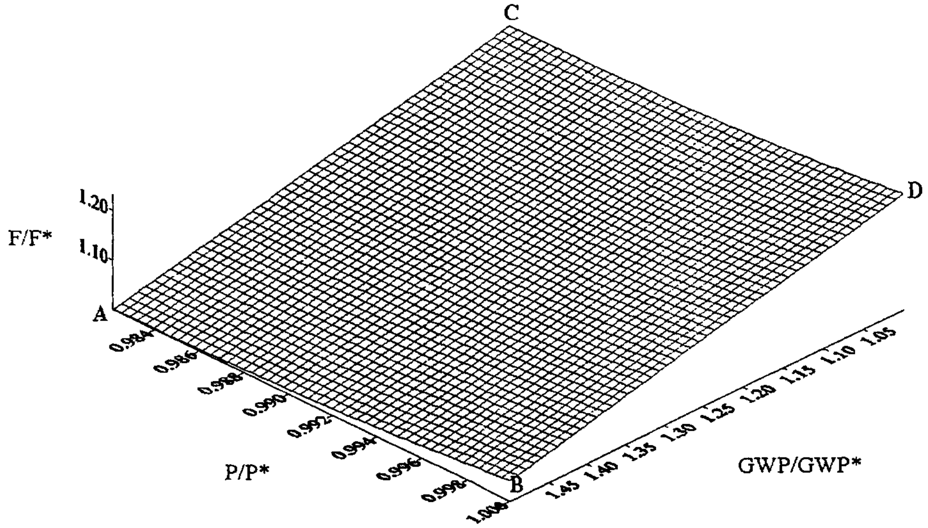

Fig. 4: Non-inferior solutions obtained in multiobjective optimisation

Furthermore, if for instance the system were to be operated at point C, GWP would be 3.3% above its optimum value obtained in the single-objective optimisation. The production would be at the minimum, and the costs would increase by 14%. The effect of GWP and Production on Costs is similar to that found at point A: an improvement in GWP of one tonne would worsen the Costs objective by £100, while a tonne increase in P would result in £36 increase in costs. Here, 93% of the steam is generated by the Cogeneration plant and the rest is produced in the Steam plant. The conveyors still remain the best transport option in the mine.

However, if for example, point D were to be chosen as the best compromise solution, then for the same value of GWP as at point C, the production would reach the maximum; however, costs would have to increase by 17%. It can be noticed here that both GWP and Production exhibit similar effect on the Costs: a decrease in GWP by one tonne increases Costs by £3600. If the production is increased by one tonne, the costs increase by £3500. At this solution, the best practicable environmental option is defined by truck transport in the mine and steam production in the Cogeneration plant. It may be noted that, for instance, the marginal costs of reducing GWP at point D are by a factor of 100 higher than at point C due to the different environmental options chosen at these two points.

These results demonstrate how optimum solutions, and therefore, BPEO, change with the operating state of the system. The same analysis can be carried out for other points on the non-inferior surface which are all optimal in the Pareto sense. By trading-off the values of different objectives at these points, decision-makers can select any solution on the surface, depending on how much of one objective they are prepared to give up in order to gain in another. If all objectives are considered to be of equal importance, then one of the possible ways to choose the best compromise solution is to identify operation by which all objectives differ from their optima by the same percentage. However, should some objectives be considered more important than the others, any other solution on the non-inferior curve can be chosen as the best compromise. The value of multiobjective optimisation in the context of LCA, therefore, lies in offering a range of choices for environmental and economic improvements of the system and so enabling preferences to be identified after analysing all the trade-offs among objectives.

4 Conclusions

Linear Programming (LP) is a powerful mathematical technique that can be successfully combined with Life Cycle Assessment (LCA). Marginal analysis, which is an integral part of LP, can be used to solve the problem of allocation in multiple-function systems. LP is also useful in identifying opportunities for environmental improvements in a product system. Since improvements cannot be carried out on the basis of environmental LCA only, LP can also be used to quantify the compromise between environmental and economic performance by using multiobjective optimisation. The advantage of multiobjective optimisation in environmental system management in the context of LCA lies in providing a set of alternative options for system improvements rather than a single optimum solution, and enabling the choice of the Best Practicable Environmental Options (BPEO) by revealing the relationships between environmental improvements and costs.

Nomenclature

$a_{c,i}$	Input/output or proportionality coefficients in constraints
$bc_{j,i}$	Environmental burden coefficients
B_j	Environmental burdens
C	Total number of constraints in LP model
C_i	Capacity of activity i
D_q	Demand for product q
e_c	Right-hand side coefficients in constraints
$ec_{k,j}$	Environmental impact coefficients
E_k	Environmental impacts
F	(Economic) Objective function (profit or costs)
F*	Optimum value of the cost function obtained in single objective optimisation
f_i	Coefficients in economic objective function
GWP	Global Warming Potential objective function
GWP*	Optimum value of the GWP objective function obtained in single objective optimisation
H_z	Heat requirement
I	Total number of activities
J	Total number of burdens
K	Total number of impacts
M	Total number of primary and raw materials
P	Total production objective function defined by the products output P_q
P*	Optimum value of the P objective function obtained in the single objective optimisation
P_q	Product outputs (functional units)
R_m	Primary and raw materials
Q	Total number of products from the boron system
Q_w	Heat lost or dissipated from process
Q_z	Heat supply
S_m	Supply of primary and raw materials
x_i	Activity level (operation level of process)
$\lambda_{j,c}$	Marginal values in LP model and marginal allocation coefficients for burden j and constraint c
$\mu_{k,c}$	Marginal values in LP model and marginal allocation coefficients for impact k and constraint c

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Appendix – An Example of a LP Model

For readers less familiar with the mathematical concepts of LP, a simplified example showing how to formulate and solve a LP model in the context of LCA is presented here.

Consider a multiple-function system which produces two products and uses five alternative feedstock materials (→ *Fig. A.1*). The inputs into the system are denoted by x_1 to x_5 and the outputs by x_6 and x_7. In the LP terms, the variables x_1 to x_7 are known as activities, or more precisely, activity levels. For the purposes of this example, it is assumed that the system produces only one environmental burden, B, and consumes a certain amount of energy, Q. The system produces 1000 kg of each product and they are taken to be the functional units of the system.

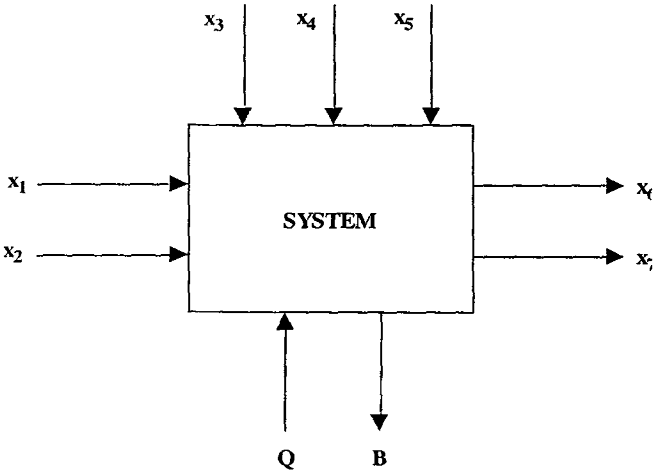

Fig. A.1: Multiple-function system with two products

The raw materials are in two categories: x_1 and x_2 from the first and the other three feeds from the second category. The amounts of these two feed categories are limited and it is not possible to process more than 1500 kg of the first category and more than 1200 kg of the second. The costs (per tonne) of the raw materials are:

	x_1	x_2	x_3	x_4	x_5
Cost (£/t)	110	120	130	110	115

The product x_6 sells for £150 per tonne and x_7 for £120/t. The unit energy used in the system costs 0.3 pence/kg of each product.

In conventional LP modelling, the most common type of questions asked is: How should the manufacturer make these products in order to maximise their profit? LCA, though, is not so much concerned with the economics of the system as with the question of how to minimise the environmental impacts. The rest of this example demonstrates how LP can provide answers to both of these questions. However, the model building is demonstrated first.

Model building

A number of constraints can be defined according to the specifications given above. First of all, the mass and energy balance is carried out, to define the mass and energy balance constraints. This step is equivalent to what is normally done in the Inventory Analysis phase. Therefore, we have:

Input = Output

Mass balance: $x_1 + x_2 + x_3 + x_4 + x_5 = x_6 + x_7 + B$

In the case where emissions are small compared with the process flows, as is the case in this example, B can be neglected. In the LP terms, the above equation is normally written in the following way:

Mass balance constraint:

$$x_1 + x_2 + x_3 + x_4 + x_5 - x_6 - x_7 = 0 \qquad \text{(A.1)}$$

The mass balance constraints are therefore always defined as equalities.

The energy balance around the system in its most basic form is defined by the following equation:

Energy balance: $Q_Z = H_{\text{product}} - H_{\text{feed}} + Q_w$

where H_{product} and H_{feed} are the enthalpies of the product and the feed, respectively, expressed in kJ; Q_Z is the enthalpy used in the system and Q_w is the enthalpy lost. As all constraints and objective functions in LP models are expressed per unit level of the activity, the enthalpies in the above equation can be substituted by the specific enthalpies, $h_1, h_2, \ldots, h_7$ of the feed the and products. They are shown below:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
h (kJ/kg)	1200	1300	1100	1200	1300	1200	1400

Thus, the energy balance constraint is equal to:

Energy balance constraint:

$$\begin{aligned} &1200x_6 + 1400x_7 - 1200x_1 - 1300x_2 - \\ &1100x_3 - 1200x_4 - 1300x_5 - (Q_z - Q_w) = 0 \end{aligned} \qquad \text{(A.2)}$$

Similar to the mass balance, the energy balance constraints are also always defined as equality constraints. It may be noted that the energy balance is not normally carried out in LCA, mainly because of the difficulties in calculating the specific enthalpies for many of the substances and the enthalpy dissipated; the energy consumed in each system, Q_z, is specified instead.

The other constraints in this example include the limitation on the processing of the two categories of the feed, as defined by:

Feed Category 1 availability: $x_1 + x_2 \leq 1500$ (A.3)

Feed Category 2 availability: $x_3 + x_4 + x_5 \leq 1200$ (A.4)

Therefore, the raw material availability can be defined as either equality or inequality constraints. The same is true for capacity, quality, heat etc. constraints.

Finally, output of the functional units is subject to the following constraints:

Product 1 constraint: $x_6 = 1000$ (A.5)

Product 2 constraint: $x_7 = 1000$ (A.6)

and $x_1, x_2, \ldots, x_7 > 0$ (A.7)

The last constraint means that no activity can have negative values. Obviously, constraints (A.1)-(A.7) are equivalent to the general form of the LP model defined by eqns. (2) and (3) in the main text. The specific enthalpies in (A.2), for example, represent the factors of proportionality (or input/output coefficients) $a_{c,i}$ in eqn. (2).

The next step in formulating a LP model is definition of the objective functions. Here, profit and environmental burdens are defined as the objectives on which the system will be optimised. Taking the feed and energy cost, and product value coefficients listed above, the profit objective in this example is defined as:

F= Product value – feed cost – energy cost

Profit objective:

$$\begin{aligned} F = {} &150x_6 + 120x_7 - 110x_1 - 120x_2 - 130x_3 - \\ &110x_4 - 115x_5 - 3\cdot10^{-3}(x_6 + x_7) \end{aligned} \qquad \text{(A.8)}$$

Equation (A.8) is equivalent to eqn. (1) defined in the main text.

The burden is defined as the sum of the individual contributions from each activity from cradle to the point of their entry into the system. In addition, the system itself produces a certain amount of the burden, equal to 5 kg for the total production of 2000 kg of products. The individual contributions to the total burden, expressed per unit level of activity, are listed below:

	x_1	x_2	x_3	x_4	x_5	x_6+x_7
Burden (kg/kg)	0.01	0.015	0.01	0.03	0.02	0.0025

Therefore:

Burden objective:

$$B = 0.01x_1 + 0.015x_2 + 0.01x_3 + 0.03x_4 + 0.02x_5 + 0.0025(x_6 + x_7) \qquad \text{(A.9)}$$

This equation corresponds to the general form of the burden objective function defined by eqn. (4). Similarly, if the analysis were carried out at the Impact Assessment level, the burden objective function would be replaced by the impact objective, as defined by eqn. (5).

System optimisation

As mentioned above, LP optimisation can be used to identify both economic and environmental optima of the system. If the focus of the analysis is the former, then the system in this example is optimised on the profit objective defined by (A.8), subject to constraints (A.1)-(A.7), to give the solution shown in **Table A.1**. Total profit of £50000 is achieved by using feeds x_1 and x_4 only, while the other feeds are not being used in the system. The active constraints for this operating state of the system are defined by eqns. (A.3) and (A.5)-(A.6). The constraint (A.4) is non-active and has a slack value, i.e. unused amount of feed, of 700 kg.

The total burden is also calculated at the solution of the model and is equal to 35 kg. This is equivalent to a conventional way of calculating the burdens in the Inventory stage, as most economic systems are operated around the point of optimum profit. Moreover, the marginal values are also calculated with the total burden to give the marginal allocation coefficients, as discussed in the main text. As only active constraints have non-zero marginal values, this means that the burdens are allocated among the Product 1, Product 2 and the Feed category 1 availability constraints. Their respective values (not shown in *Table A.1*) are 0.0325, 0.0325 and -0.02. This means that an increase in the production of either of the two products by one kg would increase the total burden by 0.0325 kg. However, as the marginal value of the Feed category 1 availability constraint has negative value, an increase of 1 kg in the feed availability would cause a 0.02 kg decrease of the burden. Thus, this simple example demonstrates that the burden does not have to be allocated fully to the products, but to the other constraints as well, depending on what determines the system operation.

Table A.1: Optimisation results for the LP example

Activity	Optimised on Profit	Burden
Profit, F (£)	50000	26000
Burden, B (kg)	35	25
Feed 1, x_1 (kg)	1500	800
Feed 2, x_2 (kg)	0	0
Feed 3, x_3 (kg)	0	1200
Feed 4, x_4 (kg)	500	0
Feed 5, x_5 (kg)	0	0
Product 1, x_6 (kg)	1000	1000
Product 2, x_7 (kg)	1000	1000
Heat, Q (MJ)	200	320

If the goal of the study is to identify options for environmental improvements as part of Improvement Assessment, the system is optimised on the burden (or impact) objective function. The results of this optimisation are also shown in **Table A.1**. For the same output of the products, the system operation is now determined by the active constraints (A.5) and (A.6). The total burden is equal to 25 kg, which represents an improvement of 29% over the burden obtained in the optimisation on the profit. This reduction is achieved by using 700 kg less of the feed x_1 compared to the previous case and using x_3 instead of x_4. However, the profit is reduced by about 48%. In addition, the total energy used in the system increases from 200 MJ to 320 MJ. Therefore, in this particular case, the reduction in one burden leads to an increase in another. To help resolve these conflicting situation, the system can be optimised on a number of environmental and socio-economic objectives function. The resulting Pareto or non-inferior surface can then be used to trade-off the environmental and economic performance between the points of minimum burden and maximum profit to identify the BPEO not entailing excessive costs.

Erratum

Int. J. LCA (5) 266 – 272 (1998) "Einstein's Lessons for Energy Accounting in LCA"
by Rolf Frischknecht, Reinout Heijungs and Patrick Hofstetter
on page 286 in the headline of section 4: The correct text is "$E = mc^2$"