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Limit Cycles for Planar Piecewise Linear Differential Systems via First Integrals

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Using first integrals we introduce an alternative tool to the Poincaré maps in order to study the limit cycles of piecewise linear differential systems. It is also shown that an usual relaxation oscillation phenomenon occurrs in a two–parameter family of planar piecewise linear vector fields.

Key Words: Limit cycles, first integrals, piecisewise linear differential systems.

1. INTRODUCTION

The existence, number and localization of limit cycles is one of the most important problems in the qualitative theory of planar differential equations, and the more usual tool for studying them is the Poincaré map. In the particular case of planar piecewise linear differential systems of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + \varphi\left(\mathbf{k}^T\mathbf{x}\right)\mathbf{b} + \mathbf{a}, \tag{1}$$

where A is a 2×2 real matrix, $\mathbf{x}, \mathbf{a}, \mathbf{b}, \mathbf{k} \in \mathbb{R}^2$, and φ is a continuous piecewise linear function, the Poincaré map is defined by the flow of the system when we take as transversal sections the boundaries of the regions where the system is linear.

For the piecewise linear differential systems having continuous characteristic functions φ formed by two–pieces of linear functions or by three–symmetric–pieces of linear functions, this problem has been completely solved, see [5] and [12]. But this technique becomes difficult to handle if we add more pieces or if we avoid the symmetry of φ. Even in these special cases, since the Poincaré map of system (1) depends on the real canonical form of the matrices A and $B = A + \mathbf{b}\mathbf{k}^T$, the number of different Poincaré maps that appear in the study of system (1) force that the proofs of the results must be divided in cases and subcases, which is tedious and not elegant.

This paper is divided in two different parts. In the first one, from Section 2 to Section 4, we provide an alternative way to the Poincaré map for analysing the limit cycles of planar piecewise linear differential systems. Using the Darboux theory of integrability and the election of an adequate coordinate system we present the first integrals of different planar linear differential systems in a unified way that we call first integrals related to a point. By means of the closing equations we show how the notion of first integral related to a point becomes very useful for studying the limit cycles of piecewise linear differential systems, see Section 4.

There is the feeling that piecewise linear diferential systems can present all the complex dynamics one can see in the nonlinear differential systems. Following this line of thought, at the second part of the paper, in Section 5, we apply the technique developed at the first part, to characterize the bifurcation diagram of the following two–parameters family of piecewise linear systems

$$\dot{x} = y - \varphi\left(x\right), \qquad \dot{y} = -\varepsilon\left(x + \alpha\right), \tag{2}$$

where $\alpha \in \mathbb{R}$, $\varepsilon \geq 0$ small, and

$$\varphi\left(\sigma\right) = \begin{cases} c\sigma + 1 + c & \text{if } \sigma < -1, \\ -\sigma & \text{if } -1 \leq \sigma \leq 0, \\ c\sigma & \text{if } 0 < \sigma, \end{cases} \tag{3}$$

where c is a fixed constant greater than 1, but close to 1.

System (2) is a piecewise linear differential version of the differential system studied by Dumortier and Roussarie in [4]. In that work the authors, taking as characteristic function $\varphi\left(\sigma\right) = \sigma^2/2 + \sigma^3/3$, characterize the "canard phenomenon" via the shape of the limit periodic sets of system (2), when ε decreases to zero, and they prove that there are "canard limit cycles" for these systems.

In Section 5, for $\varepsilon > 0$ we prove that if $\alpha \notin (0, 1)$, then system (2) has no limit cycles; and if $\alpha \in (0, 1)$ then system (2) has exactly one limit cycle, $\Gamma_{\varepsilon,\alpha}$, which is hyperbolic and stable. For a fixed $\varepsilon > 1/4$ the limit cycle is created and destroyed at a Hopf bifurcation at the values of the parameter $\alpha = 0$ and $\alpha = 1$, respectively. For a fixed $\varepsilon \leq 1/4$ the limit cycle is created and destroyed at a bifurcation of a homoclinic loop at the values of the parameter $\alpha = 0$ and at $\alpha = 1$, respectively. For sufficiently small ε, the limit cycle $\Gamma_{\varepsilon,\alpha}$ has the shape of the usual relaxation oscillation of the Van der Pol equation. Thus, we conclude that there are no "canard limit cycles" for systems (2).

2. DARBOUX THEORY OF INTEGRABILITY

The aim of this section is to introduce the terminology and some of the most relevant results of the Darboux theory of integrability for real planar polynomial differential systems. In fact the natural background for this theory are the complex polynomial differential systems. For a detailed discussion of this theory we refer the reader to [3, 2, 8].

A *real planar polynomial differential system* or simply a *polynomial system* will be a differential system of the form

$$\frac{dx}{ds} = \dot{x} = P\left(x, y\right), \quad \frac{dy}{ds} = \dot{y} = Q\left(x, y\right), \tag{4}$$

where x and y are real variables, the independent one (the time) s is real, and P and Q are polynomials in the variables x and y with real coefficients. The *degree* of polynomial system (4) is defined as $m = \max\left\{\deg P, \deg Q\right\}$. The *vector field* X associated to system (4) is defined by

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

System (4) is *integrable* on an open subset U of $\mathbb{R}^2$ if there exists a nonconstant analytic function $H : U \to \mathbb{C}$, called a *first integral* of (4) on U, which is constant on all orbits of system (4) contained on U.

Let $\mathbb{C}\left[x, y\right]$ be the ring of polynomials in the variables x and y with coefficients in $\mathbb{C}$, and let $f \in \mathbb{C}\left[x, y\right]$. The algebraic curve $f\left(x, y\right) = 0$ is an *invariant algebraic curve* of the real system (4) if for some polynomial $K \in \mathbb{C}\left[x, y\right]$ we have

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. Since the polynomial system has degree m, then a cofactor has at most degree $m - 1$.

In what follows we summarize the two results on Darboux theory of integrability that we shall need in the next section.

PROPOSITION 1. *For a polynomial system (4), $f = 0$ is an invariant algebraic curve with cofactor K if and only if $\overline{f} = 0$ is an invariant algebraic curve with cofactor $\overline{K}$. Here, conjugation denotes only conjugation of the coefficients of the polynomials.*

THEOREM 2. *Suppose that a polynomial system (4) admits p invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, 2, \ldots, p$. If there exist*

$\alpha_i \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \alpha_i K_i = 0$, then the (multi–valued) function $f_1^{\alpha_1} \cdots f_p^{\alpha_p}$ is a first integral of system (4).

3. FIRST INTEGRALS RELATED TO A POINT

By definition a *planar linear differential system* or simply a *linear system* is a differential system of the form

$$\dot{\mathbf{x}} = B\mathbf{x} + \mathbf{b}, \tag{5}$$

where B is a 2×2 real matrix, and $\mathbf{x}$ and $\mathbf{b}$ are in $\mathbb{R}^2$. When $\mathbf{b} = \mathbf{0}$, the linear system is called *homogeneous.*

In this paper we restrict our attention to linear systems with a singular point (not necessarily unique) denoted by $\mathbf{e}$. If the linear system is homogeneous, then we take $\mathbf{e} = \mathbf{0}$.

Let $\mathbf{p}$ and $\mathbf{q}$ be two linearly independent vectors of $\mathbb{R}^2$. Since $\{\mathbf{p}, \mathbf{q}\}$ is a base of $\mathbb{R}^2$, for any orbit γ of system (5) there exist differentiable functions $u, v : \mathbb{R} \to \mathbb{R}$ such that

$$\gamma = \{\mathbf{e} + u(s)\,\mathbf{p} + v(s)\,\mathbf{q} : s \in \mathbb{R}\}.$$

A non-constant differentiable function $H : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{C}$ is a *first integral of system* (5) *related to the base* $\{\mathbf{p}, \mathbf{q}\}$ if $H(u, v)$ is constant on all orbits of (5). Notice that for a homogeneous linear system, the definition of first integral related to the canonical base of $\mathbb{R}^2$, is equivalent to the standard definition of first integral. Furthermore, if H is a first integral of system (5) related to $\{\mathbf{p}, \mathbf{q}\}$, then for any $h \in \mathbb{C}$ the set

$$I_h := \{\mathbf{e} + u\mathbf{p} + v\mathbf{q} : H(u, v) = h\}$$

is empty or formed by orbits of system (5).

The following result shows that in other to find a first integral of system (5) related to a base it is sufficient to consider the homogeneous system

$$\dot{\mathbf{x}} = B\mathbf{x}. \tag{6}$$

LEMMA 3. *A function H is a first integral of system (5) related to $\{\mathbf{p}, \mathbf{q}\}$ if and only if H is a first integral of system (6) related to $\{\mathbf{p}, \mathbf{q}\}$.*

*Proof***:** The translation $\mathbf{y} = \mathbf{x} - \mathbf{e}$ transforms system (5) into system (6), and any orbit $\{\mathbf{e} + u(s)\,\mathbf{p} + v(s)\,\mathbf{q} : s \in \mathbb{R}\}$ of (5) in an orbit $\{u(s)\,\mathbf{p} + v(s)\,\mathbf{q} : s \in \mathbb{R}\}$ of (6). Thus H remains constant on the orbits. The converse implication follows in a similar way. ∎

If $\mathbf{p} \in \mathbb{R}^2$ is not an eigenvector of the matrix B, then $\mathbf{p}$ and $B\mathbf{p}$ are two linearly independent vectors. A function H is a *first integral of system* (5) *related to a point* $\mathbf{p}$ if H is a first integral of system (5) related to the base $\{\mathbf{p}, B\mathbf{p}\}$.

Let M be a regular 2×2 real matrix. The change of coordinates $\mathbf{y} = M\mathbf{x}$ transforms system (6) into system

$$\dot{\mathbf{y}} = B^*\mathbf{y}, \tag{7}$$

where $B^* = MBM^{-1}$. The following result proves that first integrals related to a point are invariant by linear transformations.

LEMMA 4. *If H is a first integral of system (6) related to $\mathbf{p}$ and M is a regular matrix, then H is a first integral of system (7) related to $M\mathbf{p}$.*

Proof: Since $B^* = MBM^{-1}$, then $MB\mathbf{p} = B^*M\mathbf{p}$, and the lemma follows. ∎

THEOREM 5. *Let B be a 2×2 real matrix, λ_1 and λ_2 the eigenvalues of B, $\lambda_1 \neq \lambda_2$, and $\mathbf{p} \in \mathbb{R}^2$ not an eigenvector of B.*

(a) The function $H(u,v) = (u+\lambda_1 v)^{\lambda_2}(u+\lambda_2 v)^{-\lambda_1}$ is a first integral of system (5) related to $\mathbf{p}$.

(b) If $\lambda_1, \lambda_2 \notin \mathbb{R}$, $D = \lambda_1\lambda_2$ and $T = \lambda_1 + \lambda_2$, then

$$H(u,v) = \left(u^2 + Tuv + Dv^2\right) e^{-\dfrac{2T}{\sqrt{4D-T^2}}\arctan\left(\dfrac{v\sqrt{4D-T^2}}{2u+vT}\right)}$$

is a real first integral of system (5) related to $\mathbf{p}$. Moreover, if $\gamma = \{\mathbf{e}+u(s)\,\mathbf{p} + v(s)\,\dot{\mathbf{p}} : s \in \mathbb{R}\}$ is an orbit of system (5), then

$$\arctan\left(\frac{v(s)\sqrt{4D-T^2}}{2u(s)+Tv(s)}\right) - \arctan\left(\frac{v(0)\sqrt{4D-T^2}}{2u(0)+Tv(0)}\right) = s\frac{\sqrt{4D-T^2}}{2}.$$

Proof: From Lemma 3 it is sufficient to prove the theorem for the homogeneous linear system $\dot{\mathbf{x}} = B\mathbf{x}$.

(a) Let $\gamma = \{u(s)\,\mathbf{p} + v(s)\,B\mathbf{p} : s \in \mathbb{R}\}$ be an orbit of the linear system $\dot{\mathbf{x}} = B\mathbf{x}$. If $D = \det(B)$, $T = \operatorname{trace}(B)$ and Id is the identity matrix, by the Cayley–Hamilton Theorem, we have $B^2 - TB + DId = \mathbf{0}$. Hence, $BB\mathbf{p} = -D\mathbf{p} + TB\mathbf{p}$. Thus, the expression of the linear map defined by B in the base $\{\mathbf{p}, B\mathbf{p}\}$ is

$$\begin{pmatrix} 0 & -D \\ 1 & T \end{pmatrix},$$

and the functions $u(s)$, $v(s)$ satisfy the following linear system

$$\begin{aligned} \dot{u} &= -Dv, \\ \dot{v} &= u + Tv. \end{aligned} \tag{8}$$

Let $f_1 = u + \lambda_1 v$ and $f_2 = u + \lambda_2 v$. Then, it is easy to check that $f_1 = 0$ and $f_2 = 0$ are invariant algebraic curves of system (8), with cofactors λ_1 and λ_2, respectively. From Theorem 2, we obtain that

$$H(u, v) = (u + \lambda_1 v)^{\lambda_2} (u + \lambda_2 v)^{-\lambda_1}$$

is a first integral of system (8). Therefore, H is a first integral of system (5) related to $\mathbf{p}$.

(b) Let $f = u + \lambda_1 v$. Since $f = 0$ is an invariant algebraic curve of system (8) with cofactor λ_1, from Proposition 1, $\bar{f} = u + \lambda_2 v = 0$ is an invariant algebraic curve of (8) with cofactor $\bar{\lambda}_1 = \lambda_2$.

Let $\alpha = \mathrm{Im}(\lambda_1) + \mathrm{Re}(\lambda_1)\, i$. Then $\alpha\lambda_1 + \overline{\alpha}\lambda_2 = 0$, and from Theorem 2 we obtain that $f^{\alpha}\overline{f}^{\overline{\alpha}}$ is a first integral of system (8). Since

$$f^{\alpha}\overline{f}^{\overline{\alpha}} = \left[||f||^2 \exp\left(-2\frac{\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)} \arctan\left(\frac{\mathrm{Im}(f)}{\mathrm{Re}(f)} \right) \right) \right]^{\mathrm{Re}(\alpha)},$$

then the function

$$\begin{aligned} H(u, v) &= ||f||^2 \exp\left(-2\frac{\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)} \arctan\left(\frac{\mathrm{Im}(f)}{\mathrm{Re}(f)} \right) \right) \\ &= \left(u^2 + Tuv + Dv^2 \right) \exp\left(-2\frac{\mathrm{Re}(\lambda_1)}{\mathrm{Im}(\lambda_1)} \arctan\left(\frac{\mathrm{Im}(\lambda_1)\, v}{u + \mathrm{Re}(\lambda_1)\, v} \right) \right), \end{aligned}$$

is a real first integral of system (8). Therefore, H is a first integral of system (5) related to $\mathbf{p}$. From

$$\lambda_1 = \frac{T}{2} + \frac{\sqrt{4D - T^2}}{2} i,$$

we obtain the expression of H given in (b).

The last statement follows noting that

$$\arctan\left(\frac{v(s)\sqrt{4D-T^2}}{2u(s)+Tv(s)}\right)-\arctan\left(\frac{v(0)\sqrt{4D-T^2}}{2u(0)+Tv(0)}\right)$$

$$=\int_0^s \frac{d}{dr}\arctan\left(\frac{v(r)\sqrt{4D-T^2}}{2u(r)+Tv(r)}\right)$$

$$=\frac{\sqrt{4D-T^2}}{2}\int_0^s \frac{\frac{dv}{dr}u(r)-\frac{du}{dr}v(r)}{u(r)^2+u(r)v(r)T+v(r)^2 D}dr,$$

where $(u(s), v(s))$ is a solution of the differential system (8). ∎

Remark 6. If $\lambda_1, \lambda_2 \notin \mathbb{R}$ and B is in the Jordan canonical form, then $Im(\lambda_1)$ is the angular velocity of a solution of system (5). From Theorem 5(b), we have that

$$\arctan\left(\frac{v(s)\sqrt{4D-T^2}}{2u(s)+Tv(s)}\right)-\arctan\left(\frac{v(0)\sqrt{4D-T^2}}{2u(0)+Tv(0)}\right) \tag{9}$$

provides the angle $\theta(s)$ run by the solution between the points $u(0)\mathbf{p}+v(0)B\mathbf{p}$ and $u(s)\mathbf{p}+v(s)B\mathbf{p}$.

When the matrix B is not in the Jordan canonical form, then (9) is not exactly the angle $\theta(s)$, but in this case it is easy to check that

$$\arctan\left(\frac{v(s)\sqrt{4D-T^2}}{2u(s)+Tv(s)}\right)-\arctan\left(\frac{v(0)\sqrt{4D-T^2}}{2u(0)+Tv(0)}\right)\in(k\pi,(k+1)\pi)$$

for some $k \in \mathbb{Z}$ if and only if $\theta(s)\in(k\pi,(k+1)\pi)$.

4. CLOSING EQUATIONS FOR PIECEWISE LINEAR SYSTEMS

In this section we obtain the *closing equations*, see [10], whose zeros determine the number of limit cycles that appear in a piecewise linear differential system of the form

$$\dot{\mathbf{x}} = A\mathbf{x}+\varphi\left(\mathbf{k}^T\mathbf{x}\right)\mathbf{b}+\mathbf{a}, \tag{10}$$

where A is a 2×2 real matrix, $\mathbf{x}=(x,y)^T$, $\mathbf{k}$ and $\mathbf{b}$ are in $\mathbb{R}^2\setminus\{\mathbf{0}\}$, $\mathbf{a}$ is in $\mathbb{R}^2$, and

$$\varphi(\sigma)=\begin{cases}1 & \text{if } \sigma\geq 1,\\ \sigma & \text{if } \sigma<1.\end{cases}$$

Since the characteristic function φ is continuous and Lipschitz in $\mathbb{R}^2$, but is no differentiable in $\mathbb{R}^2$, then the same holds for the vector field defined by (10). The characteristic function φ splits the phase space into two open half–planes $S_+=\left\{\mathbf{x}\in\mathbb{R}^2:\mathbf{k}^T\mathbf{x}>1\right\}$ and $S_-=\left\{\mathbf{x}\in\mathbb{R}^2:\mathbf{k}^T\mathbf{x}<1\right\}$ with a common boundary $\Gamma=\left\{\mathbf{x}\in\mathbb{R}^2:\mathbf{k}^T\mathbf{x}=1\right\}$. Using this notation, system (10) can be rewritten as follows

$$\dot{\mathbf{x}}=\begin{cases}B\mathbf{x}+\mathbf{a} & \text{if } \mathbf{x}\in S_-\cup\Gamma,\\ A\mathbf{x}+\mathbf{c} & \text{if } \mathbf{x}\in\Gamma\cup S_+,\end{cases}\tag{11}$$

where $B=A+\mathbf{b}\mathbf{k}^T$ and $\mathbf{c}=\mathbf{b}+\mathbf{a}$.

Let $\mathbf{e}_-$ and $\mathbf{e}_+$ be the solutions of the equations $B\mathbf{x}+\mathbf{a}=\mathbf{0}$ and $A\mathbf{x}+\mathbf{c}=\mathbf{0}$, respectively. Recall that in this paper we suppose that any linear system has singular points. If $\mathbf{e}_-\in S_-\cup\Gamma$ (respectively, $\mathbf{e}_+\in\Gamma\cup S_+$) then $\mathbf{e}_-$ (respectively, $\mathbf{e}_+$) is a singular point. On the contrary we said that $\mathbf{e}_-$ (respectively, $\mathbf{e}_+$) is a *virtual singular point*. In the rest of this section we assume that *system* (11) *has no singular points on the straight line* Γ. In particular $\mathbf{e}_-$ and $\mathbf{e}_+$ do not belong to Γ.

A point $\mathbf{p}\in\Gamma$ is called a *contact point* of the flow defined by (11) with the straight line Γ if the vector $\dot{\mathbf{p}}=A\mathbf{p}+\mathbf{c}=B\mathbf{p}+\mathbf{a}$ is parallel to Γ, that is, $\mathbf{k}^T\dot{\mathbf{p}}=0$. To simplify notation we denote by $\dot{\mathbf{q}}$ the value of the vector field at the point $\mathbf{q}$.

Let γ be a limit cycle of system (11). Since linear systems have no limit cycles, then γ intersect the half–planes $\Gamma\cup S_+$ and $S_-\cup\Gamma$. Let $\gamma_A=\gamma\cap(\Gamma\cup S_+)$ and $\gamma_B=\gamma\cap(S_-\cup\Gamma)$, see Figure 1. Hence, by continuity there exists a contact point $\mathbf{p}$ of the flow defined by (11) with the straight line Γ.

Under the assumption that there are no singular points on Γ, the vector $\mathbf{p}_-=\mathbf{p}-\mathbf{e}_-$ is not an eigenvector of B and $\mathbf{p}_+=\mathbf{p}-\mathbf{e}_+$ is not an eigenvector of A. Since $B\mathbf{p}_-=\dot{\mathbf{p}}$ and $A\mathbf{p}_+=\dot{\mathbf{p}}$, then $\gamma_B=\{\mathbf{e}_-+u\mathbf{p}_-+v\dot{\mathbf{p}}\}$ and $\gamma_A=\{\mathbf{e}_++u\mathbf{p}_++v\dot{\mathbf{p}}\}$. Let $H_A(u,v)$ be a first integral of the linear system $\dot{\mathbf{x}}=A\mathbf{x}+\mathbf{c}$ related to $\mathbf{p}_+$, and let $H_B(u,v)$ be a first integral of the linear system $\dot{\mathbf{x}}=B\mathbf{x}+\mathbf{a}$ related to $\mathbf{p}_-$. Then the closing equations of system (11) are

$$\begin{aligned}H_A(1,v)&=H_A(1,-w),\\ H_B(1,v)&=H_B(1,-w),\end{aligned}\tag{12}$$

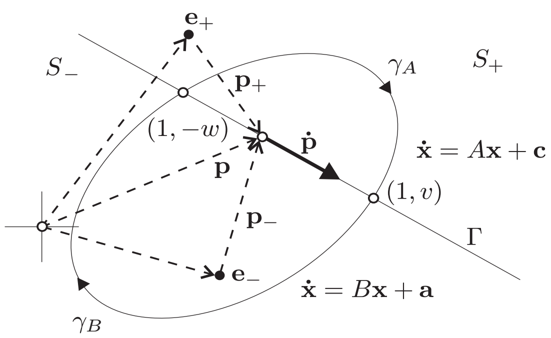

FIG. 1. The contact point $\mathbf{p}$ associated to a limit cycle γ.

where, $v > 0$ and $w > 0$. Therefore, any solution (v_0, w_0) of (12) corresponds to a periodic orbit of (11), which intersect with the straight line Γ at points $\mathbf{p} + v_0\dot{\mathbf{p}}$ and $\mathbf{p} - w_0\dot{\mathbf{p}}$.

By the Implicit Function Theorem, each equation in (12) defines a differentiable function $v_A(w)$ and $v_B(w)$, respectively, in such a way that $\pi = v_A \circ v_B$ is the Poincaré map of system (11) when we take as transversal section the straight line Γ.

As the semi–orbit γ_A only depends on the linear system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{c}$ then the Poincaré map $v_A(w)$ only depends on matrix A. Similarly, the Poincaré map $v_B(w)$ only depends on matrix B.

Qualitative and quantitative information about Poincaré maps for any linear systems can be found in [12].

5. "CANARD PHENOMENON" FOR PIECEWISE LINEAR SYSTEMS

In this section we consider the following two–parameters family of piecewise linear systems

$$X_{\varepsilon,\alpha} = \begin{cases} \dot{x} = y - \varphi(x), \\ \dot{y} = -\varepsilon(x+\alpha), \end{cases} \tag{13}$$

where $\alpha \in \mathbb{R}$, $\varepsilon \geq 0$ but small, and

$$\varphi(\sigma) = \begin{cases} c\sigma + 1 + c & \text{if } \sigma < -1, \\ -\sigma & \text{if } -1 \leq \sigma \leq 0, \\ c\sigma & \text{if } 0 < \sigma, \end{cases}$$

where c is a fixed constant greater than 1, but close to 1.

We can observe that system (13), for each ε, is invariant under the symmetry around the point $x = -1/2, y = 1/2, \alpha = 1/2$ given by $(x, y, \alpha) \to (-x-1, -y+1, -\alpha+1)$. So it is sufficient to study system (13) for values $\alpha \leq 1/2$, and to complete the bifurcation diagram using the symmetry.

A simple computation shows that $X_{\varepsilon,\alpha}$ can be written has follows.

$$\dot{\mathbf{x}} = \begin{cases} A\mathbf{x} - \mathbf{a} & \text{if } x < -1, \\ B\mathbf{x} - \mathbf{b} & \text{if } -1 \leq x \leq 0, \\ A\mathbf{x} - \mathbf{b} & \text{if } 0 < x, \end{cases} \tag{14}$$

where

$$A = \begin{pmatrix} -c & 1 \\ -\varepsilon & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 1 \\ -\varepsilon & 0 \end{pmatrix}, \; \mathbf{a} = \begin{pmatrix} 1+c \\ \varepsilon\alpha \end{pmatrix} \; \text{and} \; \mathbf{b} = \begin{pmatrix} 0 \\ \varepsilon\alpha \end{pmatrix}.$$

PROPOSITION 7. *For $\varepsilon > 0$, system $X_{\varepsilon,\alpha}$ has a unique singular point $(-\alpha, \varphi(-\alpha))$.*

(a)Suppose $\alpha < 0$.

(a.1)If $\varepsilon > c^2/4$, then it is a hyperbolic stable focus.

(a.2)If $\varepsilon \leq c^2/4$, then it is a hyperbolic stable node.

(b)Suppose $\alpha = 0$.

(b.1)If $\varepsilon > 1/4$, then it is stable.

(b.2)If $\varepsilon \leq 1/4$, then a neighborhood of the origin is union of an elliptic sector with a hyperbolic one.

(c)Suppose $0 < \alpha \leq 1/2$.

(c.1)If $\varepsilon > 1/4$, then it is a hyperbolic unstable focus.

(c.2)If $\varepsilon \leq 1/4$, then it is a hyperbolic unstable node.

Proof: We shall prove Statement (b.2), the other statements follows easier using similar arguments. If $\alpha = 0$, then the origin is a singular point. In a neighborhood U of the origin, the system $X_{\varepsilon,0}$ is $\dot{\mathbf{x}} = B\mathbf{x} - \mathbf{b}$ if $x \leq 0$ and $\dot{\mathbf{x}} = A\mathbf{x} - \mathbf{b}$ if $x \geq 0$, where $T = \text{trace}(B) = 1$, $D = \det(B) = \varepsilon$, $t = \text{trace}(A) = -c$ and $d = \det(A) = \varepsilon$. Hence, the origin is repelling on $\{(x, y) : x < 0\}$ and attracting on $\{(x, y) : x > 0\}$.

Let $\gamma = \{(x(s), y(s)) : s \in \mathbb{R}\}$ be the orbit of system $X_{\varepsilon,0}$ which intersects with the y–axis at $(0, y_0)$. Since $dx/ds|_{(0,y_0)} = y_0$, then $dx/ds|_{(0,y_0)} > 0$ when $y_0 > 0$ and $dx/ds|_{(0,y_0)} < 0$ when $y_0 < 0$.

Suppose that $|y_0|$ sufficiently small. If $y_0 > 0$, then the α– and the ω–limit set of γ is the origin. If $y_0 < 0$, then the α– and the ω–limit set of

γ is not contained in U. Therefore, U is union of an elliptic sector with a hyperbolic one. ∎

5.1. On the uniqueness of limit cycles in $X_{\varepsilon,\alpha}$

An extension to piecewise linear Lienard's systems of the Cherkas–Zhilevich criteria [1] for the uniqueness of limit cycles is proved in the next theorem. The proof follows the steps of the smooth case with the adequate changes.

THEOREM 8. *Consider the system*

$$\begin{aligned} \frac{dx}{ds} &= -\psi(y) - F(x), \\ \frac{dy}{ds} &= g(x), \end{aligned} \tag{15}$$

where

$$F(x) = \int_0^x f(s)\, ds$$

and $f(x)$ is a piecewise linear function (non necessarily continuous). Assume that the following conditions hold for $x \in (a,b)$, $a < 0$, $b > 0$ and $y \in \mathbb{R}$:

(i) The origin is a singular point.

(ii) $xg(x) > 0$ for $x \neq 0$ and $y\psi(y) > 0$ for $y \neq 0$.

(iii) Functions $g(x)$ and $\psi(y)$ are continuously differentiable, $g'(0) > 0$, $\psi'(0) > 0$, $\psi'(y) \geq 0$ and $f(0) < 0$ (respectively, $f(0) > 0$).

(iv) There are real numbers u, v such that the function

$$f_1(x) = f(x) + g(x)\left[u + vF(x)\right]$$

has simple zeros $x_1 < 0$ and $x_2 > 0$, and $f_1(x) \leq 0$ (respectively, $f_1(x) \geq 0$) in (x_1, x_2).

(v) Outside $[x_1, x_2]$, the function $f_1(x)/g(x)$ is nondecreasing (respectively, nonincreasing).

(vi) Every limit cycle of the system is intersected by the straight lines $x = x^*$, with $x^* \in [x_1, x_2]$.

Then, system (15) has at most one limit cycle in the strip $\{(x,y) : a < x < b\}$ and, when it exists, it is hyperbolic and stable (respectively, unstable).

Proof: We prove the theorem for $f(0) < 0$, the other case follows similarly. In the strip $\{(x,y) : a < x < b\}$ system (15) has a unique singular point $(0,0)$, which is an unstable focus or node. Suppose that there are more than one limit cycles surrounding the origin. Let γ_1 be the closest limit cycle surrounding the origin, and let γ_2 be the closest limit cycle surrounding γ_1.

Let $\gamma_1 = \{(x(s), y(s)) : s \in [0, T_1]\}$, $x_l = \min\{x(s) : s \in [0, T_1]\}$, $x_r = \max\{x(s) : s \in [0, T_1]\}$, $y_v = \min\{y(s) : s \in [0, T_1]\}$ and $y_u = \max\{y(s) : s \in [0, T_1]\}$. Since γ_1 surrounds the origin, then $x_l < 0 < x_r$ and $y_v < 0 < y_u$, see Figure 2.

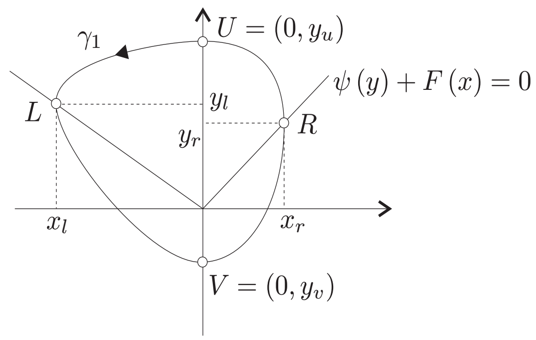

FIG. 2. Shape of the limit cycle γ_1.

Since the function $g(x)$ only vanishes at $x = 0$, the points $U = (0, y_u)$ and $V = (0, y_v)$ are the unique ones on γ_1 for which $dy/ds = 0$. Moreover, since $dy/ds = g(x) < 0$ if $x < 0$, and $dy/ds > 0$ if $x > 0$, the cycle γ_1 intersects the graph of $\psi(y) + F(x) = 0$ in exactly the two points $L = (x_l, y_l)$ and $R = (x_r, y_r)$, see Figure 2. The shape of the cycle γ_2 is similar.

Let $f(x) = a_k x + b_k$ if $x \in I_k = (c_k, c_{k+1})$ for $k = 1, 2, ..., n$, and let $\gamma_p^k = \gamma_p \cap I_k \times \mathbb{R}$ for $k = 1, 2, ..., n$ and $p = 1, 2$. As it is proved in [7] also for planar piecewise linear differential systems, the characteristic exponent of γ_p is given by

$$c_p = -\oint_{\gamma_p} f(x)\, ds, \quad \text{for } p = 1,2,$$

where we denote by $\oint_{\gamma_p} f(x)\, ds$ the value of $\sum_{k=1}^{n} \int_{\gamma_p^k} f(x)\, ds$. Since, along each limit cycle γ_p we have

$$\oint_{\gamma_p} g(x)\, ds = 0, \quad \oint_{\gamma_p} g(x)\, \psi(y)\, ds = 0, \text{ and } \quad \oint_{\gamma_p} g(x)\, [\psi(y) + F(x)]\, ds = 0,$$

then

$$c_p = -\oint_{\gamma_p} f_1(x)\, ds, \quad \text{for } p = 1, 2.$$

We shall prove that $c_2 < c_1$. To this end consider Figure 3. Application of Green's formula yields

$$\begin{aligned} \oint_{E_1 E} f_1(x)\,ds - \oint_{DA} f_1(x)\,ds &= \oint_{E_1 E A D E_1} \frac{f_1(x)}{g(x)} dy \\ &= \int\int_{R_1} \frac{d}{dx}\left(\frac{f_1}{g}\right) dxdy \geq 0, \end{aligned}$$

$$\begin{aligned} \oint_{A_1 B_1} f_1(x)\,ds - \oint_{AB} f_1(x)\,ds &= -\oint_{A_1 B_1 B A A_1} \frac{f_1(x)}{\psi(y) + F(x)} dx \\ &= \int\int_{R_2} \frac{f_1(x)\,\psi'(y)}{(\psi(y) + F(x))^2} dxdy > 0. \end{aligned}$$

Similary, we obtain

$$\oint_{K K_1} f_1(x)\,ds - \oint_{BC} f_1(x)\,ds \geq 0,$$

$$\oint_{C_1 D_1} f_1(x)\,ds - \oint_{CD} f_1(x)\,ds > 0.$$

We also have the inequalities

$$\oint_{B_1 K} f_1(x)\,ds \geq 0, \ \oint_{E A_1} f_1(x)\,ds \geq 0,$$

$$\oint_{K_1 C_1} f_1(x)\,ds \geq 0 \ \text{ and } \ \oint_{D_1 E_1} f_1(x)\,ds \geq 0.$$

Hence, $c_2 < c_1$.

Since the origin is unstable, γ_1 is stable from the inside. So $c_1 \leq 0$. Collecting together the above inequalities we conclude that

$$c_2 < c_1 \leq 0. \tag{16}$$

We now prove that γ_1 is stable from the outside. Assume that γ_1 is unstable from the outside. Then system (15) has a solution $P_1 P_2 P_3$ as in Figure 4.

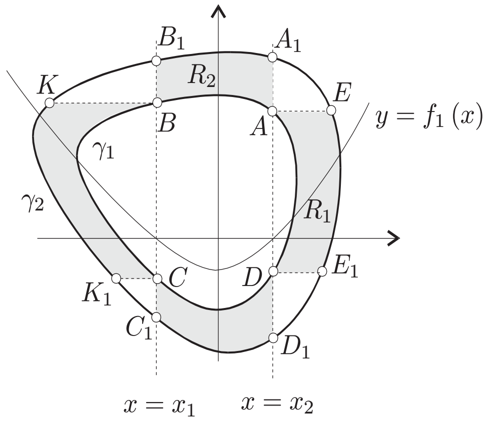

FIG. 3. The limit cycles γ_1 and γ_2.

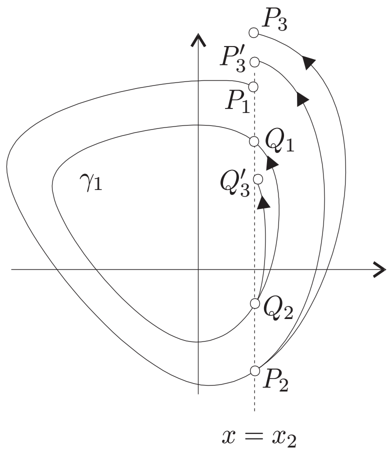

FIG. 4. Two orbits by the flows (15) and (17).

Consider the system

$$\begin{aligned} \frac{dx}{ds} &= -\psi(y) - \overline{F}(x), \\ \frac{dy}{ds} &= g(x), \end{aligned} \tag{17}$$

where

$$\overline{F}\,(x) = \begin{cases} F\,(x) & \text{if } x \leq x_2, \\ F\,(x) + \int_{x_2}^{x} r\,(s - x_2)^2\, g\,(s)\, ds & \text{if } x > x_2, \end{cases}$$

with r a positive real number. Notice that system (17) is like system (15) and satisfies the six conditions of the theorem.

Comparing the vector fields (17) and (15) we get that, for sufficiently small r, system (17) has the orbit arcs $P_1P_2P_3'$ and $Q_1Q_2Q_3'$. Hence, in the region bounded by the curves $P_1P_2P_3'P_1$ and $Q_1Q_2Q_3'Q_1$, system (17) has a limit cycle γ_2^* with characteristic exponent, $c_2^* \geq 0$, and inside $Q_1Q_2Q_3'Q_1$ a limit cycle γ_1^* with $c_1^* \leq 0$ (because the origen is unstable). Since γ_2^* surrounds γ_1^* and $c_2^* \geq c_1^*$, this contradicts inequality (16). Then, γ_1 is stable from the outside, and $c_1 \leq 0$. But two limit cycles γ_1 and γ_2 having identical stability cannot be consecutive. Thus, system (15) has not two limit cycles. ∎

Corollary 9. *If $\varepsilon > 0$, then $X_{\varepsilon,\alpha}$ has at most one limit cycle which, if it exists, is hyperbolic and unstable (respectively, stable) when $\alpha \leq 0$ (respectively, $\alpha \in (0, 1/2]$).*

*Proof***:** The change of coordinates $(x, y) \to (x + a, -y + \varphi(-a))$ transforms system $X_{\varepsilon,\alpha}$ into system

$$\begin{aligned} \dot{x} &= -y - F\,(x)\,, \\ \dot{y} &= g\,(x)\,, \end{aligned}$$

where $g\,(x) = \varepsilon x$ and $F\,(x) = \varphi\,(x - \alpha) - \varphi\,(-\alpha)$.

Suppose $\alpha \in (0, 1/2]$. It is easy to check that

$$F\,(x) = \begin{cases} cx + (1-\alpha)\,(1+c) & \text{if } x < \alpha - 1, \\ -x & \text{if } \alpha - 1 \leq x \leq \alpha, \\ cx - \alpha\,(1+c) & \text{if } \alpha < x, \end{cases}$$

$$F'\,(x) \;=\; f\,(x) = \begin{cases} c & \text{if } x < \alpha - 1, \\ -1 & \text{if } \alpha - 1 \leq x \leq \alpha, \\ c & \text{if } \alpha < x, \end{cases}$$

and

$$f_1\,(x) = \begin{cases} c + \varepsilon x\,[u + v\,(cx + (1-\alpha)\,(1+c))] & \text{if } x < \alpha - 1, \\ -1 + \varepsilon x\,(u - vx) & \text{if } \alpha - 1 \leq x \leq \alpha, \\ c + \varepsilon x\,[u + v\,(cx - \alpha\,(1+c))] & \text{if } \alpha < x, \end{cases}$$

satisfy the hypotheses of Theorem 8 with $f\,(0) < 0$. Hence, system $X_{\varepsilon,\alpha}$ has at most one limit cycle and, when it exists, it is hyperbolic and stable. For $\alpha \leq 0$ the proof follows similarly. ∎

PROPOSITION 10. *Consider* $\varepsilon > 0$.

(a)If $\alpha \leq 0$, *then* $X_{\varepsilon,\alpha}$ *has no limit cycles.*

(b)If $0 < \alpha \leq 1/2$, *then* $X_{\varepsilon,\alpha}$ *has a unique limit cycle,* $\Gamma_{\varepsilon,\alpha}$, *and it is hyperbolic and stable.*

Proof: The system $X_{\varepsilon,\alpha}$ can be written as $\dot{\mathbf{x}} = A\mathbf{x} - \Phi(\mathbf{x})$, where

$$A = \begin{pmatrix} -c & 1 \\ -\varepsilon & 0 \end{pmatrix} \text{ and } \Phi(\mathbf{x}) = \begin{cases} (c+1,\alpha)^T & \text{if } x < -1, \\ (-x(c+1),\alpha)^T & \text{if } -1 \leq x \leq 0, \\ (0,\alpha)^T & \text{if } 0 < x. \end{cases}$$

Thus, if $\mathbf{x}(s)$ is a solution of $X_{\varepsilon,\alpha}$, then

$$\mathbf{x}(s) = e^{sA}\mathbf{x}(0) + \int_0^s e^{(s-r)A}\Phi(\mathbf{x}(r))\,dr.$$

Since the eigenvalues of A are negative, there exist positive constants M, m such that $\left|\left|e^{sA}\right|\right| \leq Me^{-ms}$, for every $s \in \mathbb{R}$, see for instance [9], page 56. Furthermore, there exists a positive contant K such that $||\Phi(\mathbf{x})|| \leq K$, for every $\mathbf{x} \in \mathbb{R}^2$. Hence

$$||\mathbf{x}(s)|| \leq Me^{-ms}\left(||\mathbf{x}(0)|| - \frac{KM}{m}\right) + \frac{KM}{m} \text{ for all } s > 0.$$

Then, there exists a compact set in $\mathbb{R}^2$ which contains the ω–limit set of any orbit of $X_{\varepsilon,\alpha}$.

(a) Suppose that $X_{\varepsilon,\alpha}$ has a limit cycle. From Corollary 9 it is unique and unstable. In contradiction with the fact that there exists a compact set which contains the ω–limit set of any orbit of $X_{\varepsilon,\alpha}$.

(b) Since the unique singular point of $X_{\varepsilon,\alpha}$ is unstable (see Proposition 7) and there exists a compact set which contains the ω–limit set of any orbit of $X_{\varepsilon,\alpha}$, by the Poincaré–Bendixson Theorem we conclude that there exists a limit cycle. From Corollary 9 it is unique and stable. ∎

5.2. Behaviour of $\Gamma_{\varepsilon,\alpha}$ when ε tends to 0

For $\varepsilon = 0$, every point belonging to the curve $L = \{y = \varphi(x)\}$ is a singular point of $X_{0,\alpha}$ and, except $(-1,1)$ and $(0,0)$, they are normally hyperbolic singular points (see for more details [6]). Outside L, the orbits of $X_{0,\alpha}$ are contained in horizontal straight lines.

The phase portrait of $X_{0,\alpha}$ is given in Figure 5.

We can recognize four different kinds of closed curves formed by solutions, see Figure 6. The question is to identify which of these *limit periodic sets*,

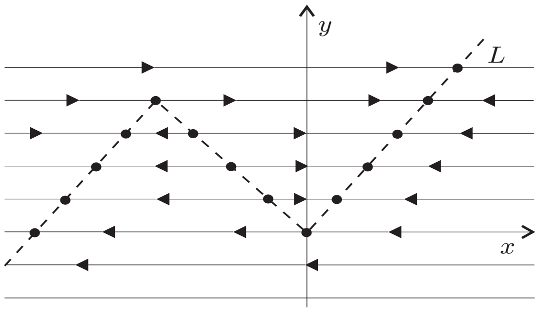

FIG. 5. Phase portrait of $X_{0,\alpha}$.

see for a definition [11], can be approached by the one–parameter family of limit cycles $\Gamma_{\varepsilon,\alpha(\varepsilon)}$ of the one–parameter family of systems $X_{\varepsilon,\alpha(\varepsilon)}$. The limit periodic sets (a), (b) and (c) are commonly called "canards" (ducks), because of the shape of the (c) case. The limit periodic set (d) is called "big".

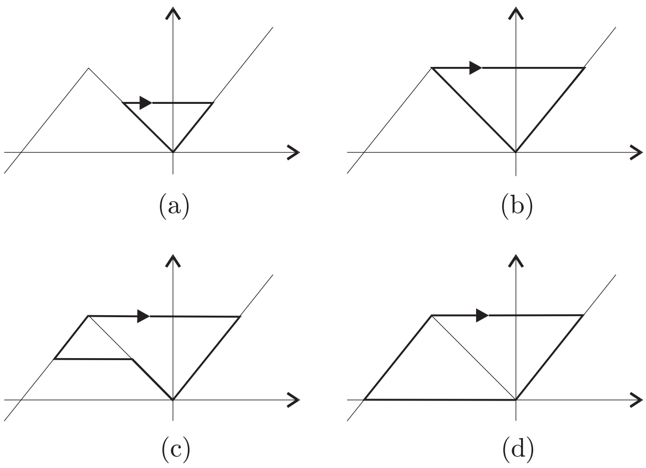

FIG. 6. Possible limit periodic sets of $X_{\varepsilon,\alpha}$ when $\varepsilon \searrow 0$.

Using the closing equations we obtain the following quantitative information about the limit cycles $\Gamma_{\varepsilon,\alpha}$.

THEOREM 11. *The following statements hold.*

(a)If $\varepsilon > 1/4$, the limit cycle $\Gamma_{\varepsilon,\alpha}$ borns from a Hopf bifurcation at the origin when $\alpha = 0$.

(b)If $\varepsilon \leq 1/4$, the limit cycle $\Gamma_{\varepsilon,\alpha}$ emerges from the boundary of an elliptic sector when $\alpha = 0$. Moreover, $\Gamma_{\varepsilon,\alpha}$ is not contained in the half–plane $\mathcal{R} = \{(x, y) : x \geq -1\}$.

(c)There exists a differentiable function $\alpha : (1/4, +\infty) \to (0, 1/2)$ such that $\Gamma_{\varepsilon,\alpha(\varepsilon)}$ is contained in $\mathcal{R}$ and $\Gamma_{\varepsilon,\alpha(\varepsilon)}$ is tangent to the straight line $x = -1$. Moreover,

$$\alpha(\varepsilon) \approx \frac{1}{1 + Ke^{\overline{\sqrt{4\varepsilon - 1}}}{\pi}}$$

when ε is close to $1/4$. Therefore, $\lim_{\varepsilon \searrow 1/4} \alpha(\varepsilon) = 0$ and $\lim_{\varepsilon \searrow 1/4} \alpha'(\varepsilon) = 0$.

(d)The limit of $\Gamma_{\varepsilon,\alpha}$ when ε tends to 0 is the big limit periodic set.

Proof: (a) Assume that $\varepsilon > 1/4$. From Proposition 10 when $\alpha \leq 0$ the system $X_{\varepsilon,\alpha}$ has no limit cycles, and for $\alpha \in (0, 1/2]$ there exists a unique limit cycle $\Gamma_{\varepsilon,\alpha}$. Therefore a limit cycle bifurcates at $\alpha = 0$. We want to study the kind of bifurcation that takes place at $\alpha = 0$.

The points $\mathbf{e}_+ = (-\alpha, -\alpha c)^T$ and $\mathbf{e}_- = (-\alpha, \alpha)^T$ are the solutions of the equations $A\mathbf{x} - \mathbf{b} = 0$ and $B\mathbf{x} - \mathbf{b} = 0$ respectively, and the origin, $\mathbf{p}$, is the contact point of the flow of system $X_{\varepsilon,\alpha}$ with the straight line $x = 0$. Set $\mathbf{p}_+ = \mathbf{p} - \mathbf{e}_+$ and $\mathbf{p}_- = \mathbf{p} - \mathbf{e}_-$, and let $H_A(u, v)$ and $H_B(u, v)$ be the first integrals of systems $\dot{\mathbf{x}} = A\mathbf{x} - \mathbf{b}$ and $\dot{\mathbf{x}} = B\mathbf{x} - \mathbf{b}$ related to $\mathbf{p}_+$ and $\mathbf{p}_-$, respectively, see (14).

If $\Gamma_{\varepsilon,\alpha} \subset \mathcal{R}$, then the closing equations

$$\begin{aligned} H_A(1, v) &= H_A(1, -w), \\ H_B(1, v) &= H_B(1, -w), \end{aligned}$$

have exactly one solution (v_0, w_0), which determines the limit cycle $\Gamma_{\varepsilon,\alpha}$. Moreover $\Gamma_{\varepsilon,\alpha}$ intersects the y–axis at points $v_0\dot{\mathbf{p}}$ and $-w_0\dot{\mathbf{p}}$, where $\dot{\mathbf{p}} = (0, -\varepsilon\alpha)^T$.

The eigenvalues of the matrices A and B are denoted by λ_k and Λ_k, respectively. From Theorem 5(a), $H_A(1, v) = (1 + \lambda_1 v)^{\lambda_2} (1 + \lambda_2 v)^{-\lambda_1}$ and $H_B(1, v) = (1 + \Lambda_1 v)^{\Lambda_2} (1 + \Lambda_2 v)^{-\Lambda_1}$. Since λ_k and Λ_k only depends on ε (see (14) and remember that c is fixed), then v_0 and w_0 only depends on ε. Thus, we write $v_0(\varepsilon)$ and $w_0(\varepsilon)$. Furthermore, when α tends to 0, the intersection points, $v_0(\varepsilon)\dot{\mathbf{p}}$ and $-w_0(\varepsilon)\dot{\mathbf{p}}$, of $\Gamma_{\varepsilon,\alpha}$ with the y–axis tends to the origin. Hence, we conclude that the limit cycle emerges from a Hopf bifurcation at the origin when $\alpha = 0$.

(b) If $0 < \varepsilon \leq 1/4$, then $0 > \lambda_1 > \lambda_2$ and $1 > \Lambda_1 \geq \Lambda_2 > 0$. It is easy to check that the eigenvectors of the matrix A associated to λ_1 and λ_2 are respectively $(1, \lambda_1 + c)^T$ and $(1, \lambda_2 + c)^T$, and the eigenvectors of the matrix B associated to Λ_1 and Λ_2 are respectively $(-1,1-\Lambda_1)^T$ and $(-1,1-\Lambda_2)^T$.

Suppose $\alpha=0$ and let γ_1 be the orbit which intersects the straight line $x=-1$ at $(-1,1-\Lambda_1)^T$, see Figure 7(a). Let $r_1=\{(x,y): y-x(\lambda_1+c)-1-c=0\}$ and $r_2=\{(x,y): y-x(\lambda_1+c)=0\}$. The half–lines r_1 with $x<-1$ and r_2 with $x>0$ are contained into two orbits of $X_{\varepsilon,\alpha}$. Hence, it is easy to check that the origin is the α– and the ω–limit set of γ_1. Thus, $\gamma_1\cup\{\mathbf{p}\}$ is a homoclinic loop. Let Σ_{γ_1} be the open region bounded by the homoclinic loop, then any orbit contained in Σ_{γ_1} is a homoclinic loop.

Since $\Lambda_1\geq\Lambda_2>0$, the point $(-1,1-\Lambda_2)^T$ is contained in Σ_{γ_1} and each orbit γ of the system having the origin as α–limit set is tangent to the vector $(-1,1-\Lambda_2)^T$ at the origin, see Figure 7(a). Therefore, γ is contained in Σ_{γ_1} and γ_1 is the boundary of an elliptic sector.

Suppose now that $\alpha\in(0,1/2]$. From Proposition 10(b) there exists a unique limit cycle, $\Gamma_{\varepsilon,\alpha}$. Since the singular point $\mathbf{e}_-$ is contained in $\mathcal{R}$ and it is a linear node, then the limit cycle $\Gamma_{\varepsilon,\alpha}$ cannot be contained in $\mathcal{R}$.

Let $\Sigma_{\Gamma_{\varepsilon,\alpha}}$ be the open region bounded by the limit cycle $\Gamma_{\varepsilon,\alpha}$. Since the origin is the contact point of the flow of the system with the straight line $x=0$, then the origin is contained in $\Sigma_{\Gamma_{\varepsilon,\alpha}}$.

Let $\mathbf{q}_1$ be the intersection point of the limit cycle $\Gamma_{\varepsilon,\alpha}$ with the straight line $x=0$, see Figure 7(b). Suppose now that the point $(-1,1-\Lambda_1)^T$ is not contained in $\Sigma_{\Gamma_{\varepsilon,\alpha}}$. Then the limit cycle $\Gamma_{\varepsilon,\alpha}$ and the straight line, r, passing through the origin and the point $(-1,1-\Lambda_1)^T$ intersect at a point $\mathbf{q}_2=(x_0,y_0)^T$ with $-1\leq x_0\leq 0$. Thus, there exists a point $\mathbf{q}_3$ contained in the orbit arc $\mathbf{q}_1\mathbf{q}_2$ such that the vector field on $\mathbf{q}_3$ is parallel to the eigenvector $(-1,1-\Lambda_1)^T$. Since the system in this region is linear, the point $\mathbf{q}_3$ is on a repelling invariant linear subspace of the singular point, in contradiction with the fact that $\mathbf{q}_3$ is on the limit cycle. Therefore, the point $(-1,1-\Lambda_1)^T$ is contained in $\Sigma_{\Gamma_{\varepsilon,\alpha}}$.

Let $r_1=\{(x,y): y-x(\lambda_1+c)-1-c-\alpha\lambda_1=0\}$ and $r_2=\{(x,y): y-x(\lambda_1+c)-\alpha\lambda_1=0\}$. The half–lines r_1 with $x\leq-1$ and r_2 with $x>0$ are contained into two orbits of $X_{\varepsilon,\alpha}$, see Figure 7(b). When α tends to 0, the half–lines r_1 and r_2 tends to the half–lines r_1 and r_2, respectively, in Figure 7(a). Thus, when α tends to 0 the limit cycle $\Gamma_{\varepsilon,\alpha}$ tends to the boundary of the elliptic sector.

(c) Since $\mathbf{q}=\left(1-\frac{1}{\alpha}\right)\mathbf{p}_-$ is the contact point of the flow of system $X_{\varepsilon,\alpha}$ with the straight line $x=-1$, the existence of a limit cycle, $\Gamma_{\varepsilon,\alpha}$, of $X_{\varepsilon,\alpha}$ contained in the half–plane $\mathcal{R}$ and tangent to $x=-1$ is determined by the equation

$$\mu(\varepsilon,\alpha)=H_B\left(1-\frac{1}{\alpha},0\right)-H_B(1,v_0(\varepsilon))=0. \tag{18}$$

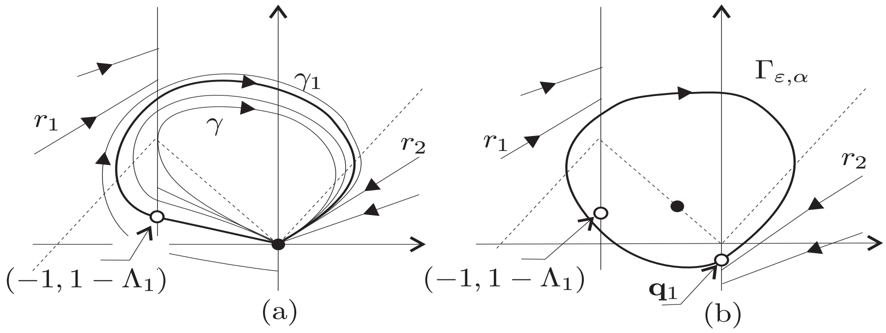

FIG. 7. (a) Boundary of the elliptic sector when $\alpha = 0$. (b) Limit cycle $\Gamma_{\varepsilon,\alpha}$ when $\alpha \in (0, 1/2]$.

Where $(v_0(\varepsilon), w_0(\varepsilon))$ is the solution of the closing equations

$$\begin{aligned} H_A(1, v) &= H_A(1, -w), \\ H_B(1, v) &= H_B(1, -w). \end{aligned}$$

In this case, the limit cycle intersect with the straight line $x = 0$ at points $v_0(\varepsilon)\dot{\mathbf{p}}$ and $-w_0(\varepsilon)\dot{\mathbf{p}}$, for more details see the proof of the statement (a).

Assume $\varepsilon_0 > 1/4$ and $\alpha > 0$. When α is sufficiently small the limit cycle $\Gamma_{\varepsilon_0,\alpha}$ is contained in $\mathcal{R}$, see Statement (a); and when α is sufficiently close to $1/2$, by the symmetry around the point $x = -1/2, y = 1/2, \alpha = 1/2$, the cycle $\Gamma_{\varepsilon_0,\alpha}$ intersects with the three open regions of the phase space $\{(x, y) : x < -1\}$, $\{(x, y) : -1 < x < 0\}$ and $\{(x, y) : 0 < x\}$. Hence, there exists $\alpha_0 \in (0, 1/2)$ such that $\mu(\varepsilon_0, \alpha_0) = 0$.

Since

$$\begin{aligned} \left.\frac{\partial \mu}{\partial \alpha}\right|_{(\varepsilon_0,\alpha_0)} &= \left.\frac{\partial H_B}{\partial u}\right|_{\left(1-\frac{1}{\alpha_0},0\right)} \frac{1}{\alpha_0^2} \\ &= (\Lambda_2 - \Lambda_1)\left(\frac{\alpha_0 - 1}{\alpha_0}\right)^{\Lambda_2-\Lambda_1-1} \frac{1}{\alpha_0^2} \neq 0, \end{aligned}$$

by the Implicit Function Theorem there exists a differentiable function $\alpha : (1/4, \infty) \to (0, 1/2)$ such that $\mu(\varepsilon, \alpha(\varepsilon)) = 0$. From expression (18) and Theorem 5(b) we obtain that

$$(1 + Tv_0(\varepsilon) + Dv_0(\varepsilon)^2) = e^{\frac{2T}{\sqrt{4D-T^2}}\varrho(\varepsilon)} \left(1 - \frac{1}{\alpha(\varepsilon)}\right)^2,$$

where the angle

$$\varrho(\varepsilon) = \arctan\left(\frac{v_0(\varepsilon)\sqrt{4D - T^2}}{2 + Tv_0(\varepsilon)}\right) - \arctan(0)$$

belongs to the same domain that the angle θ covered by the solution between the points $v_0(\varepsilon)\,\dot{\mathbf{p}}$ and $\mathbf{q}$, see Remark 6.

Isolating $\alpha(\varepsilon)$ we obtain that

$$\alpha(\varepsilon) = \frac{1}{1 + \sqrt{1 + Tv_0(\varepsilon) + Dv_0(\varepsilon)^2}\; e^{\frac{-T}{\sqrt{4D-T^2}}\varrho(\varepsilon)}}.$$

Since $H_A(u, v) = (u + \lambda_1 v)^{\lambda_2}(u + \lambda_2 v)^{-\lambda_1}$ with $0 > \lambda_1 > \lambda_2$, $v_0(\varepsilon) > 0$ and $w_0(\varepsilon) > 0$, from equation

$$(1 + \lambda_1 v_0(\varepsilon))^{\lambda_2}(1 + \lambda_2 v_0(\varepsilon))^{-\lambda_1} = (1 - \lambda_1 w_0(\varepsilon))^{\lambda_2}(1 - \lambda_2 w_0(\varepsilon))^{-\lambda_1}$$

follows that $1 + \lambda_1 v_0(\varepsilon) > 0$, and then $v_0(\varepsilon) < 2/\left(c - \sqrt{c^2 - 1}\right)$. Thus, when ε tends to $1/4$ the intersection point $v_0(\varepsilon)\,\dot{\mathbf{p}}$ tends to the origin, the angle θ tends to $-\pi$ and $\lim\limits_{\varepsilon \searrow 1/4} \varrho(\varepsilon) = -\pi$.

Therefore, it is easy to check that

$$\alpha(\varepsilon) \approx \frac{1}{1 + Ke^{\frac{\pi}{\sqrt{4\varepsilon - 1}}}}$$

when ε is close to $1/4$.

(d) If ε tends to zero, the eigenvalue Λ_1 of the matrix B tends to 1. Therefore, $(-1, 1 - \Lambda_1)^T$ tends to $(-1, 0)^T$. Since $(-1, 1 - \Lambda_1)^T$ is contained in $\Sigma_{\Gamma_{\varepsilon,\alpha}}$ (the open region bounded by $\Gamma_{\varepsilon,\alpha}$), see Figure 7, then $\Gamma_{\varepsilon,\alpha}$ tends to the curve represented in Figure 5(d). ∎

From Theorem 11, it follows that the bifurcation set for the limit cycles of systems $X_{\varepsilon,\alpha}$ when $\varepsilon > 0$ and $\alpha \in (-\infty, 1)$ are the half–line $H_0 = \{(0, \varepsilon) : \varepsilon > 1/4\}$ and the segment $H_{ES} = \{(0, \varepsilon) : 0 < \varepsilon \leq 1/4\}$. The half–line H_0 corresponds to a Hopf bifurcation at the origin, and the segment H_{ES} corresponds to a bifurcation of a limit cycle from a homoclinic loop, see Figure 8. The dashed lines are not bifurcation lines of limit cycles, but they represent the transition of the singular point from focus to node. For parameters on the shading region bounded by the graph of $\alpha(\varepsilon)$, the limit cycle $\Gamma_{\varepsilon,\alpha}$ is not contained in the half–plane $\mathcal{R}$.

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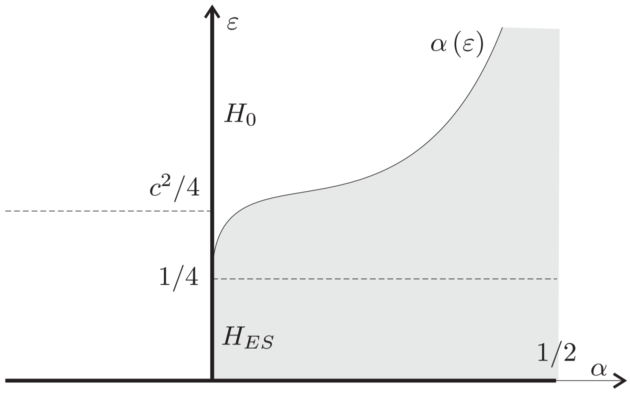

FIG. 8. Bifurcation set of $X_{\varepsilon,\alpha}$, where $\varepsilon \geq 0$ and $\alpha \in (-\infty, 1/2]$.

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