The Coupled Einstein-Weyl Field Equations with Cosmological Constant and Role of Two Higgs Phenomena in Weyl's Gauge Model Coupled to a Higgs Field.

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(ricevuto il 28 Marzo 1983)

Summary. — We study, under the full action of gauge invariance including two Higgs phenomena, Weyl's gauge model coupled to a Higgs field. As a result, we obtain the coupled Einstein-Weyl field equations with the cosmological constant, which are analogous to the Einstein-Maxwell equations apart from difficulties inherent to Weyl's geometry. The vacuum solution of a Higgs field will be discussed in connection with a manifestly gauge-invariant formulation of the Lagrangian density of gravitation. A generalized action which is a function of the gauge-invariant scalar curvature is examined.

PACS. 11.10. – Field theory.

1. – Introduction.

Recent studies on Weyl's gauge model coupled to a Higgs field have been promoted in connections with the appearance of the cosmological constant and the vacuum solution of a Higgs field [1-3]. They are mainly concerned with the Lagrangian density of a Higgs field which is, in terms of a manifestly gauge-invariant formulation, the Lagrangian density of gravitation.

(1) G. DOMOKOS: DESY preprint, DESY 76/24 (1976).
(2) V. DE ALFARO, S. FUBINI and G. FURLAN: *Nuovo Cimento A*, **50**, 523 (1979).
(3) S. L. ADLER: *Phys. Rev. Lett.*, **44**, 1567 (1980).

On the other hand, the study on Weyl's gauge field has not been developed except for Utiyama's work (4); he investigated, several years ago, in good earnest Weyl's gauge field by introducing a new scalar field called a measure field to avoid the two defects of Weyl's gauge theory which are: *a*) an invariant distance d*s* at any world point located in a gauge field cannot have a definite magnitude, *b*) Einstein equation in the conventional form violates the gauge invariance. Because of the complicated formulation of the theory, his work did not rigorously lead to definite results.

One defect (*b*)) can be resolved by making use of a manifestly gauge-invariant formulation which corresponds to the full action of gauge invariance and which was used by UTIYAMA (4). The other (*a*)) can also be avoided by the use of a measure field, but we are in this case faced with a complicated formulation, so we are so far from obtaining rigorous solutions (5). In this paper we are at the sacrifice of avoiding defect *b*), but another attempt of avoiding defect *b*) is seen in (6).

In general, a Higgs field ψ is massless and complex, but the phase of ψ (the Goldstone boson) can be removed by a gauge transformation (the Higgs phenomenon) (1). So, if we introduce a polar decomposition of ψ, we can treat the radial part of ψ as the Higgs field, which is nonnegative and massless, and which is denoted by φ.

The affine connection $\Gamma^\lambda_{\mu\nu}$ in Weyl's geometry is defined by

$$\Gamma^\lambda_{\mu\nu}=\frac{1}{2}\,g^{\lambda\sigma}[\nabla_\mu g_{\nu\sigma}+\Delta_\nu g_{\mu\sigma}-\nabla_\sigma g_{\mu\nu}]\,, \tag{1.1}$$

where

$$\nabla_\mu g_{\nu\sigma}=\partial_\mu g_{\nu\sigma}+fA_\mu g_{\nu\sigma}\,, \tag{1.2}$$

which is transformed as

$$\nabla_\mu g_{\nu\sigma}\to\Lambda^2\nabla_\mu g_{\nu\sigma}$$

under the gauge transformation

$$\varphi\to\Lambda^{-1}\varphi\,,\qquad A_\mu\to A_\mu-\frac{2}{f}\frac{\partial_\mu\Lambda}{\Lambda}\,,\qquad g_{\mu\nu}\to\Lambda^2 g_{\mu\nu}\,,\qquad g^{\mu\nu}\to\Lambda^{-2}g^{\mu\nu}\,, \tag{1.3}$$

where Λ is an arbitrary function of x.

Here A_μ and $\frac{1}{2}f$ are Weyl's gauge field and the charge carried by the Higgs field φ, respectively.

(4) R. UTIYAMA: *Prog. Theor. Phys.*, **53**, 565 (1975).
(5) M. NISHIOKA: *Nuovo Cimento A*, **75**, 80 (1983).
(6), M. NISHIOKA: *Lett. Nuovo Cimento*, **36**, 266 (1983).

Therefore, $\Gamma^{\lambda}_{\mu\nu}$ is gauge invariant and is expressed in the following way:

$$\Gamma^{\lambda}_{\mu\nu} = \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} + \frac{1}{2} f\{\delta^{\lambda}_{\mu} A_{\nu} + \delta^{\lambda}_{\nu} A_{\mu} - A^{\lambda} g_{\mu\nu}\}, \tag{1.4}$$

where $\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}$ is the Christoffel symbol.

The curvature tensor $W_{\mu\nu}$ derived from the $\Gamma^{\lambda}_{\mu\nu}$ is given by

$$W_{\mu\nu} = \partial_{\mu}\Gamma^{\lambda}_{\nu\lambda} - \partial_{\lambda}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\mu\varrho}\Gamma^{\varrho}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\varrho}_{\lambda\varrho}, \tag{1.5}$$

the corresponding scalar curvature W is

$$W = R + \frac{3}{2} f^2 A_{\mu} A^{\mu} + 3f A^{\mu}_{;\mu}, \tag{1.6}$$

where $A^{\mu}_{;\mu}$ is the covariant derivative of A^{μ} with respect to x^{μ}.

The usual form of the gravitational part of the Lagrangian density $\sqrt{-g}\, W$ is not gauge invariant, so we make it gauge invariant by making use of the Higgs field φ

$$L_{G} = -\frac{1}{12}\sqrt{-g}\, \varphi^2 W. \tag{1.7}$$

The Lagrangian density of the Higgs field is given by

$$L_{H} = -\sqrt{-g}\left\{\frac{1}{2} \nabla_{\mu}\varphi \nabla_{\nu}\varphi g^{\mu\nu} + \frac{\lambda}{4!}\varphi^4\right\}, \tag{1.8}$$

and the Lagrangian density of Weyl's gauge field is

$$L_{A} = -\frac{1}{4}\sqrt{-g}\, g^{\mu\nu} g^{\varrho\sigma} F_{\mu\varrho} F_{\nu\sigma}, \tag{1.9}$$

where

$$\nabla_{\mu}\varphi = \partial_{\mu}\varphi - \frac{f}{2} A_{\mu}\varphi, \qquad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \tag{1.10}$$

By making use of (1.10), the total Lagrangian density $L = L_{G} + L_{H} + L_{A}$ is given by

$$L = -\sqrt{-g}\left\{\nabla_{\mu}\varphi\nabla_{\nu}\varphi g^{\mu\nu} + \frac{\lambda}{4!}\varphi^4 + \frac{1}{4} g^{\mu\nu} g^{\varrho\sigma} F_{\mu\varrho} F_{\nu\sigma} + \right.$$
$$\left. + \frac{1}{12}\varphi^2 R - \frac{1}{2} g^{\mu\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi + \frac{1}{2}\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}\, \varphi g^{\mu\nu}\partial_{\nu}\varphi)\right\}. \tag{1.11}$$

We divide the total Lagrangian density into two parts, L_m and L_g, as follows:

$$(1.12)\quad \begin{cases} L = L_m + L_g\,, \\ L_m = -\sqrt{-g}\left\{\nabla_\mu\varphi\nabla_\nu\varphi g^{\mu\nu} + \frac{1}{4}\, g^{\mu\nu} g^{\varrho\sigma} F_{\mu\varrho} F_{\nu\sigma}\right\}, \\ L_g = -\sqrt{-g}\left\{-\frac{1}{2}\, g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{12}\,\varphi^2 R + \frac{\lambda}{4!}\,\varphi^4 + \right. \\ \qquad\qquad \left. + \frac{1}{2}\,\frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\,\varphi g^{\mu\nu}\partial_\nu\varphi)\right\}, \end{cases}$$

where the last term of L_g is the divergent term in the calculus of variation, so it disappears. Then we neglect it hereafter.

L_g is often used in a scalar-tensor theory of gravitation [7,8] in connection with broken symmetry and the appearance of the cosmological constant. Moreover, recent studies on Weyl's gauge model coupled to a Higgs field [1,2] exhausted much effort in obtaining the vacuum solution of the Higgs field.

In sect. **2**, using L_g, we study the connection between a manifestly gauge-invariant formulation and the Higgs-Kibble transformation [9] based on spontaneous symmetry breaking. In sect. **3**, we will obtain the coupled Einstein-Weyl equations for Weyl's gauge field and their solutions. In the final section, by making use of the gauge-invariant scalar curvature, we study an action which is expressed as a function of the gauge-invariant scalar curvature.

2. – A manifestly gauge-invariant formulation and the Higgs-Kibble transformation.

Using L_g (1.12), we study the connection between a manifestly gauge-invariant formulation and the Higgs-Kibble transformation based on spontaneous symmetry breaking. We assume the Higgs field φ to have nonvanishing vacuum expectation value (we may work in the tree approximation).

From (1.12), we have the field equation for φ and the gravitational equation, respectively,

$$(2.1)\qquad \Box\varphi + \frac{1}{6}\,\varphi R - \frac{\lambda}{6}\,\varphi^3 = 0\,,$$

$$(2.2)\qquad \frac{\varphi^2}{6}\, G_{\mu\nu} = T_{\mu\nu}\,,$$

(7) Y. Fujii: *Ann. Phys.* (*N. Y.*), **69**, 494 (1972).
(8) A. Zee: *Phys. Rev. Lett.*, **42**, 417 (1979).
(9) F. Gürsey: *Ann. Phys.* (*N.Y.*), **24**, 211 (1963).

where $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$, $T_{\mu\nu}$ is the energy-momentum tensor of the field φ [1,2].

Taking the trace of (2.2), we obtain

$$\varphi\left(\frac{1}{6}\varphi R+\Box\varphi-\frac{\lambda}{6}\varphi^3\right)=0\,. \tag{2.3}$$

As we assume for φ nonvanishing vacuum expectation value (spontaneous symmetry breaking), (2.1) is a consequence of (2.2).

φ is a radial part of ψ and is assumed to be nonzero, so φ is considered to be positive. We consider the following gauge transformation:

$$\bar{g}_{\mu\nu}=\varphi^2 g_{\mu\nu}\,,\qquad \bar{g}^{\mu\nu}=\varphi^{-2}g^{\mu\nu}\,,\qquad B_\mu=A_\mu-\frac{2}{f}\partial_\mu\varphi/\varphi\,. \tag{2.4}$$

This gauge transformation was fully used by UTIYAMA [4] in his theory of Weyl's gauge field. The gauge transformation (2.4) except for the last relation of (2.4) is equivalent to the finite Weyl transformation which is used by DOMOKOS [1]

$$g'_{\mu\nu}=\exp[-\varrho]g_{\mu\nu}\,,\qquad \varphi'=\exp\left[\frac{1}{2}\varrho\right]\varphi\,, \tag{2.5}$$

where $-\infty<\varrho(x)<\infty$, $\varphi'=\text{const}$ (which can be chosen to be unity). FUBINI *et al.* [2] considered the case in which $g'_{\mu\nu}=\eta_{\mu\nu}$ (Minkowski metric), in our notation (2.4), $\bar{g}_{\mu\nu}=\eta_{\mu\nu}$.

We notice here that, in Utiyama's original usage of (2.4), φ is not limited to be positive.

By making use of (2.4) or (2.5), we can absorb the co-ordinate dependence of φ into the metric, this is the Higgs-Kibble transformation. Moreover, we would like to stress that any quantity like, say, L_m or L_g, if written in terms of $\bar{g}_{\mu\nu}$, B_μ, etc., is a gauge-invariant quantity.

In terms of $\bar{g}_{\mu\nu}$, $\bar{g}^{\mu\nu}$, B_μ (which are also gauge invariant), we express L_m and L_g as

$$L_m=-\sqrt{-\bar{g}}\left\{\frac{1}{4}f^2B_\mu B_\nu\bar{g}^{\mu\nu}+\frac{1}{4}\bar{g}^{\mu\nu}\bar{g}^{\varrho\sigma}F_{\mu\varrho}F_{\nu\sigma}\right\}, \tag{2.6}$$

$$L_g=-\frac{1}{12}\sqrt{-\bar{g}}\left(\bar{R}+\frac{1}{2}\lambda\right), \tag{2.7}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu=\partial_\mu B_\nu-\partial_\nu B_\mu$, $\bar{R}$ is the scalar curvature in terms of $\bar{g}_{\mu\nu}$.

From L_g, we obtain from the variational principle the Einstein equation with the cosmological constant on the assumption that $\varphi'=\text{unity}\neq 0$

$$\bar{G}_{\mu\nu}-\frac{1}{4}\lambda\bar{g}_{\mu\nu}=a\bar{T}_{\mu\nu}\,, \tag{2.8}$$

where $\bar{G}_{\mu\nu}$ is the Einstein tensor in terms of $\bar{g}_{\mu\nu}$, $\bar{T}_{\mu\nu}$ is the energy-momentum tensor made from L_m, a is a constant.

If we neglect $\bar{T}_{\mu\nu}$, as for the solution of (2.8), we obtain a space of constant curvature. With regard to the vacuum solution of φ including the case of a space of constant curvature, see [1,2,5].

3. – The coupled Einstein-Weyl equations.

From (2.6) and (2.7), we obtain from the variational principle the coupled Einstein-Weyl equations. For the variation of B_μ we have

$$\partial_\nu(\sqrt{-\bar{g}}\,F^{\mu\nu}) + \frac{1}{2} f^2\sqrt{-\bar{g}}B^\mu = 0\,. \tag{3.1}$$

For the variation of $\bar{g}_{\mu\nu}$ we obtain (2.8). $\bar{T}_{\mu\nu}$ is given by

$$\begin{aligned}\bar{T}_{\mu\nu} &= 2\,\frac{\partial L_m}{\partial \bar{g}^{\mu\nu}}\Big/\sqrt{-\bar{g}} = \\ &= -\bar{g}^{\varrho\sigma}F_{\mu\varrho}F_{\nu\sigma} - \frac{1}{2} f^2 B_\mu B_\nu + \bar{g}_{\mu\nu}\left(\frac{1}{4} f^2 B_\alpha B_\beta \bar{g}^{\alpha\beta} + \frac{1}{4}\bar{g}^{\alpha\beta}\bar{g}^{\varrho\sigma}F_{\alpha\varrho}F_{\beta\sigma}\right).\end{aligned} \tag{3.2}$$

The trace of $\bar{T}_{\mu\nu}$ is

$$\bar{g}^{\mu\nu}\bar{T}_{\mu\nu} = \frac{1}{2} f^2 B_\mu B^\mu\,; \tag{3.3}$$

it is generally known [4,10] that the trace of $\bar{T}_{\mu\nu}$ vanishes, then we have

$$B_\mu B^\mu = 0\,. \tag{3.4}$$

As $F_{\mu\nu}$ is antisymmetric with respect to μ and ν, we obtain from (3.1)

$$B^\mu_{\ ;\mu} = 0\,, \tag{3.5}$$

where $B^\mu_{\ ;\mu}$ is the covariant derivative with respect to x^μ.

From (3.2) and (3.4) $\bar{T}_{\mu\nu}$ becomes

$$\bar{T}_{\mu\nu} = \left(\frac{1}{4}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\bar{g}^{\varrho\sigma}F_{\alpha\varrho}F_{\beta\sigma} - \bar{g}^{\varrho\sigma}F_{\mu\varrho}F_{\nu\sigma}\right) - \frac{1}{2} f^2 B_\mu B_\nu\,. \tag{3.6}$$

Comparing our results with the coupled Einstein-Maxwell equations [11], first, as for the gravitational equation (2.8) with (3.6), we have an extra term which

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[10] L. GIRARDELLO and S. PALLUA: *Nuovo Cimento A*, **41**, 377 (1977).
[11] M. CARMELI: *Classical Fields* (New York, N. Y., 1982), p. 110, 189, 311.

is the last term of the r.h.s. of (3.6), second, Weyl's gauge field B_μ in the second term of the l.h.s. of (3.1) corresponds to the current density in the Einstein-Maxwell equation. Then the coupled Einstein-Weyl equations are analogous to the Einstein-Maxwell equations, but not one to one.

As for behaviour of B_μ in this framework, B_μ behaves normally, because in Minkowski metric eqs. (3.1), (3.4) and (3.5) become

$$\left(\Box - \frac{1}{2} f^2\right) B_\mu = 0\,, \qquad B_\mu B^\mu = 0\,, \qquad \partial_\mu B^\mu = 0\,,$$

respectively, then $B_\mu = a_\mu \exp[ikx]$ with the properties

$$a_\mu a^\mu = 0 \text{ (null vector)}\,, \qquad a_\mu k^\mu = 0\,, \qquad k^0 = \sqrt{(k^i)^2 + \frac{1}{2} f^2}\,.$$

4. – A generalized action.

By making use of the gauge-invariant quantities (2.4), we rewrite $\varphi^2 W$ in (1.6) or (1.7) as follows:

$$W' = \varphi^2 W = \bar{R} + \frac{3}{2} f^2 B_\mu B^\mu + 3f B^\mu_{;\mu}: \tag{4.1}$$

this is the gauge-invariant scalar curvature.

We consider the following Lagrangian density L':

$$L' = \sqrt{-g}\left\{f(W') - a\,\frac{1}{4}\,\bar{g}^{\mu\nu}\bar{g}^{\varrho\sigma} F_{\mu\varrho} F_{\nu\sigma}\right\}, \tag{4.2}$$

where L' is, of course, gauge invariant, $f(z)$ is a differentiable function, a is constant.

The field equations derived from (4.2) contain in general derivatives of $\bar{g}_{\mu\nu}$ higher than second order, then we must introduce some conditions. WEYL [12] considered the case for $f(z) = z^2$, but his W' was not gauge invariant, correspondingly the Lagrangian density did not. In his case he considered the gauge condition $W' = \text{const}$.

According to WEYL, we impose the gauge condition

$$W' = \omega = \text{const} \tag{4.3}$$

on the Lagrangian density L'.

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[12] H. WEYL: *Sitzungsber. Preuss. Akad. Wiss.*, 465 (1918).

Using (4.3), we have the formula which will be useful for later discussion

$$\delta\{\sqrt{-\bar g}\,f(W')\} = f'(\omega)\,\delta(\sqrt{-\bar g}\,W') + \{f(\omega) - \omega f'(\omega)\}\,\delta(\sqrt{-\bar g})\,, \tag{4.4}$$

where $f'(z) = \mathrm{d}f/\mathrm{d}z$.

Using (4.4) and considering $-L'/f'(\omega)$ to be the Lagrangian density for our system, we have

$$L = -L'/f'(\omega) = -\sqrt{-\bar g}\left\{W' + \lambda + \frac{1}{4}\,\bar g^{\mu\nu}\bar g^{\varrho\sigma}F_{\mu\varrho}F_{\nu\sigma}\right\}, \tag{4.5}$$

where $\lambda = \{f(\omega) - \omega f'(\omega)\}/f'(\omega)$, $-a = f'(\omega)$.

From (4.5) and (4.1) the Lagrangian density L is expressed as

$$L = -\sqrt{-\bar g}\left\{\bar R + \frac{3}{2}\,f^2 B_\mu B^\mu + \lambda + \frac{1}{4}\,\bar g^{\mu\nu}\bar g^{\varrho\sigma}F_{\mu\varrho}F_{\nu\sigma}\right\}, \tag{4.6}$$

where the last term of $\sqrt{-\bar g}\,W'$ vanishes because of the divergence in the calculus of variation.

Comparing (4.6) with (2.6) + (2.7), we find that they have the same form except for the numerical value of coefficients. Therefore, the coupled Einstein-Weyl equations for (4.6) have the same forms as (2.8) and (3.1) except for the numerical value of the coefficients, and the equation of Weyl's gauge field also produces (3.5). The vanishing of the trace of the energy-momentum tensor obtained from (4.6) introduces again (3.4). We must notice that the cosmological constant appears as the consequence of the gauge condition (4.3); in case that $f(z) \neq z$, the cosmological constant always appears. Finally we may conclude that (4.6) and (2.6) + (2.7) will be physically equivalent in case that the Higgs-Kibble transformation (2.5) or (2.4) can apply; in other words, the Higgs field is absorbed into the metric and the Lagrangian density can be written in terms of the gauge-invariant quantities.

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We would like to thank Prof. G. DOMOKOS for sending us the preprint on broken Weyl symmetry.

● RIASSUNTO (*)

Si studia, sotto l'azione piena dell'invarianza di gauge che include due fenomeni di Higgs, il modello di gauge di Weyl accoppiato ad un campo di Higgs. Come risultato si ottengono le equazioni di campo accoppiate di Einstein-Weyl con la costante cosmologica, che sono analoghe alle equazioni di Einstein-Maxwell tranne per difficoltà inerenti alla geometria di Weyl. La soluzione nel vuoto di un campo di Higgs sarà discussa in connessione con una formulazione palesemente invariante di gauge della densità lagrangiana di gravitazione. Si esamina un'azione generalizzata che è funzione della curvatura scalare invariante di gauge.

(*) *Traduzione a cura della Redazione.*

Связанные полевые уравнения Эйнштейна-Вейля с космологической постоянной и роль двух явлений Хиггса в калибровочной модели Вейля, связанной с полем Хиггса.

Резюме (*). — Мы исследуем калибровочную модель Вейля, связанную с полем Хиггса. Мы получаем связанные полевые уравнения Эйнштейна-Вейля с космологической постоянной, которые аналогичны уравнениям Эйнштейна-Максвелла, за исключением трудностей, присущих геометрии Вейля. Вакуумное решение для поля Хиггса обсуждается в связи с явной калибровочно инвариантной формулировкой плотности Лагранжиана для гравитации. Исследуется обобщенное действие, которое является функцией калибровочно инвариантной скалярной кривизны.

(*) *Переведено редакцией.*