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Note on Nonstability of the Linear Recurrence

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Abstract. We prove a non-stability result for linear recurrences with constant coefficients in Banach spaces. As a consequence we obtain a complete solution of the problem of the Hyers-Ulam stability for those congruences in the complex Banach space.

Throughout this note, denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$, as usual, the sets of positive integers, nonnegative integers, reals and complex numbers, respectively. Moreover $\mathbb{K}$ is either the field $\mathbb{R}$ or $\mathbb{C}$, X is a Banach space over $\mathbb{K}$, and $\mathbb{S} := \{a \in \mathbb{C} \mid |a| = 1\}$.

The stability problem of functional equations was originally raised by S. M. ULAM [13] in 1940. He posed the following problem: under what conditions does there exist an additive mapping near an approximately additive mapping? In 1941, this problem was solved by D. H. HYERS [8] in the case of Banach spaces.

After HYERS's result a great deal of papers on this subject have been published, generalizing ULAM's problem and HYERS's theorem in various directions and to other functional equations (see e.g. [1, 2, 3, 4, 7, 10, 12]). Furthermore, many surveys, specially on stability of functional equations and their systems in several variables, were given successively (see e.g. [5, 6, 9]).

The Hyers-Ulam stability of linear recurrence with constant coefficients, a discrete case of equations in a single variable, was investigated in [11], where the second author of the present paper has proved the following theorem.

Theorem 1. *Let $p \in \mathbb{N}$, $\delta > 0$, $a_1, \dots, a_p \in \mathbb{K}$ be such that the equation*

$$r^p - \sum_{i=1}^{p} a_i r^{p-i} = 0 \tag{1}$$

admits the roots $r_1, \dots, r_p \in \mathbb{K} \setminus \mathbb{S}$ and $(b_n)_{n \in \mathbb{N}_0}$ be a sequence in X. Suppose that $(y_n)_{n \in \mathbb{N}_0}$ is a sequence in X with the property

$$\|y_{n+p} - a_1 y_{n+p-1} - \cdots - a_p y_n - b_n\| \le \delta, \quad \forall n \in \mathbb{N}_0. \tag{2}$$

Then there exists a sequence $(x_n)_{n \in \mathbb{N}_0}$ in X, given by the recurrence

$$x_{n+p} = a_1 x_{n+p-1} + \cdots + a_p x_n + b_n, \quad \forall n \in \mathbb{N}_0, \tag{3}$$

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such that

$$\|y_n - x_n\| \leq \frac{\delta}{|(1-|r_1|)\cdots(1-|r_p|)|}, \quad \forall n \in \mathbb{N}_0. \tag{4}$$

Roughly speaking, we say that a functional equation is stable in the Hyers-Ulam sense if for every solution of the perturbed equation, there exists a solution of the equation that differs from the solution of the perturbed equation with a small error. However, because of many later restatements (see e.g. [1, 9]) of the original problem, raised by S. M. ULAM in 1940, we introduce the following two definitions (cf. [10], p. 290).

Definition 1. Recurrence (3) is said to be weakly stable in the Hyers-Ulam sense provided, for every unbounded sequence $(y_n)_{n\in\mathbb{N}_0}$ in X with

$$\sup_{n\in\mathbb{N}} \|y_{n+p} - a_1 y_{n+p-1} - \cdots - a_p y_n - b_n\| < \infty,$$

there exists a sequence $(x_n)_{n\in\mathbb{N}_0}$ in X such that (3) holds and

$$\sup_{n\in\mathbb{N}_0} \|y_n - x_n\| < \infty.$$

Definition 2. Recurrence (3) is said to be strongly stable in the Hyers-Ulam sense provided for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every sequence $(y_n)_{n\in\mathbb{N}_0}$ in X satisfying (2), there exists a sequence $(x_n)_{n\in\mathbb{N}_0}$ in X such that (3) holds and $\sup_{n\in\mathbb{N}_0} \|y_n - x_n\| \leq \varepsilon$.

Of course some other definitions are possible, but the first one seems to be reasonably weak, while the second one is quite strong. It is easily seen that a recurrence that is strongly stable is also weakly stable and a recurrence which is not weakly stable is not strongly stable either. Therefore the definitions suit well the statement of the next theorem, where we supplement the result of Theorem 1 by showing that, in each case where at least one of the roots of the characteristic equation (1) is in $\mathbb{S}$, the congruence is not (weakly or strongly) stable in the Hyers-Ulam sense. Thus we completely solve the problem of the Hyers-Ulam stability for the linear congruences with constant coefficients in the complex Banach spaces, in the sense of either of the definitions.

Theorem 2. *Let* $\mathbb{K} = \mathbb{C}$, $a_1, \ldots, a_p \in \mathbb{K}$, *and* $(b_n)_{n\in\mathbb{N}_0}$ *be a sequence in* X. *The recurrence* (3) *is weakly (strongly, respectively) stable in the Hyers-Ulam sense if and only if the characteristic equation* (1) *does not have any root in* $\mathbb{S}$.

In the real case we have only the following.

Theorem 3. *Let $\mathbb{K} = \mathbb{R}$, $a_1, \ldots, a_p \in \mathbb{K}$, and $(b_n)_{n\in\mathbb{N}_0}$ be a sequence in X. Assume that the characteristic equation* (1) *admits roots $r_1, \ldots, r_p \in \mathbb{R}$. The recurrence* (3) *is weakly (strongly, respectively) stable in the Hyers-Ulam sense if and only if $r_1, \ldots, r_p \notin \{-1, 1\}$.*

Corollary 1. *Let $\mathbb{K} = \mathbb{R}$, $a \in \mathbb{K}$, and $(b_n)_{n\in\mathbb{N}_0}$ be a sequence in X. The recurrence*

$$x_{n+1} = ax_n + b_n, \quad \forall n \in \mathbb{N}_0 \tag{5}$$

is weakly (strongly, respectively) stable in the Hyers-Ulam sense if and only if $|a| \neq 1$.

It is easily seen that Theorems 2 and 3 are immediate consequences of Theorem 1 and the given below Theorem 4 (Corollary 1 follows from Theorem 3).

Theorem 4. *Let $a_1, \ldots, a_p \in \mathbb{K}$ and $(b_n)_{n\in\mathbb{N}_0}$ be a sequence in X. Suppose that the characteristic equation* (1) *admits the roots $r_1, r_2, \ldots, r_p \in \mathbb{K}$ and at least one of them is of module* 1. *Then for any $\delta > 0$ there exists an unbounded sequence $(y_n)_{n\in\mathbb{N}_0}$, satisfying the inequality* (2), *such that for every sequence $(x_n)_{n\in\mathbb{N}_0}$ in X, fulfilling the linear recurrence* (3), *we have*

$$\sup_{n\in\mathbb{N}_0} \|y_n - x_n\| = \infty. \tag{6}$$

For the proof of Theorem 4 we need two lemmas.

Lemma 1. *Let $a \in \mathbb{K}$, $(b_n)_{n\in\mathbb{N}_0}$ be a sequence in X, $x_0 \in X$, and*

$$x_{n+1} = ax_n + b_n, \quad \forall n \in \mathbb{N}_0, \tag{7}$$

Then

$$x_n = a^n x_0 + \sum_{k=1}^{n} a^{n-k} b_{k-1}, \quad \forall n \in \mathbb{N}. \tag{8}$$

Proof. Induction on n. □

Lemma 2. *Let $\delta > 0$, $a \in \mathbb{K}$, $|a| = 1$, and $(b_n)_{n\in\mathbb{N}_0}$ be a sequence in X. Then for each $y_0 \in X$ there exists an unbounded sequence $(y_n)_{n\in\mathbb{N}_0}$, satisfying the inequality*

$$\|y_{n+1} - ay_n - b_n\| \leq \delta, \quad \forall n \in \mathbb{N}_0, \tag{9}$$

such that for arbitrary sequence $(x_n)_{n\in\mathbb{N}_0}$ in X, satisfying (7), *we have*

$$\sup_{n\in\mathbb{N}_0} \|y_n - x_n\| = \infty. \tag{10}$$

Proof. Let $u \in X$, $\|u\| = 1$,

$$\varepsilon := \begin{cases} 1, & \text{if } \sup_{n\in\mathbb{N}_0} \left\|\sum_{k=1}^{n} a^{n-k} b_{k-1} + na^n \delta u\right\| = \infty; \\ -1, & \text{otherwise,} \end{cases}$$

$y_0 \in X$, and $(y_n)_{n\in\mathbb{N}_0}$ be given by

$$y_{n+1} = ay_n + b_n + \varepsilon a^{n+1} \delta u, \quad \forall n \in \mathbb{N}_0.$$

Take $x_0 \in X$ and define $(x_n)_{n\in\mathbb{N}_0}$ by (7). Then, according to Lemma 1,

$$y_n - x_n = a^n(y_0 - x_0) + n\varepsilon a^n \delta u, \quad \forall n \in \mathbb{N}_0.$$

Since, for $n \in \mathbb{N}$,

$$\|y_n - x_n\| \geq \big|\|a^n(y_0 - x_0)\| - \|na^n\delta u\|\big| = |\|y_0 - x_0\| - n\delta|$$

it is easily seen that

$$\lim_{n\to\infty} \|y_n - x_n\| = \infty.$$

To complete observe that, according to Lemma 1, for every $n \in \mathbb{N}$,

$$y_n = a^n y_0 + \sum_{k=1}^{n} a^{n-k} b_{k-1} + n\varepsilon a^n \delta u,$$

whence, in the case $\varepsilon = 1$,

$$\|y_n\| \geq \left\|\sum_{k=1}^{n} a^{n-k} b_{k-1} + na^n\delta u\right\| - \|y_0\|,$$

and, in the case $\varepsilon = -1$,

$$\|y_n\| = \left\|a^n y_0 + \sum_{k=1}^{n} a^{n-k} b_{k-1} - na^n\delta u\right\| \geq 2n\delta - \left\|\sum_{k=1}^{n} a^{n-k} b_{k-1} + na^n\delta u\right\| - \|y_0\|,$$

which, in either case, means that $(y_n)_{n\in\mathbb{N}_0}$ is unbounded. □

Now we are in a position to prove Theorem 4.

Proof of Theorem 4. For $p = 1$, the conclusion of Theorem 4 is true in virtue of Lemma 2.

For $p \geq 2$, without loss of generality, assume that $|r_1| = 1$. From Lemma 2 it follows that there exists an unbounded sequence $(\bar{y}_n)_{n\in\mathbb{N}_0}$ in X, satisfying the inequality

$$\|\bar{y}_{n+1} - r_1\bar{y}_n - b_n\| \leq \delta, \quad \forall n \in \mathbb{N}_0, \tag{11}$$

such that for every sequence $(\bar{x}_n)_{n\in\mathbb{N}_0}$ with

$$\bar{x}_{n+1} = r_1\bar{x}_n + b_n, \quad \forall n \in \mathbb{N}_0, \tag{12}$$

we have

$$\sup_{n\in\mathbb{N}_0} \|\bar{y}_n - \bar{x}_n\| = \infty. \tag{13}$$

Further, there exists a sequence $(y_n)_{n\in\mathbb{N}_0}$ in X with

$$y_{n+p-1}-(r_2+\cdots+r_p)y_{n+p-2}+\cdots+(-1)^{p-1}r_2\cdots r_p y_n=\bar{y}_n, \quad \forall n\in\mathbb{N}_0 \quad (14)$$

(it suffices to take $y_0=\cdots=y_{p-2}=0$, $y_{p-1}=\bar{y}_0$; then $(y_n)_{n\in\mathbb{N}_0}$ can be determined step by step).

Inequality (11) implies that the sequence $(y_n)_{n\in\mathbb{N}_0}$ satisfies the inequality

$$\|y_{n+p}+(-1)(r_1+\cdots+r_p)y_{n+p-1}+\cdots+(-1)^p r_1\cdots r_p y_n-b_n\|\le\delta, \quad \forall n\in\mathbb{N}_0, \quad (15)$$

which is equivalent with (2). Let now $(x_n)_{n\in\mathbb{N}_0}$ be an arbitrary sequence defined by (3) and $(\bar{x}_n)_{n\in\mathbb{N}_0}$ be the sequence given by

$$\bar{x}_n=x_{n+p-1}+(-1)(r_2+\cdots+r_p)x_{n+p-2}+\cdots+(-1)^{p-1}r_2\cdots r_p x_n, \quad \forall n\in\mathbb{N}_0. \quad (16)$$

Then (12) and (13) holds.

We have to prove that $\sup_{n\in\mathbb{N}_0}\|x_n-y_n\|=\infty$. Suppose the contrary. Then there exists $M>0$ such that

$$\|y_n-x_n\|\le M, \quad \forall n\in\mathbb{N}_0.$$

From (14) and (16) it follows that

$$\begin{aligned}\|\bar{y}_n-\bar{x}_n\| &\le \|y_{n+p-1}-x_{n+p-1}\|+|r_2+\cdots+r_p|\cdot\|y_{n+p-2}-x_{n+p-2}\|\\ &\quad+\cdots+|r_2\ldots r_p|\cdot\|y_n-x_n\|\\ &\le(1+|r_2+\cdots+r_p|+\cdots+|r_2\ldots r_p|)M\end{aligned}$$

for every $n\in\mathbb{N}_0$, which contradicts (13).

To complete the proof observe that, in view of (14), $(y_n)_{n\in\mathbb{N}_0}$ must be unbounded. □

Remark 1. Let $\mathbb{K}=\mathbb{R}$. Consider the linear recurrence

$$x_{n+2}=-x_n, \quad \forall n\in\mathbb{N}_0. \quad (17)$$

Its characteristic equation

$$r^2=-1$$

have roots $r_1=i$, $r_2=-i$. We show that (17) is not stable.

Let $u\in X$, $\|u\|=1$, $y_0\in X$, and $(y_n)_{n\in\mathbb{N}_0}$ be given by

$$y_{n+2}=-y_n+(-1)^{[\frac{n}{2}]}\delta u, \quad \forall n\in\mathbb{N}_0,$$

where $\delta>0$ and $[a]$ is the biggest integer which is not greater than a. Clearly, we have

$$\|y_{n+2}+y_n\|\le\delta.$$

For $n=2k$, we have

$$y_{2(k+1)}=-y_{2k}+(-1)^k\delta u, \quad \forall k\in\mathbb{N}_0. \quad (18)$$

We claim that

$$y_{2k}=(-1)^k y_0+(-1)^{k-1}k\delta u, \quad \forall k\in\mathbb{N}. \quad (19)$$

In fact, it is trivial for $k = 1$. Take $m \in \mathbb{N}$ and suppose that it is true for $k = m$. By (18),

$$\begin{aligned} y_{2(m+1)} &= -y_{2m} + (-1)^m \delta u \\ &= -\big((-1)^m y_0 + (-1)^{m-1} m\delta u\big) + (-1)^m \delta u \\ &= (-1)^{m+1} y_0 + (-1)^m (m+1)\delta u. \end{aligned}$$

So we have proved (19). Hence, for every sequence $(x_n)_{n\in\mathbb{N}_0}$ in X, satisfying (17), we have

$$\begin{aligned} \|y_{2k} - x_{2k}\| &= \|(-1)^k y_0 + (-1)^{k-1} k\delta u - (-1)^k x_0\| \\ &\geq \big|\|(-1)^k (y_0 - x_0)\| - \|(-1)^{k-1} k\delta u\|\big| = |\|y_0 - x_0\| - k\delta| \end{aligned}$$

for every $k \in \mathbb{N}$, which implies that

$$\sup_{n\in\mathbb{N}_0} \|y_n - x_n\| = \infty.$$

Thus we have shown that in the case where $\mathbb{K} = \mathbb{R}$ and the characteristic equation of (3) (i.e. (1)) has no real roots, the recurrence can be not stable. Therefore the following problem seems to be of interest.

Problem. Let $\mathbb{K} = \mathbb{R}$, $a_1, \ldots, a_p \in \mathbb{R}$ and $(b_n)_{n\in\mathbb{N}_0}$ be a sequence in X. Suppose that $r_1, r_2, \ldots, r_p \in \mathbb{C}$ are the roots of the characteristic equation (1). Is it true that recurrence (3) is weakly (strongly, resp.) stable in the Hyers-Ulam sense if and only if $|r_i| \neq 1$ for $i = 1, \ldots, p$? (In other words: does Theorem 2 remain valid if $\mathbb{K} = \mathbb{R}$?)

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