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Semigroup-Valued Solutions of the Gołąb-Schinzel Type Functional Equation

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Abstract. Let $(S, \circ)$ be a semigroup. We determine all solutions of the functional equation

$$f(x + g(x)y) = f(x) \circ f(y)$$

under the assumption that $g : \mathbb{R} \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to S$.

Let $(S, \circ)$ be a semigroup. The functional equation

$$f(x + g(x)y) = f(x) \circ f(y) \quad \text{for } x, y \in \mathbb{R}, \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to S$ are unknown functions is a generalization of the equations of the form

$$f(x + f(x)^k y) = tf(x)f(y), \tag{2}$$

where t is a non-zero real number and k is a positive integer. Equations of that type are known in a literature as the Gołąb-Schinzel type functional equations and have been intensively studied in the last half-century. Solutions of (2) under various regularity assumptions (continuity, continuity at a point, measurability) have been considered e.g. in [5], [8], [9], [11], [12], [18] and [23]. It is worthy to mention that the equations of the form (2) play an important role in the determination of substructures of various algebraical structures (see e.g. [2], [9] and [10]). There are also the close connections between (2) (in the case $t = k = 1$) and a problem of classification of near-rings (cf. [4]) and quasialgebras (cf. [22]). For more informations concerning (2), its generalizations and applications we refer to [1], [3], [6], [7], [9], [13]–[17], [19]–[22] and [24]–[26].

In the present paper we determine all solutions (f, g) of (1) under the assumption that g is continuous. As a consequence we obtain a generalization of some results concerning continuous solutions of (2).

In the sequel we introduce the following notation. For a given set $\emptyset \neq A \subset S$,

$$Z_L(A) := \{s \in S \mid s \circ a = s \quad \text{for} \quad a \in A\}$$

and

$$Z(A) := \{s \in S \mid s \circ a = a \circ s = s \quad \text{for} \quad a \in A\}.$$

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Moreover, by $E(S)$ we denote a set of all idempotents of the semigroup S, i.e. $E(S) := \{s \in S \mid s \circ s = s\}$.

We begin with the following

Remark 1. Let $(S, \circ)$ be a semigroup. The equation (1) has a solution if and only if $E(S) \neq \emptyset$. In fact, if (f, g) is a solution of (1), then $f(0) = f(0) \circ f(0)$, so $f(0) \in E(S)$. Conversely, if $s \in E(S)$, then a pair (f, g), where g is an arbitrary function and $f \equiv s$, is a solution of (1).

For a proof of the main result we need two lemmas.

Lemma 1. *Assume that $(S, \circ)$ is a semigroup, $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to S$ are non-constant functions and g is continuous. If (f, g) is a solution of (1), then $\emptyset \neq g^{-1}(\{0\}) = f^{-1}(Z_L(f(\mathbb{R})))$.*

Proof. Assume that (f, g) is a solution of (1). If $x \in g^{-1}(\{0\})$, then by (1), $f(x) = f(x) \circ f(y)$ for $y \in \mathbb{R}$, whence $x \in f^{-1}(Z_L(f(\mathbb{R})))$. Conversely, if $x \in f^{-1}(Z_L(f(\mathbb{R})))$, then in view of (1), $f(x + g(x)y) = f(x)$ for $y \in \mathbb{R}$. Thus $x \in g^{-1}(\{0\})$, because otherwise f would be constant. Therefore $g^{-1}(\{0\}) = f^{-1}(Z_L(f(\mathbb{R})))$. It remains to show that $g^{-1}(\{0\}) \neq \emptyset$. For the proof by contradiction suppose that $g^{-1}(\{0\}) = \emptyset$. Then, by (1), we have

$$f(0) = f(x) \circ f\Big(-\frac{x}{g(x)}\Big) \quad \text{for } x \in \mathbb{R}. \tag{3}$$

Let $A := f^{-1}(\{f(0)\})$ and $B := g^{-1}(\{1\})$. Note that

$$A \cup B = \mathbb{R}. \tag{4}$$

In fact, if $t \in \mathbb{R} \setminus B$, then taking $x_t := \frac{t}{1-g(t)}$, on account of (1) and (3), we obtain

$$\begin{aligned} f(t) &= f(t) \circ f(0) = f(t) \circ f(x_t) \circ f\Big(-\frac{x_t}{g(x_t)}\Big) \\ &= f(t + g(t)x_t) \circ f\Big(-\frac{x_t}{g(x_t)}\Big) = f(x_t) \circ f\Big(-\frac{x_t}{g(x_t)}\Big) = f(0), \end{aligned}$$

whence $t \in A$. Next, by (1), for every $a \in A$ and $b \in B$, we have

$$f(b + a) = f(b + g(b)a) = f(b) \circ f(a) = f(b) \circ f(0) = f(b).$$

Thus

$$f(b + a) = f(b) \quad \text{for } a \in A, b \in B. \tag{5}$$

Since f is non-constant and (4) holds, there exists $b \in B \setminus A$. We prove by induction that for every $n \in \mathbb{N}$

$$(b + nA) \cap A = \emptyset. \tag{6}$$

If $b + a \in A$ for some $a \in A$, then by (5), we get $f(0) = f(b + a) = f(b)$. Thus $b \in A$, which gives a contradiction. Therefore $(b + A) \cap A = \emptyset$. Assume that (6) holds for some $n \in \mathbb{N}$. Then $f(b + na) \neq f(0)$ for $a \in A$ and, by (4), $b + nA \subset B$. Thus using (5), we obtain

$$f(b + (n + 1)a) = f((b + na) + a) = f(b + na) \neq f(0) \quad \text{for } a \in A.$$

Hence $(b + (n + 1)A) \cap A = \emptyset$, so we have proved (6). Now, if $0 \in \operatorname{int} A$, then in view of (6), we have $A = A \cap \mathbb{R} = A \cap \bigcup_{n=1}^{\infty}(b + nA) = \emptyset$. Thus, by (4), $B = \mathbb{R}$, whence $g \equiv 1$. This yields a contradiction. Therefore there exists a sequence $(\alpha_n : n \in \mathbb{N})$ of elements of $\mathbb{R} \setminus A \subset B$ converging to 0. Since $\mathbb{R} \setminus B$ is a non-empty open set, then by (4), for every $a \in \mathbb{R} \setminus B$ and sufficiently large $n \in \mathbb{N}$, we have $\alpha_n + a \in \mathbb{R} \setminus B \subset A$. Thus, in virtue of (5), for sufficiently large $n \in \mathbb{N}$, $f(0) = f(\alpha_n + a) = f(\alpha_n)$, whence $\alpha_n \in A$. This gives is a contradiction, so the proof is completed. □

Lemma 2. *Assume that* $(S, \circ)$ *is a semigroup,* $g : \mathbb{R} \to \mathbb{R}$ *and* $f : \mathbb{R} \to S$ *are non-constant functions and* g *is continuous. If* (f, g) *is a solution of* (1)*, then there exists* $c \in \mathbb{R} \setminus \{0\}$ *such that*

$$g(x) = cx + 1 \quad \text{for } x \in \mathbb{R} \tag{7}$$

or

$$g(x) = \max\{cx + 1, 0\} \quad \text{for } x \in \mathbb{R}. \tag{8}$$

Proof. Assume that (f, g) is a solution of (1). Let for every $x \in \mathbb{R}$, a function $g_x : \mathbb{R} \to \mathbb{R}$ be given by $g_x(y) = x + g(x)y$ for $y \in \mathbb{R}$. At first we show that a set $G := g^{-1}(\{0\})$ is strongly invariant with respect to a family of functions $\{g_x \mid x \in \mathbb{R} \setminus G\}$, i.e. that for every $x \in \mathbb{R} \setminus G$

$$g_x(y) \in G \quad \text{if and only if } y \in G. \tag{9}$$

Let $x \in \mathbb{R} \setminus G$. If $y \in G$, then according to Lemma 1, $f(y) \in Z_L(f(\mathbb{R}))$. Thus, by (1), we get

$$\begin{aligned} f(g_x(y)) \circ f(z) &= f(x + g(x)y) \circ f(z) = f(x) \circ f(y) \circ f(z) \\ &= f(x) \circ f(y) = f(x + g(x)y) = f(g_x(y)) \quad \text{for } z \in \mathbb{R}. \end{aligned}$$

Hence $f(g_x(y)) \in Z_L(f(\mathbb{R}))$, so in view of Lemma 1, $g_x(y) \in G$.

Now, suppose that $g_x(y) \in G$ for some $y \notin G$. Then, applying Lemma 1 and using (1), we obtain

$$\begin{aligned} Z_L(f(\mathbb{R})) \ni f(g_x(y)) &= f(g_x(y)) \circ f\Big(-\frac{y}{g(y)}\Big) = f(x + g(x)y) \circ f\Big(-\frac{y}{g(y)}\Big) \\ &= f(x) \circ f(y) \circ f\Big(-\frac{y}{g(y)}\Big) = f(x) \circ f(0) = f(x). \end{aligned}$$

Thus $x \in f^{-1}(Z_L(f(\mathbb{R})))$, so according to Lemma 1, $x \in G$. This is a contradiction. Therefore we have proved (9).

Now, let $G^- := (-\infty, 0] \cap G$ and $G^+ := [0, \infty) \cap G$. In view of Lemma 1, at least one of the sets G^- and G^+ is non-empty. Suppose that both of them are non-empty. As they are closed, there exist $z_1 := \max G^-$ and $z_2 := \min G^+$. Obviously $z_1 < 0 < z_2$ and

$$(z_1, z_2) \cap G = \emptyset. \tag{10}$$

Since g is continuous and $g(z_2) = 0$, there is an $x \in (0, z_2)$ such that $g(x) \in \big(\frac{z_1}{z_2}, 1\big)$. If $g(x) \in \big(\frac{z_1}{z_2}, 0\big)$, then $z_1 < x + z_1 < x + g(x)z_2 < x < z_2$, so $g_x(z_2) \in (z_1, z_2)$. On the other hand, by (9), $g_x(z_2) \in G$. This contradicts (10). If $g(x) \in (0, 1)$, then

$z_1 < x + z_1 < x + g(x)z_1 < x$, whence $g_x(z_1) \in (z_1, x) \subset (z_1, z_2)$, which again contradicts (10). Consequently, exactly one of the sets G^- and G^+ is non-empty. Since the proof in both cases is analogous, assume that $G^- \neq \emptyset$ and $G^+ = \emptyset$. Then

$$z := \max G < 0. \tag{11}$$

Fix an $x \in (z, \infty)$. By (11), $g(x) \neq 0$. If $g(x) < 0$, then taking into account (9) and (11), we obtain $z < x < x + g(x)z = g_x(z) \leq z$, which brings a contradiction. Therefore $g(x) > 0$ and so g_x is strictly increasing. Moreover, in view of (9) and (11), we get that $g_x(z) \leq z$ and ${g_x}^{-1}(z) \leq z$. Hence $g_x(z) = z$, which implies that $g(x) = 1 - \frac{x}{z}$. In this way we have proved that

$$g(x) = 1 - \frac{x}{z} \quad \text{for } x \in [z, \infty). \tag{12}$$

Now, suppose that $g(x) \neq 0$ for some $x < z$. If $g(x) > 0$, then according to (9), $G \ni {g_x}^{-1}(z) = \frac{z-x}{g(x)} > 0$, which contradicts (11). Consequently $g(x) < 0$, g_x is strictly decreasing and, in virtue of (9) and (11), we obtain

$$(-\infty, g_x(z)) \cap G = g_x((z, \infty)) \cap G = \emptyset.$$

Thus $g_x(z) \leq z$ and $G \subset [g_x(z), z]$. As G is closed, this means that there exists $z_0 := \min G$. Since $-z \in (z, \infty) \subset \mathbb{R} \setminus G$, by (9) and (12), we get

$$G \ni g_{(-z)}(z_0) = -z + g(-z)z_0 = -z + \left(1 - \frac{-z}{z}\right)z_0 = -z + 2z_0.$$

Hence $z_0 \leq -z + 2z_0$, so $z \leq z_0$. Thus $z_0 = z$ and $G = \{z\}$. Therefore, we have proved that either $G = (-\infty, z]$ or $G = \{z\}$.

If the first possibility occurs, then using (12), we obtain that g has the form (8) with $c := -\frac{1}{z} \neq 0$. If the second one is valid, then according to (9), we get that

$$g_x(z) = z \quad \text{for } x \in \mathbb{R} \setminus G = \mathbb{R} \setminus \{z\}.$$

Hence

$$g(x) = 1 - \frac{x}{z} \quad \text{for } x \in \mathbb{R} \setminus \{z\}.$$

Since $g(z) = 0$, this means that g has the form (7) with $c := -\frac{1}{z} \neq 0$. This completes the proof. □

The next theorem is the main result of the paper.

Theorem 1. *Assume that $(S, \circ)$ is a semigroup, $g : \mathbb{R} \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to S$. Then (f, g) is a solution of (1) if and only if one of the following conditions holds:*

(i) *there is $s \in E(S)$ such that $f \equiv s$;*
(ii) *$g \equiv 0$ and there is a subsemigroup S_0 of S such that $u \circ v = u$ for $u, v \in S_0$ and $f(\mathbb{R}) \subset S_0$;*
(iii) *$g \equiv 1$ and f is a homomorphism of an additive group of real numbers into S;*

(iv) *there exist a non-trivial homomorphism ϕ of a multiplicative semigroup of real numbers into S and $c \in \mathbb{R} \setminus \{0\}$ such that*

$$g(x) = cx + 1 \quad \text{for } x \in \mathbb{R},$$
$$f(x) = \phi(cx + 1) \quad \text{for } x \in S;$$

(v) *there exist a non-trivial homomorphism ψ of a multiplicative semigroup of non-negative real numbers into S, $c \in \mathbb{R} \setminus \{0\}$ and $l \in Z_L(\psi([0, \infty)))$ such that*

$$g(x) = \max\{cx + 1, 0\} \quad \text{for } x \in \mathbb{R},$$
$$f(x) = \begin{cases} \psi(cx + 1) & \text{for } x \in D_c^+ \\ \psi(-(cx + 1)) \circ l & \text{for } x \in D_c^-, \end{cases}$$

where $D_c^+ := \{x \in \mathbb{R} \mid cx + 1 \geq 0\}$ and $D_c^- := \mathbb{R} \setminus D_c^+$.

Proof. Assume that (f, g) is a solution of (1). If f is constant, then taking into account Remark 1, we get (i). Now, assume that f is non-constant and g is constant. If $g \equiv 0$, then (ii) holds with $S_0 := f(\mathbb{R})$. In the case, where $g \equiv 1$, we get (iii). Suppose that $g \equiv a \notin \{0, 1\}$. Then, by (1)

$$f(x + ay) = f(x) \circ f(y) \quad \text{for } x, y \in \mathbb{R},$$

whence

$$f(ay) = f(0) \circ f(y) \quad \text{for } y \in \mathbb{R}.$$

Thus

$$\begin{aligned} f(ax + ay) &= f(ax) \circ f(y) = f(0) \circ f(x) \circ f(y) = \\ &= f(0) \circ f(x + ay) = f(ax + a^2 y) \quad \text{for } x, y \in \mathbb{R}, \end{aligned}$$

so for every $x, y \in \mathbb{R}$, we have

$$f(x) = f\Big(a\frac{ax - y}{a^2 - a} + a\frac{y - x}{a^2 - a}\Big) = f\Big(a\frac{ax - y}{a^2 - a} + a^2\frac{y - x}{a^2 - a}\Big) = f(y).$$

Hence f is constant, which yields a contradiction.

Therefore it remains to consider the case where f and g are non-constant. Then according to Lemma 2, g is either of the form (7) or (8). If the first possibility occurs, then we define a function $\phi : \mathbb{R} \to S$ by $\phi(x) = f\big(\frac{x-1}{c}\big)$ for $x \in \mathbb{R}$. Note that in view of (1) and (7), we have

$$\begin{aligned} \phi(x) \circ \phi(y) &= f\Big(\frac{x - 1}{c}\Big) \circ f\Big(\frac{y - 1}{c}\Big) = f\Big(\frac{x - 1}{c} + g\Big(\frac{x - 1}{c}\Big)\frac{y - 1}{c}\Big) = \\ &= f\Big(\frac{x - 1}{c} + \Big(c\frac{x - 1}{c} + 1\Big)\frac{y - 1}{c}\Big) = f\Big(\frac{xy - 1}{c}\Big) \\ &= \phi(xy) \quad \text{for } x, y \in \mathbb{R}. \end{aligned}$$

As f is non-constant, this means that ϕ is a non-trivial homomorphism of a multiplicative semigroup of real numbers into S. Furthermore, $f(x) = \phi(cx + 1)$ for $x \in \mathbb{R}$, so (iv) holds.

Now, assume that g is of the form (8). Let $\psi : [0, \infty) \to S$ be given by $\psi(x) = f\left(\frac{x-1}{c}\right)$ for $x \in [0, \infty)$. Then, arguing analogously as in the previous case, we get that ψ is a non-trivial homomorphism of a multiplicative semigroup of non-negative real numbers into S and

$$f(x) = \psi(cx + 1) \quad \text{for } x \in D_c^+. \tag{13}$$

Moreover, since $-\frac{2}{c} - x \in D_c^+$ for $x \in D_c^-$, then taking into account (1), (8) and (13), we get

$$\begin{aligned} f(x) &= f\Big(-\frac{2}{c} - x + g\Big(-\frac{2}{c} - x\Big)\Big(-\frac{2}{c}\Big)\Big) = \\ &= f\Big(-\frac{2}{c} - x\Big) \circ f\Big(-\frac{2}{c}\Big) = \psi(-(cx+1)) \circ f\Big(-\frac{2}{c}\Big) \quad \text{for } x \in D_c^-. \end{aligned}$$

Furthermore, making sequentially use of (13), (1) and (8), for every $x \in D_c^+$, we obtain

$$f\Big(-\frac{2}{c}\Big) \circ \psi(cx+1) = f\Big(-\frac{2}{c}\Big) \circ f(x) = f\Big(-\frac{2}{c} + g\Big(-\frac{2}{c}\Big)x\Big) = f\Big(-\frac{2}{c}\Big).$$

Hence $f\left(-\frac{2}{c}\right) \in Z_L(\psi([0, \infty)))$. Consequently (v) is valid with $l := f\left(-\frac{2}{c}\right)$.

Since the converse is easy to check, the proof is completed. □

Corollary 1. *Assume that $(S, \circ)$ is a comutative semigroup, $g : \mathbb{R} \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to S$. Then (f, g) is a solution of* (1) *if and only if either one of the conditions* (i), (iii) *and* (iv) *of Theorem* 1 *holds; or*

(v′) *there exist a homomorphism ψ of a multiplicative semigroup of non-negative real numbers into S, $c \in \mathbb{R} \setminus \{0\}$ and $l \in Z(\psi([0, \infty)))$ such that*

$$g(x) = \max\{cx + 1, 0\} \quad \textit{for } x \in \mathbb{R},$$

$$f(x) = \begin{cases} \psi(cx+1) & \textit{for } x \in D_c^+ \\ l & \textit{for } x \in D_c^-. \end{cases}$$

Proof. Since S is commutative, every subsemigroup S_0 of S such that $u \circ v = u$ for $u, v \in S_0$, has a form $S_0 = \{s\}$ with some $s \in E(S)$. Furthermore, for every $\emptyset \neq A \subset S$, $Z_L(A) = Z(A)$. Therefore, applying Theorem 1, we get the assertion. □

Corollary 2. *Assume that $(S, \circ)$ is a group, $g : \mathbb{R} \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to S$. Then (f, g) is a solution of* (1) *if and only if either $f \equiv e$, where e is a neutral element of the group $(S, \circ)$; or $g \equiv 1$ and f is a homomorphism of an additive group of real numbers into S.*

Proof. Assume that (f, g) is a solution of (1). Then, according to Theorem 1, one of the conditions (i)–(v) is valid. Since S is a group, we have $E(S) = \{e\}$. Moreover, the only subsemigroup S_0 of S such that $u \circ v = u$ for $u, v \in S_0$, is $S_0 = \{e\}$. Thus each of the conditions (i) and (ii) implies that $f \equiv e$. Furthermore, as every homomorphism of a multiplicative semigroup of reals (non-negative reals, resp.) into a group is trivial, neither (iv) nor (v) occur. So the proof is completed. □

The next proposition generalizes some results from [8] and [10].

Proposition 1. *Assume that* $\circ$ *is an associative binary operation on* $\mathbb{R}$ *and* k *is a positive integer. Then* $f : \mathbb{R} \to \mathbb{R}$ *is a non-constant continuous solution of the equation*

$$f(x + f(x)^k y) = f(x) \circ f(y) \quad \text{for } x, y \in \mathbb{R} \tag{14}$$

if and only if one of the following three conditions holds:

(a) k *is even,* $u \circ v = uv$ *for* $u, v \in [0, \infty)$ *and there exists* $c \in \mathbb{R} \setminus \{0\}$ *such that*

$$f(x) = (\max\{cx + 1, 0\})^{\frac{1}{k}} \quad \text{for } x \in \mathbb{R}; \tag{15}$$

(b) k *is odd,* $u \circ v = uv$ *for* $u, v \in \mathbb{R}$ *and there exists* $c \in \mathbb{R} \setminus \{0\}$ *such that* f *has a form* (15) *or*

$$f(x) = (cx + 1)^{\frac{1}{k}} \quad \text{for } x \in \mathbb{R}; \tag{16}$$

(c) k *is even,* $u \circ v = -uv$ *for* $u, v \in (-\infty, 0]$ *and there exists* $c \in \mathbb{R} \setminus \{0\}$ *such that*

$$f(x) = -(\max\{cx + 1, 0\})^{\frac{1}{k}} \quad \text{for } x \in \mathbb{R}. \tag{17}$$

Proof. It is easy to check that each of conditions (a)–(c) implies (14). So, assume that f is a non-constant continuous solution of (14). Then (f, g), where $g = (f)^k$, is a solution of (1). Hence, according to Lemma 2, there exists $c \in \mathbb{R} \setminus \{0\}$ such that either $f(x)^k = cx + 1$ for $x \in \mathbb{R}$; or $f(x)^k = \max\{cx + 1, 0\}$ for $x \in \mathbb{R}$. Assume that the first possibility holds. Then k is odd and f has the form (16). Moreover, for every $u, v \in \mathbb{R}$ there exist $x_1, x_2 \in \mathbb{R}$ such that $u = f(x_1) = (cx_1 + 1)^{\frac{1}{k}}$ and $v = f(x_2) = (cx_2 + 1)^{\frac{1}{k}}$. Thus, by (14), we get

$$\begin{aligned} u \circ v &= f(x_1) \circ f(x_2) = f(x_1 + f(x_1)^k x_2) = f(x_1 + x_2 + cx_1 x_2) \\ &= (cx_1 + 1)^{\frac{1}{k}} (cx_2 + 1)^{\frac{1}{k}} = uv. \end{aligned}$$

If the second possibility is valid, then either f has the form (15), or k is even and f is of the form (17). Furthermore, arguing as previously, we obtain that $u \circ v = uv$ for $u, v \in [0, \infty)$ in the first case, and $u \circ v = -uv$ for $u, v \in (-\infty, 0]$ in the second one. This completes the proof. □

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