THE FRATTINI SUBLATTICE OF A DISTRIBUTIVE LATTICE

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1. Introduction

The Frattini sublattice of a lattice L is defined to be the intersection of all its maximal proper sublattices and will be denoted $\Phi(L)$. Koh [6] considers the problem of characterisations of lattices L such that $\Phi(L)=\emptyset$ and proves that for every lattice L there exists a lattice L_1 such that $L\cong\Phi(L_1)$. In this paper we consider the Frattini sublattice of a distributive lattice in the category of distributive lattices. Our approach is from a topological point of view.

We prove that for any distributive lattice L there exists a distributive lattice L_1 such that $L\cong\Phi(L_1)$. Further consideration of the method employed enables us to answer questions that developed from work by Koh [6].

The topology used is that introduced by H. A. Priestley [7]. We restate the Representation Theorem.

DEFINITION. A topological space S with a partial order $(\leq^*)$ defined on it is said to be totally order disconnected provided that for $x, y\in S$ $x\not\geq^* y$ there exists disjoint clopen sets X, Y such that $x\in X$, $y\in Y$, X is decreasing and Y is increasing.

REPRESENTATION THEOREM. *Every* (0, 1) *distributive lattice L has a topological dual space L′ which is a compact totally order disconnected space. The elements of L are isomorphic with the clopen decreasing sets in L′. Conversely every compact totally order disconnected space is homeomorphic and order isomorphic with the dual space of some* (0, 1) *distributive lattice.*

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2. The topology

Any (0, 1) distributive lattice L has a topological representation as a compact totally order disconnected space L', the elements of L being isomorphic with the clopen decreasing sets in L'. Thus we may think of L as being the collection of clopen decreasing sets in L', denoted $\mathbf{L}$, and any proper sublattice L_1 of L as a proper subset $\mathbf{L}_1\subset\mathbf{L}$ (sublattices are by definition allowed to be void). We begin by showing that for any sublattice L_1 of L a separating set S can be uniquely defined over the topo-

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logical space L', after which we consider how S behaves for a maximal sublattice.

We remark that the essence of Lemmas 2 and 3 is contained in Section 9 of J. Hashimoto [5]. However our presentation in terms of the Priestley topology is quite different. Because of this and for completeness we give proofs.

We denote topological duals with a prime symbol.

DEFINITION 1. (a) For a (0, 1) distributive lattice L with a sublattice L_1 we may associate three sets S_i $i \in \{\alpha, \beta, \gamma\}$, viz.

$S_\alpha = \{x \mid x = \langle a \rangle$ for $a \in L'$ and $a \in A$ for all $A \in \mathbf{L}_1\}$

$S_\beta = \{x \mid x = \langle\langle a \rangle\rangle$ for $a \in L'$ and $a \notin A$ for any $A \in \mathbf{L}_1\}$

$S_\gamma = \{x \mid x = \langle a, b \rangle$ where $a, b \in L'$, $a \ngeq^* b$ and for any $A \in \mathbf{L}_1$ $(a \in A \rightarrow b \in A)\}$.

Let the *separating set* be $S = S_\alpha \cup S_\beta \cup S_\gamma$.

(b) A subset X of L' is said to be *compatible* with a separating set S providing (i) $\langle a \rangle \in S$ implies $a \in x$, (ii) $\langle\langle a \rangle\rangle \in S$ implies $a \notin X$ and (iii) $\langle a, b \rangle \in S$ and $a \in X$ implies $b \in X$.

LEMMA 2. *Let L be a* (0, 1) *distributive lattice having a sublattice L_1 with separating set S. Then for a clopen decreasing set $A \subseteq L'$ we have $A \in L_1$ iff A is compatible with S. Thus for sublattices L_1, L_2 with separating sets S_1, S_2 respectively, $L_1 \neq L_2$ implies $S_1 \neq S_2$.*

Proof. Given a clopen decreasing set A that is not compatible with S then by definition $A \notin \mathbf{L}_1$.

Conversely suppose A is compatible with S. When there exist $x, y \in L'$ such that $x \in A$ and $y \in L' - A$ we must have $x \ngeq^* y$. But then by compatibility $x \ngeq^* y$ implies the existence of $B_x \in \mathbf{L}_1$ such that $x \in B_x$ and $y \notin B_x$. For a given y we can form an open cover $\{B_x \mid x \in A\}$ for A, which by compactness has a finite subcover $B_{x_1}, \dots, B_{x_n}$. Set $B_y = \bigcup_{1 \leq i \leq n} B_{x_i}$. Then $B_y \supseteq A$, $y \notin B_y$ and $B_y \in \mathbf{L}_1$ since $\mathbf{L}_1$ represents a sublattice. But $\bigcap_{y \in L' - A} B_y = A$. Hence by compactness again we have a finite subset such that $\bigcap_{1 \leq i \leq m} B_{y_i} = A$. Since L_1 is a lattice $A \in \mathbf{L}_1$. If A is empty then just choose B_y such that $B_y \in \mathbf{L}_1$ and $y \notin B_y$, or if $A = L'$ then simply cover A with $B_x \in \mathbf{L}_1$ such that $x \in B_x$; and then treat as before.

LEMMA 3. *If a* (0, 1) *distributive lattice L has a maximal proper sublattice L_1 with separating set S, then S consists of one element of the form*

(i) *$\langle a \rangle$ where $a \in L'$ is a fixed element and $a \leq^* x$ for all $x \in L'$*

(ii) *$\langle\langle a \rangle\rangle$ where $a \in L'$ is a fixed element and $a \geq^* x$ for all $x \in L'$*

or

(iii) *$\langle a, b \rangle$ a fixed pair $a, b \in L'$, $a \ngeq^* b$ such that $(a_1 >^* a \rightarrow a_1 \geq^* b)$ and $(b_1 <^* b \rightarrow b_1 \leq^* a)$.*

Conversely a set containing an element of the form (i), (ii) *or* (iii) *is the separating set of a maximal proper sublattice.*

Proof. First suppose that L_1 is a maximal proper sublattice so that S is not empty. Then there are three cases to consider $(\alpha)\exists\langle a\rangle\in S$, $(\beta)\exists\langle\!\langle a\rangle\!\rangle\in S$ and $(\gamma)\exists\langle a, b\rangle\in S$.

(α) If $\langle a\rangle\in S$ and $a\nleq^* x$ for some $x\in L'$, then $\langle x, a\rangle\in S$ since $a\in A$ for all $A\in\mathbf{L}_1$. But since $x\ngeq^* a$ there is a clopen decreasing set C such that $x\in C$ and $a\notin C$; $C\notin\mathbf{L}_1$. Consider the sublattice of L given by $\{\emptyset\}\cup\mathbf{L}_1$. This is a proper extension of L_1 since by hypothesis $\emptyset\notin\mathbf{L}_1$. However $C\notin\{\emptyset\}\cup\mathbf{L}_1$ contradicting the maximality of L_1.

Thus $\langle a\rangle\in S$ implies a is minimum $(\geq^*)$ and hence $\langle x\rangle\in S$ implies $x=a$.

Suppose in additon to $\langle a\rangle$ there are points $\langle\!\langle x\rangle\!\rangle$ or $\langle x, y\rangle\in S$. Then the sublattice $\{\emptyset\}\cup\mathbf{L}_1$ is again a proper extension of $\mathbf{L}_1$ that omits L' in the case $\langle\!\langle x\rangle\!\rangle\in S$, or a clopen decreasing set C with $x\in C$ and $y\notin C$ in the case $\langle x, y\rangle\in S$. Either of these cases contradicts the maximality of $\mathbf{L}_1$.

Hence $\langle a\rangle\in S$ implies a is minimum $(\leq^*)$ and $\bar{\bar{S}}=1$.

(β) If $\langle\!\langle a\rangle\!\rangle\in S$ we can use similar arguments to those in case (α) to show a is a maximum $(\leq^*)$ and $\bar{\bar{S}}=1$.

(γ) If there is a pair $\langle a, b\rangle\in S$ then by (α) and (β) the only other elements of S are of the form $\langle x, y\rangle$.

We begin by showing that if $\bar{\bar{S}}=1$ then the conditions in (iii) are satisfied. Suppose there exists a different ordered pair $\langle a_1, b_1\rangle$ such that $a_1\ngeq^* b_1$, $a_1\geq^* a$ and $b_1\leq^* b$. Then a clopen decreasing set C with $a_1\in C$ and $b_1\notin C$ certainly exists. However for any such C we have $a_1\in C$ implies $a\in C$, and $b_1\notin C$ implies $b\notin C$. Thus C is incompatible with $\langle a, b\rangle$ and is not a member of $\mathbf{L}_1$. Thus $\langle a_1, b_1\rangle$ is compatible with every $C\in\mathbf{L}_1$. We deduce the following statement:

(A) $((\langle a, b\rangle\in S\wedge a_1\ngeq^* b_1\wedge a_1\geq^* a\wedge b_1\leq^* b)\rightarrow\langle a_1, b_1\rangle\in S$.

Hence we have that if $S=\{\langle a, b\rangle\}$ for a given pair $a, b\in L'$, then $a_1>^* a$ implies $a_1\geq^* b$, and $b_1<^* b$ implies $b_1\leq^* a$.

If S contains at least two different pairs $\langle a, b\rangle$, $\langle a_1, b_1\rangle$ then for one pair, say $\langle a_1, b_1\rangle$, we must have either $a\ngeq^* a_1$ or $b\nleq^* b_1$. Suppose $a\ngeq^* a_1$. Then there is a set $C_1\in\mathbf{L}$ such that $a\in C_1$ and $a_1\notin C_1$. There is also a $C_2\in\mathbf{L}$ such that $a\in C_2$ and $b\notin C_2$. Setting $C=C_1\cap C_2$ we see that $a\in C$ while $a_1, b\notin C$. Thus we consider the lattice $\mathbf{L}_2$ generated by C and $\mathbf{L}_1$ and see that $\langle a, b\rangle$ is not an element of its separating set. However $(a_1\in X\rightarrow b_1\in X)$ is preserved under finite joins and meets of members of $\{C\}\cup\mathbf{L}_1$. Thus $\langle a_1, b_1\rangle$ is a member of the separating set of L_2 contradicting the maximality of L_1. Similarly for $b\nleq^* b_1$.

This shows that for a maximal proper sublattice S is of the form described. Next suppose a separating set S consists of one element of the form (i), (ii) or (iii).

If the set consists of a single pair $\langle a, b\rangle$ that satisfies condition (iii) then to prove the Lemma we must show that for any $C_1\in\mathbf{L}$ that is incompatible with $\langle a, b\rangle$ together with all $C\in\mathbf{L}$ that are compatible with $\langle a, b\rangle$ we can generate $\mathbf{L}$.

Let X be the member of $\mathbf{L}$ we elect to generate. For $x\in X$ and $y\in L'-X$ we must have $x\ngeq^* y$ and by condition (A) if $\langle x, y\rangle\neq\langle a, b\rangle$ then either $x\ngeq^* a$ or $y\nleq^* b$. Thus by the argument given above there is a $C \in \mathbf{L}$ such that $x \in C$, $y \notin C$ but $a \in C$ implies $b \in C$. If $\langle x, y\rangle = \langle a, b\rangle$ then we take C_1. Hence for a fixed y we may cover X with allowable elements none of which contain y. By compactness this has a finite subcover the union of which, say X_y, is suitable generated. Now as y varies we have $\bigcap_{y \in L' - X} X_y = X$. Choose a finite subcover by compactness. Thus X is finitely generated by C_1 and allowable members of $\mathbf{L}$. If X is $\emptyset$ or L' then it is automatically compatible with $\langle a, b\rangle$.

The only remaining cases are when the given set consists of a minimum $(\leq^*)$ element $\langle a\rangle$, or a maximum $(\leq^*)$ element $\langle\langle a\rangle\rangle$. These automatically correspond to proper maximal sublattices that hold all of L except a meet irreducible zero or a join irreducible unit respectively. This completes the proof of Lemma 3.

We point out that although Lemmas 2 and 3 are in terms of (0, 1) distributive lattices this restriction is not essential. For example if the distributive lattice lacks a zero then we add one in order to obtain a topological representation. This is equivalent to a minimum $(\leq^*)$ point m in the space, so that the elements of the lattice are clopen decreasing sets that contain m. Thus separating sets for sublattices in Lemma 2 all contain $\langle m\rangle$, and in Lemma 3 we adapt by considering only cases (ii) and (iii). The case when L consists of a single point must be dealt with separately. However it presents no real problem.

3. The Frattini sublattice

The Frattini sublattice of a lattice L is the intersection of all its proper maximal sublattices and is denoted by $\Phi(L)$. If we let $\Phi^2(L) = \Phi(\Phi(L))$ we can inductively define $\Phi^{\alpha+1}(L) = \Phi(\Phi^{\alpha}(L))$ for successor ordinals $\alpha+1$ and $\Phi^{\alpha}(L) = \bigcap_{\beta<\alpha} \Phi^{\beta}(L)$ for limit ordinals. This means that for each lattice L we have a sequence:

$$L \supseteq \Phi(L) \supseteq \Phi^2(L) \cdots \supseteq \Phi^{\alpha}(L) \supseteq \cdots$$

In [6] Koh asks whether there is always an ordinal α such that $\Phi^{\alpha}(L) = \emptyset$. This is equivalent to showing that there exists a lattice with no maximal proper subalgebras, which with the aid of Lemma 3 we are now in a position to do.

We remark that Birkhoff [2] conjectured that every sublattice of a distributive lattice could be extended to a maximal proper sublattice. This problem was solved in 1951 by Takeuchi [9] with a counter-example, namely the sublattice $\{(x, 0) \mid x \in \omega^*\}$ of the lattice $\omega^* \times 2$. Hashimoto [5] points out that ω^* can be replaced by any distributive lattice L that has no proper maximal filter.

THEOREM 4. *There is a distributive lattice with no maximal proper sublattices.*

Proof. We consider the chain $\omega+1$ with its interval topology. This is a compact totally disconnected space C on which we will impose a $(\leq^*)$ relation in the following manner.

For a proper initial segment $I=\{0,\ldots,n\}$ of $\omega+1$ with a $(\leq^*)$ relation defined on it we let $p_1,\ldots,p_r$ be a list of pairs $\langle a,b\rangle$ $a,b\in I$ such that $a\not\geq^* b$. If $\langle a,b\rangle=p_j$ then set $x<^* n+j$ for $x\leq^* a$, $x\in I$. This gives us a new initial segment $R(I)=\{0,\ldots,n,n+1,\ldots,n+r\}$ with an extended $(\leq^*)$ relation defined on it.

Let $I_0=\{0,1\}$. I_0 is trivially ordered under $(\leq^*)$.

$$I_{n+1}=R(I_n) \quad \text{for} \quad 0\leq n<\omega.$$

Since $x\leq^* y$ $x,y\in I_n$ implies $x\leq^* y$ in I_m $m\geq n$ we have a well defined $(\leq^*)$ relation on ω. We let this, together with $x<^*\omega$ for all $x<\omega$, be the $(\leq^*)$ relation on C.

We must show that $(\leq^*)$ is a partial order and is totally order disconnected.

Since $(\leq^*)\subseteq(\leq)$ we have $(\leq^*)$ is reflexive and anti-symmetric. To see that it is transitive consider $x<^*y$ and $y<^*z$. We take an induction on n and let $n<\omega$ be the induction stage such that $z\in I_n$. At this stage we must have $x,y\in I_{n-1}$ and a pair $\langle a,b\rangle$ for which we set $z>^*k$ all $k\leq^* a$. But $y\leq^* a$ implies $x\leq^* a$ by transitivity in I_{n-1} and thus $x<^*z$. If $z=\omega$ then $x\leq^* z$ all x by definition.

To show that $C(\leq^*)$ is totally order disconnected we have two cases. First $m<^*n$, in which case $\{x\mid x\leq m\}$ will serve. Second $m)(^*n$, which implies $m,n<\omega$). Say m is to be a member of a clopen increasing set. Then $\{x\mid x\geq^* m \text{ or } x\geq n+1\}$ will do.

If $C(\leq^*)$ represents the lattice L we take $L-\{1\}$ as our example. For suppose $L-\{1\}$ has a maximal sublattice. Our construction together with Lemma 3 implies that its separating set must be a pair $\langle a,b\rangle$ of type (iii), $a,b<\omega$. Choose I_n such that $a,b\in I_n$. Since $a\not\geq^* b$ there is $c\in I_{n+1}$, $c>^*a$ and $c\not\geq^* b$ which contradicts maximality.

We now prove that every distributive lattice can be obtained as the Frattini sublattice of some other distributive lattice.

THEOREM 5. *For a distributive lattice L there is a distributive lattice L_1 such that $L=\Phi(L_1)$.*

Proof. Suppose to begin with that L is a (0, 1) distributive lattice. We build on the topological space L' a new topology that is a mixture of product topology and compactification. This new space (with slight modification in certain stated cases) will be the topological interpretation of an L_1 satisfying the Theorem.

Let $X=\{\langle n,p\rangle \mid 0\leq n<\omega \text{ and } p\in L'\}\cup\{m\}$. m here may be any symbol, not necessarily an integer.

We define a topology τ on X to have an open base given by sets of the following forms:

(i) $\{n\}\times Y$ for given $n<\omega$ and $Y\subseteq L'$, Y open;

(ii) $\{m\}\cup\bigcup_n(\{n\}\times L')$ for all n such that $N\leq n<\omega$ for some given N.

We claim that X together with the topology τ is compact. This is because for a given open cover X_i of X one set must have m as an element. If this open set is K where

$K=\{m\}\cup\bigcup_n(\{n\}\times L')$ for $n\geq N$ we have $X-K=\bigcup_n(\{n\}\times L')$ $n<N$. But $(X-K)\cap\cap X_i$ is an open cover of $X-K$ with the product topology. This yields a finite subcover $X_{i_1},\ldots,X_{i_r}$, since the product topology of two compact spaces is compact. Thus $X_{i_1},\ldots,X_{i_r}, K$ is a finite subcover for X. Hence (X,τ) is compact.

We now impose a $(\leq^*)$ relation on (X,τ) by setting

(α) $m>^*x$, $x\in X-\{m\}$.

(β) $\langle 0,p_1\rangle<^*\langle 0,p_2\rangle$ if $p_1<^*p_2$ in L'.

(γ) $\langle 0,p\rangle<^*\langle n,p\rangle$ for $0<n<\omega$.

(δ) If $\langle 0,p_1\rangle<^*\langle 0,p_2\rangle$ set $\langle 0,p_1\rangle<^*\langle n,p_2\rangle$.

Using (α)... (δ) we show $(\leq^*)$ is a partial ordering. The reflexivity is trivial. To see antisymmetry suppose $x, y\in X$, $x<^*y$ and $y<^*x$. Then since $m\nless^*x$ any $x\in X$ we may suppose $x=\langle n_1,p_1\rangle$ and $y=\langle n_2,p_2\rangle$. But $x<^*y$ implies $n_1=0$. Similarly $y<^*x$ implies $n_2=0$, a contradiction, since this reduces to case (β) which means $p_1<^*p_2$ and $p_2<^*p_1$ in L'. Hence $(\leq^*)$ is antisymmetric. Transitivity is secured by condition (δ) and is also straightforward to verify since $0<n_1, n_2<\omega$ implies $\langle n_1,p_2\rangle$ and $\langle n_2,p_2\rangle$ are $(\leq^*)$ incomparable.

In order to show $(X,\bar{\tau}\leq^*)$ is the Stone space of a distributive lattice we must show that it is totally order disconnected. That is for d_1, d_2, $d_1\nleq^*d_2$ we must find a clopen increasing set D such that $d_1\in D$ and $d_2\notin D$. Since $d_1\nleq^*d_2$ we must have $d_2=\langle n_2, p_2\rangle$ for some n_2, p_2. We consider the various cases d_1 takes.

(i) $d_1=m$.

Then $D=\bigcup_n(\{n\}\times L')\cup\{m\}$, $n_2<n<\omega$ will serve.

(ii) $d_1=\langle n_1,p_1\rangle$, $n_1>0$.

(a) $n_1\neq n_2$. Set $D=\bigcup_n(\{n\}\times L')\cup\{m\}$ where $n=n_1$ or $n_2<n<\omega$.

(b) $n_1=n_2$ and $p_1\neq p_2$. Choose a clopen set C in L' such that $p_1\in C$ and $p_2\notin C$. Then set $D=(\{n_1\}\times C)\cup\{m\}\cup\bigcup_n(\{n\}\times L')$ where $n_2<n<\omega$.

(iii) $d_1=\langle n_1,p_1\rangle$, $n_1=0$.

By definition $p_1\nleq^*p_2$. Hence there is a clopen increasing set C in L', $p_1\in C$ and $p_2\notin C$. Let $D=\bigcup_n(\{n\}\times C)\cup\bigcup_k(\{k\}\times L')\cup\{m\}$ where $n_2<k<\omega$, $0\leq n<\omega$.

We conclude that $(X,\tau,\leq^*)$ is a compact totally order disconnected space and proceed to investigate its Frattini sublattice.

We will show that a non empty clopen decreasing subset C of X is a member of every maximal sublattice iff it is contained in $\{0\}\times L'$. There are three cases to consider.

(i) $C=X$.

By definition $x\in X$ implies $x\leq^*m$. Hence $\{\langle\!\langle m\rangle\!\rangle\}$ represents a separating set of a maximal sublattice that does not contain X. Thus $X\notin\Phi((X,\tau,\leq^*))$.

(ii) $C\nsubseteq\{0\}\times L'$ and $C\neq X$.

Since $C\nsubseteq\{0\}\times L'$ we must have $\langle n_1,p_1\rangle\in C$ for some $n_1>0$. However $C\neq X$ implies that $m\notin C$. Consequently C clopen implies $\langle n_2,p_1\rangle\notin C$ for some $n_2>n_1$. But this means $\langle n_1, p_1\rangle \not\geq^* \langle n_2, p_1\rangle$. Suppose $x >^* \langle n_1, p_1\rangle$ for some $x \in X$. Then by definition $x = m$ and it follows that $x >^* \langle n_2, p_1\rangle$. Alternatively suppose $x <^* \langle n_2, p_1\rangle$. Then by definition $x \leq^* \langle 0, p_1\rangle$ and it follows that $x <^* \langle n_1, p_1\rangle$. Thus the pair $\langle\langle n_1, p_1\rangle, \langle n_2, p_2\rangle\rangle$ satisfies condition (iii) of Lemma 3 and is the separating set of a maximal sublattice that does not contain C.

(iii) $C \subseteq \{0\} \times L'$.

Since $C \neq \emptyset$ by Lemma 3 any maximal sublattice not containing C must be of the type (iii) with separating set $\{\langle a, b\rangle\}$. Suppose $\langle a, b\rangle$ is such an ordered pair. Then with C not a member of this maximal sublattice we have $a \in C$ and $b \notin C$. Thus a is of the form $\langle 0, p\rangle$ for some $p \in L'$. We show that for $b \notin C$ it is not possible for $\langle a, b\rangle$ to satisfy condition (iii) in Lemma 3 and thus $C \in \Phi((X, \tau, \leq^*))$. Firstly we cannot have $b = m$ because $\langle 1, p\rangle >^* \langle 0, p\rangle$ but $\langle 1, p\rangle \not\geq^* m$. Secondly we cannot have $b = \langle n_1, p_1\rangle$, $n_1 > 0$ since $\langle n_1+1, p\rangle >^* \langle 0, p\rangle$ but $\langle n_1+1, p\rangle \not\geq^* \langle n_1, p_1\rangle$. Finally we consider $b = \langle 0, p_1\rangle$. $\langle 0, p\rangle \not\geq^* \langle 0, p_1\rangle$ implies $p \not\geq^* p_1$ in L', thus by definition $\langle 1, p\rangle \not\geq^* \langle 0, p_1\rangle$ although $\langle 1, p\rangle >^* \langle 0, p\rangle$.

Thus we have shown, providing there is no $x \in L'$ such that $x \leq^* y$ for all $y \in L'$, that the Frattini sublattice of $(X, \tau, \leq^*)$ is precisely the clopen decreasing sets contained in $\{0\} \times L'$ which by definition is homeomorphic and order isomorphic with L'. So let L_1 be the lattice represented by $(X, \tau, \leq^*)$. In the case that a minimum $(\leq^*)$ point x exists in L' we would have $\emptyset \notin \Phi(L_1)$ with $\emptyset$ meet irreducible. But $\Phi(\omega \times 2) = \{\langle 0, 0\rangle\}$ (Koh [6]) and $\Phi(L_1 + L_2) = \Phi(L_1) + \Phi(L_2)$ (Koh [6]) where $+$ denotes ordered sum. So that in order to obtain L as the Frattini sublattice we let $L_1 = (\omega \times 2) +$ Lattice $(X, \tau, \leq^*)$.

If the given distributive lattice L lacks either a zero or a unit we add a meet irreducible zero or join irreducible unit respectively in order to obtain a topological representation which will have a minimum $(\leq^*)$ point or maximum $(\leq^*)$ point. If only a zero is lacking we let $L_1 = (X, \tau, \leq^*)$. The fly in the ointment is when L lacks a unit. Then we must ensure that $\{0\} \times L' \notin \Phi((X, \tau, \leq^*))$. To do this we add new elements to the $(\leq^*)$ relation with the following clause:

(ε) $\langle 1, m_L\rangle <^* \langle n, m_L\rangle$ $1 < n < \omega$ where $m_L \in L'$ and $x \leq^* m_L$ for all $x \in L'$.

Similar arguments to those used above give that the new $(\leq^*)$ relation is a partial order that is totally order disconnected over (X, τ). We now want to determine when a clopen decreasing set C is in $\Phi((X, \tau, \leq^*))$. If $C = X$ or $\langle 0, m_L\rangle \notin C$ then the same arguments apply. Thus we have two main cases.

(i) $C = \{0\} \times L'$.

Consider the pair $\langle 0, m_L\rangle \not\geq^* \langle 1, m_L\rangle$. Then $x >^* \langle 0, m_L\rangle$ implies $x \geq^* \langle 1, m_L\rangle$ by (ε) and $x <^* \langle 1, m_L\rangle$ implies $x \leq^* \langle 0, m_L\rangle$. That is to say $\{\langle\langle 0, m_L\rangle, \langle 1, m_L\rangle\rangle\}$ is the separating set of a maximal lattice that omits C.

(ii) $\langle 1, m_L\rangle \in C$, $C \neq X$.

Since L has no unit m_L is a maximum $(\leq^*)$ that is an accumulation point of

$L' - \{m_L\}$. Thus $\langle 1, m_L \rangle \in C$ implies $\langle 1, p \rangle \in C$ for some $p \in L' - \{m_L\}$. This reduces to the previous case (ii), giving a maximal sublattice that omits C.
Koh's example $\Phi(\omega \times 2) = \{\langle 0, 0 \rangle\}$ deals with the case when L is a single element. This concludes Theorem 5.

In Theorem 5 the distributive lattice L_1 that we construct is such that $|L_1| = |L| + \omega$. Koh's theorem in the category of lattices is such that for infinite L we have $|L_1| = |L|$ while for $1 < |L| < \omega$ we have $|L_1| < \omega$. (By necessity if $|L| = 1$ and $L = \Phi(L_1)$ we will have $|L_1| \geq \omega$.) We will show now that in general the cardinality for finite L cannot be reduced.

THEOREM 6. *Let L be a finite distributive lattice. The only members of $\Phi(L)$ to have complements are the zero and unit of $\Phi(L)$ itself.*

Proof. The unit of $\Phi(L)$ will be the greatest join reducible element of L and the zero of $\Phi(L)$ will be the least meet reducible one. We may assume these are the unit and zero of L since $\Phi(L_1 + L_2) = \Phi(L_1) + \Phi(L_2)$. Thus suppose contrary to the hypothesis of the Theorem that there exists a finite lattice L with join reducible unit and meet reducible zero and an element p such that $p \in \Phi(L)$ has a complement. By Lemma 3 every separating set of a maximal sublattice is of the form $\{\langle a, b \rangle\}$. Since $p' \subseteq L'$ and p has a complement $L' - p'$ is clopen decreasing. Thus $x \in p'$ and $y \in L' - p'$ implies $x)(^* y$. Choose maximal $(\leq^*)$ x and minimal $(\leq^*)$ y. Then $\langle x, y \rangle$ is the separating set of a maximal sublattice. But $x \in p'$, $y \notin p'$. Thus p is not a member of this sublattice. This contradicts $p \in \Phi(L)$ and concludes the proof.

Koh [6] asked whether for all distributive lattices L such that every sublattice can be extended to a maximal proper sublattice and $\Phi(L) = \emptyset$ it follows that L is a chain. And whether the only lattices for which every sublattice is the intersection of maximal sublattices are chains. The answer to both these questions is negative. By improvising on the method of proof of Theorem 5 we gain some insight as to how strong the answer for distributive lattices is.

DEFINITION 7. (i) For an ideal I of a distributive lattice define the *congruence* Θ_I by $a \equiv b(\Theta_I)$ iff $a \vee i = b \vee i$ for some $i \in I$.

(ii) For a filter F of a distributive lattice define the *congruence* Θ_F by $a \equiv b(\Theta_F)$ iff $a \wedge f = b \wedge f$ for some $f \in F$.

(iii) Let I be a relatively complemented ideal of a distributive lattice L; then I_m the maximal relatively complemented ideal exists. Similarly for F_m the maximal relatively complemented filter. Then we define the *Boolean Congruence* Θ_B of L to be $\Theta_{I_m} \vee \Theta_{F_m}$; if either I_m or F_m is undefined then Θ_{I_m} or Θ_{F_m} are replaced by the identity congruence.

THEOREM 8. *For a given* (0, 1) *distributive lattice L there is a distributive lattice L_1 such that:*

(i) *every sublattice of* L_1 *can be extended to a proper maximal one*;

(ii) $L_1/\Theta_B \cong L$;

(iii) $\Phi(L_1)=\emptyset$.

Proof. Suppose that L is a (0, 1) distributive lattice with representation L'. We consider the set X

$$X=\{\langle n,p\rangle \mid n\in\omega^*+\omega+1+\omega^*+\omega, p\in L'\}\cup\{m_1\}\cup\{m_2\}.$$

We give X a topology τ defined to have an open base consisting of the sets:

(i) $Z\times Y$, where Z is an open set of the interval topology on $(1+\omega^*+\omega+1+$ $+\omega^*+\omega+1)$ that does not contain end points, and Y is open in L',

(ii) $\{m_1\}\cup\bigcup_n(\{n\}\times L')$, $n>N$ for some fixed N n, $N\in\omega^*+\omega+1+\omega^*+\omega$, and

(iii) $\{m_2\}\cup\bigcup_n(\{n\}\times L')$, $n<N$ for some fixed N n, $N\in\omega^*+\omega+1+\omega^*+\omega$.

Since $(1+\omega^*+\omega+1+\omega^*+\omega+1)$ is a compact space we have by similar arguments to those in Theorem 7 that (X, τ) is compact.

We now impose another $(\leq^*)$ relation on (X, τ). The set $\omega^*+\omega+1+\omega^*+\omega$ partitions naturally into $P_1+P_2+P_3$ with $P_1, P_3=\omega^*+\omega$ and $P_2=1$. We let:

(α) $m_1>^*x$ for all $x\in X-\{m_1\}$
$m_2<^*x$ for all $x\in X-\{m_2\}$

(β) $\langle 0,p_1\rangle<^*\langle 0,p_2\rangle$, $0\in P_2$ if $p_1<^*p_2$

(γ) $\langle 0,p\rangle<^*\langle n,p\rangle$, $n\in P_3$
$\langle 0,p\rangle>^*\langle n,p\rangle$, $n\in P_1$

(δ) If $x<^*\langle 0,p\rangle$ let $x<^*\langle n,p\rangle$, $n\in P_3$
If $x>^*\langle 0,p\rangle$ let $x>^*\langle n,p\rangle$, $n\in P_1$

By similar arguments to those used before the relation $(\leq^*)$ is a partial order (transivity is secured by two applications of (δ)) that is totally order disconnected over (X, τ); and thus $(X, \tau, \leqslant^*)$ represents a distributive lattice L_X.

Because of (α) L_X has a join irreducible unit and a meet irreducible zero. Let $L_1=L_X-\{0, 1\}$. We begin by showing that every proper sublattice K of L_1 can be extended to a maximal one. We have that $K\cup\{0, 1\}$ is a proper sublattice in L_X and an extension to a maximal proper sublattice of L_X gives a maximal proper sublattice of L_1. Accordingly Lemma 3 gives that the separating set of $K\cup\{0, 1\}$ contains a pair $\langle a, b\rangle$, with $a\ngeq^*b$ and neither one equal to m_1 or m_2. We consider the values taken by a, b where $a=\langle n_1,p_1\rangle$, $b=\langle n_2,p_2\rangle$ and show that there is an associated maximal pair $\langle a_1, b_1\rangle$.

(i) $a=\langle n_1,p_1\rangle$, $n_1\in P_3$.

If $b=\langle n_2,p_2\rangle$, $n_2\in P_3$ and $\langle 0,p_2\rangle<^*\langle n_1,p_1\rangle$ we set $a_1=a$ and $b_1=b$. If $\langle 0,p_2\rangle\nless^*$ $\langle n_1,p_1\rangle$ or $n_2=0$, choose $n_3\in P_1$ and set $a_1=a$, $b_1=\langle n_3,p_2\rangle$. Finally in this case if $n_2\in P_1$ we leave $a_1=a$ and $b_1=b$.

(ii) $a=\langle 0,p_1\rangle$, $0\in P_2$.

$a \not\geq^* b$ means $\langle 0, p_1 \rangle \not\geq^* \langle n_2, p_2 \rangle$ which implies the existence of $n_3 \in P_3$ such that $\langle n_3, p_1 \rangle \not\geq^* \langle n_2, p_2 \rangle$. We now operate on this pair in the manner of case (i) to obtain a_1, b_1.

(iii) $a = \langle n_1, p_1 \rangle, n_1 \in P_1$.

If $\langle 0, p_1 \rangle \geq^* \langle n_2, p_2 \rangle$ then $n_2 \in P_2$ or P_1. Either way let $a_1 = a$ and $b_1 = \langle n_3, p_2 \rangle$ where $n_3 \in P_1$; $n_3 = n_2$ if $n_2 \in P_1$, $n_3 = n_1 + 1$ otherwise. For $\langle 0, p_1 \rangle \not\geq^* \langle n_2, p_2 \rangle$ we take this pair to case (ii) to find a_1, b_1.

Thus for $\langle a, b \rangle$, $a \not\geq^* b$ we obtain a new pair $\langle a_1, b_1 \rangle$, $a_1 \not\geq^* b_1$, $x >^* a_1$ implies $x \geq^* b_1$ and $x <^* b_1$ implies $x \leq^* a_1$. This means that $\langle a_1, b_1 \rangle$ represents a maximal proper sublattice in K. Since $a_1 \geq^* a$ and $b_1 \leq^* b$ statement (A) of Lemma 3 shows that this is a proper extension of K.

We now show that $L_1/\Theta_B \cong L$. In L_1 Θ_{I_m} is represented topologically by $\bigcup_n (\{n\} \times \times L') \cup \{m_2\}$ $n \in P_1$, a decreasing open set. This is because apart from m_2 (which as a representation of L_1 is a member of every clopen decreasing set) any two points are incomparable ($\leq^*$) while any other $x \in X$ implies $x >^* y$ for some $y \neq m_2$ in the set. Similarly Θ_{F_m} can be represented by the open increasing set $\bigcup_n (\{n\} \times L') \cup \{m_1\}$, $n \in P_3$. But then the dual space of $L_1/\Theta_{I_m} \vee \Theta_{F_m}$ is $X - (\bigcup_{n \in P_1} \{n\} \times L' \cup \{m_2\}) - - (\bigcup_{n \in P_3} \{n\} \times L' \cup \{m_1\})$. This is precisely the closed subset $\{0\} \times L'$ with the induced topology which by definition is homeomorphic and order isomorphic with L'. It remains only to show that $\Phi(L_1) = \emptyset$. To see this consider any clopen decreasing set $C \in \mathbf{L}_1$. By definition $m_2 \in C$ and $m_1 \notin C$. This implies that for some $n_1, n_2 \in P_1$ or for $n_1, n_2 \in P_3$. We have $\langle n_1, p \rangle \in C$ and $\langle n_2, p \rangle \notin C$ for some fixed $p \in L'$. But then $\{\langle\langle n_1, p \rangle, \langle n_2, p \rangle\rangle\}$ is the separating set of a maximal sublattice excluding C. This concludes the proof.

The proof may be further generalised to give:

THEOREM 9. *For every* (0, 1) *distributive lattice L there is a distributive lattice* L_1 *with a congruence* Θ *on it such that*:

(i) $L_1/\Theta \cong L$;

(ii) *Every sublattice is the intersection of maximal sublattices.*

We omit the proof and instead prove the following Theorem which illustrates some of the essential ideas. This Theorem may also be deduced from Hashimoto [5].

THEOREM 10. *If L is a distributive lattice without zero or unit such that every proper ideal is the intersection of maximal proper ideals and dually for filters, then every sublattice is the intersection of maximal proper sublattices.*

Proof. Given such an L we add a zero and unit in order to obtain a topological dual L' that has a maximum ($\leq^*$) m_1 and minimum ($\leq^*$) m_2. By Lemma 5 of Adams [1] the topology in such a space is such that for a given point $x \neq m_1, m_2$ if x is not covered ($\leq^*$) by m_1 then $x \in Closure\ \{x_r\}_{r \in R}$ where $x <^* x_r$ and x_r is covered ($\leq^*$) by

m_1. Similarly if x does not cover $(\leq^*)$ m_2 then $x \in \text{Closure } \{x_r\}_{r \in R}$ where $x_r <^* x$ and x_r covers $(\leq^*)$ m_2.

Suppose L_1 is a sublattice of L. Then $L_1 \cup \{0, 1\}$ has a separating set in which all members are pairs of the form $\langle a, b \rangle$. Now consider a clopen decreasing set $C \in \mathbf{L}$ which is not a member of the sublattice. Then for some pair $\langle a, b \rangle$, $a \not\geq^* b$ we have $a \in C$ and $b \notin C$. If a is not covered $(\leq^*)$ by m_1 then C clopen implies that there exists $a_1 \in C$, $a_1 >^* a$ and a_1 covered $(\leq^*)$ by m_1. If a is covered let $a_1 = a$. Similarly for b we can find $b_1 \leq^* b$, $b_1 \notin C$ and b_1 covers $(\leq^*)$ m_2. However $a_1 \in C$ and $b_1 \notin C$ implies $a_1 \not\geq^* b_1$ and $\langle a_1, b_1 \rangle$ represents a maximal sublattice that by statement (A) of Lemma 3 contains L_1 but omits C. Since this is true for any C we have every sublattice is the intersection of maximal ones.

A special case of Theorem 10 is the following:

COROLLARY 11. *If L is a relatively complemented distributive lattice without a zero or unit then every subalgebra is the intersection of the maximal ones that contain it.*

We remark that Sachs [8] has shown that every subalgebra of a Boolean algebra is the intersection of maximal subalgebras.

Finally we give a characterisation of the Frattini sulattice of the direct product $L_1 \times L_2$ of two (0, 1) distributive lattices. First a definition:

DEFINITION 12. For a (0, 1) distributive lattice L let:

(i) $D_0(L) = \{x \mid x \in L, x \text{ has a pseudo complement } x^* \text{ and } x^* = 0\}$

(ii) $D_1(L) = \{x \mid x \in L, \exists x^* \text{ such that for all } y \in L \ (x \vee y = 1 \leftrightarrow y \geq x^*) \text{ and } x^* = 1\}$.

THEOREM 13. *Let $L = L_1 \times L_2$ where L_i is a (0, 1) distributive lattice then $(x_1, x_2) \in \Phi(L)$ iff*

(a) $x_i \in \Phi(L_i) \cup \{0_i, 1_i\} \quad i = 1, 2,$

and

(b) *One of the following holds*

(i) $x_i \in D_0(L_i) \quad i = 1, 2.$

(ii) $x_i \in D_1(L_i) \quad i = 1, 2.$

Proof. The Stone space of $L_1 \times L_2$ is homeomorphic with the sum of the Stone spaces L_1' and L_2'; the $(\leq^*)$ ordering being the same within each component. But $d_1 \in L_1'$, $d_2 \in L_2'$ implies d_1 incompatible $(\leq^*)$ with d_2. Thus an element of L is topologically the sum of its components.

We prove first that (a) and (b) hold if $x = (x_1, x_2) \in \Phi(L)$.

(a) Suppose $x_i \notin \Phi(L_i)$. Then we must have a maximal sublattice of L_i that omits x_i. If $x_i \neq 0_i, 1_i$ Lemma 3 implies the existence of a pair $\langle a, b \rangle$, $a, b, \in L_i$. But in L' if $a_1 >^* a$ or $b_1 <^* b$ we would have $a_1, b_1 \in L_i'$. So $\langle a, b \rangle$ represents a maximal sublattice in L that omits x.

(b) Consider a point $x \in A$, some (0, 1) distributive lattice, with topological dual

$x' \subseteq A'$. Suppose x' has a pseudo complement C, then C is a clopen decreasing set and $C = \{y \mid y \in A' \ y \ngeq^* k$ for any $k \in x'\}$. This is true because if $y \geq^* k$ for some $k \in x'$ then $y \in C$ implies $k \in C$, a contradiction, since $k \in x' \cap C$. If $y \ngeq^* k$ any $k \in x'$ then we can find a clopen increasing set K_k such that $k \in K_k$, $y \notin K_k$. Covering x' in this manner and choosing a finite subcover we have a clopen increasing set K such that $x' \subseteq K$ and $y \notin K$. But then $(A' - K) \cap x' = \emptyset$ implies $A' - K \subseteq C$, that is $y \in C$.

Thus x' has a pseudo complement iff $\{y \mid y \in A', y \ngeq^* k$ for any $k \in x'\}$ is clopen and the pseudo complement is zero only if this set is empty. So we have:

(I) $x \in D_0(A)$ iff $y \in A' - x'$ implies $\exists k \in x'$ such that $y >^* k$.

Similarly

(II) $x \in D_1(A)$ iff $y \in x'$ implies $\exists k \in A' - x'$ such that $y <^* k$.

Suppose that neither (b) (i) nor (ii) are true and without loss of generality let us say $x_1 \notin D_0(L_1)$. Then there is $y \in L_1' - x_1'$ such that $y \ngeq^* k$ for any $k \in x_1'$. We may suppose without contradiction that y is minimal $(\leq^*)$.

If $x_2 \notin D_1(L_2)$ then there is a maximal $(\leq^*)$ $z \in x_2'$ such that $z \nleq^* k$ any $k \in L_2' - x_2'$. But since z is maximal $(\leq^*)$, y is minimal $(\leq^*)$ and $z \ngeq^* y$ we have $\langle z, y\rangle$ represents a maximal sublattice. However $z \in x'$ and $y \notin x'$ implies x is not a member, contradicting $x \in \Phi(L)$. Thus $x_2 \in D_1(L_2)$. Since (b) (ii) fails $x_1 \notin D_1(L_1)$ and there exists a maximal $(\leq^*)$ $z \in x_1'$, $z \nleq^* k$ any $k \in L_1' - x_1'$. But again $\langle z, y\rangle$ represents a maximal sublattice that omits x. Contradicting $x \in \Phi(L)$. Thus we conclude that $x \in \Phi(L)$ implies (a) and (b).

Suppose next that (a) and (b) hold. Since L_1' and L_2' are not empty we have that any maximal sublattice is of type (iii) in Lemma 3. Suppose $\langle a, b\rangle$ represents a maximal sublattice that excludes x; that is to say $a \ngeq^* b$, $a \in x'$, and $b \notin x'$. Since x satisfies (a) we must have $a \in L_1'$ and $b \in L_2'$ or vice versa; suppose $a \in L_1'$. Consider when (b) (i) is satisfied. We have $b \notin x'$, $b \in L_2'$ and $x_2 \in D_0(L_2)$. But this implies the existence of $k \in x_2'$ such that $b >^* k$. Thus $\langle a, k\rangle$ is in any separating set containing $\langle a, b\rangle$, contradicting the maximality of the sublattice. Alternately suppose (b) (ii) holds. Then $a \in x_1'$ necessitates the existence of $k \in L_1' - x_1'$, $k >^* a$ and $k \ngeq^* b$. Once again we derive a new pair $\langle k, b\rangle$ that is in any separating set containing $\langle a, b\rangle$, a contradiction.

Thus for any $x \in L$ that satisfies (a) and (b) we may deduce that x is a member of $\Phi(L)$.

Theorem 13 is in terms of (0, 1) distributive lattices; no attempts has been made to investigate the cases where a zero or unit is missing. We point out however that although the situation is less natural it is still manageable from the topological point of view.

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