RESIDUALLY SMALL VARIETIES

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In this paper we consider the question of when a variety $\mathscr{V}$ of algebras may be written as $\mathscr{V} = \mathbf{ISP}\, K$ (the class of algebras isomorphic to a subalgebra of some direct product of members of K) for some *set* $K \subseteq \mathscr{V}$. Of course there always exists a single algebra $\mathfrak{A} \in \mathscr{V}$ such that $\mathscr{V} = \mathbf{HSP}\{\mathfrak{A}\}$ (the class of homomorphic images of algebras in $\mathbf{SP}\{\mathfrak{A}\}$); $\mathfrak{A}$ may be taken as the $\mathscr{V}$-free algebra on $\aleph_0$ generators, or more generally, any algebra which is generic (*alias* functionally free) in $\mathscr{V}$ [34] [11]. Moreover G. Birkhoff proved [9] that $\mathscr{V} = \mathbf{ISP}\, K$, where K is the class of subdirectly irreducible algebras in $\mathscr{V}$. Thus the above question is equivalent to the question of when the isomorphism types of subdirectly irreducible algebras in $\mathscr{V}$ form a set, or equivalently, of when there exists an upper bound on the cardinalities of subdirectly irreducible algebras in $\mathscr{V}$. If such an upper bound exists, then we call $\mathscr{V}$ *residually small.*

An algebra $\mathfrak{A}$ is *equationally compact* [30] [38] iff it satisfies the following condition: if Σ is any set of equations (possibly with uncountably many unknowns) with constants from $\mathfrak{A}$, and if every finite subset of Σ can be satisfied in $\mathfrak{A}$, then Σ can be simultaneously satisfied in $\mathfrak{A}$. J. Mycielski asked in [30] which varieties $\mathscr{V}$ have the property that every algebra in $\mathscr{V}$ can be embedded in an equationally compact algebra in $\mathscr{V}$. I proved in [36] that if $\mathscr{V}$ has this property, then each subdirectly irreducible algebra in $\mathscr{V}$ is of power $\leqslant 2^{n}$ (where n is $\aleph_0$ + the number of operations in the algebras of $\mathscr{V}$). In our principal result (Theorem 1.2), we prove a strong converse, namely, that if there exists any upper bound on the size of subdirectly irreducible algebras in $\mathscr{V}$ (i.e. if $\mathscr{V}$ is residually small), then every algebra in $\mathscr{V}$ can be embedded in an equationally compact algebra in $\mathscr{V}$. Hence we answer at the same time this question of Mycielski and the question of the first paragraph above. As a corollary we deduce that the *Hanf number* for subdirect irreducibility in a variety is $(2^{n})^{+}$. Theorem 1.2 also gives several refinements of these ideas which we do not mention in this introduction.

In §2 we make some remarks about injective algebras in a variety $\mathscr{V}$. No deep new theorem is proved, but several known results are explained from the viewpoint of Theorem 1.2. In particular, we give in Theorem 2.10 a much simplified proof of our theorem [36] that the injective hull of an algebra $\mathfrak{A} \in \mathscr{V}$ has power $\leqslant 2^{n+|A|}$ (if it exists).

In §3 we discuss the notion of *pure subalgebra,* which has played a central rôle in the theory of equational compactness. $\mathfrak{A}$ is a pure subalgebra of $\mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$ and

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every finite set of equations with constants from $\mathfrak{A}$ which is satisfiable in $\mathfrak{B}$ is satisfiable in $\mathfrak{A}$. We prove the following counterpart of the above mentioned subdirect representation theorem of G. Birkhoff (see Theorem 3.6). Every algebra is a pure subalgebra of a product of algebras $\mathfrak{B}_i (i \in I)$, where each $\mathfrak{B}_i$ is pure-irreducible in the sense that whenever $\mathfrak{B}_i$ is a pure subdirect product of algebras $\mathfrak{C}_j$, then for some $\mathfrak{C}_j$ the projection of $\mathfrak{B}_i$ onto $\mathfrak{C}_j$ is an isomorphism. We then prove a theorem (see 3.12) similar to Theorem 1.2 (mentioned above), stating that every algebra in $\mathscr{V}$ is a pure subalgebra of an equationally compact algebra if and only if there exists an upper bound to the size of pure-irreducible algebras in $\mathscr{V}$.

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0. Preliminaries

We mainly deal with algebras $\mathfrak{A} = \langle A, F_s \rangle_{s \in S}$, but in § 3 we also deal with structures $\mathfrak{U} = \langle A, R_t, F_s \rangle_{t \in T\, s \in S}$ having both relations R_t and operations F_s. A structure or algebra is denoted by a capital German letter, with the corresponding Roman letter denoting its universe.

We assume knowledge of the following elementary notions from first order logic (see e.g. [19]): terms; (formal) equations [possibly with constants from an algebra $\mathfrak{A}$], written $\sigma \cong \tau$, where σ and τ are terms; atomic formulas [possibly with constants from a structure $\mathfrak{A}$]; satisfiability in $\mathfrak{A}$ of a set of equations [or atomic formulas] with constants in $\mathfrak{A}$; validity of a set of equations in an algebra $\mathfrak{A}$. Thus, in particular, if $\mathfrak{A}$ is a structure (or algebra), $a_0, \ldots, a_n \in A$, and φ is a formula in a vocabulary applicable to $\mathfrak{A}$ whose free variables are among $x_0, \ldots, x_n$, then $\mathfrak{A} \models \varphi [a_0, \ldots, a_n]$ means that φ holds in $\mathfrak{A}$ under the assignment of a_i for x_i $(0 \leqslant i \leqslant n)$. And $\Sigma \vdash \varphi$ means that the sentence φ follows logically from the set Σ of formulas. $\mathfrak{A}$ is a *pure substructure* of a structure $\mathfrak{B}$ iff $\mathfrak{A} \subseteq \mathfrak{B}$ and every finite set of atomic formulas with constants from $\mathfrak{A}$ which is satisfiable in $\mathfrak{B}$ is satisfiable in $\mathfrak{A}$.

The definition of *equational compactness* of algebras is stated in the introduction. More generally, a structure $\mathfrak{U}$ is *atomic-compact* iff the following is true: if Σ is a set of atomic formulas (possibly involving uncountably many unknowns) with constants from $\mathfrak{A}$ such that every finite subset of Σ is satisfiable in $\mathfrak{A}$, then Σ is simultaneously satisfiable in $\mathfrak{A}$. For this and related notions of compactness, consult [30], [38] and [36]. In our proofs of 1.2 and 3.12 we use some results of [36]; familiarity with [38] will also help the reader.

A *variety* is a class of algebras (or structures) (all of the same similarity type) closed under the formation of products, subalgebras and homomorphic images. G.

Birkhoff proved in 1933 that a class $\mathscr{V}$ of algebras of the same type is a variety if and only if there exists a set Σ of equations such that $\mathscr{V}$ consists exactly of models of Σ (i.e. algebras $\mathfrak{A}$ in which all equations of Σ are valid) [8].

An algebra $\mathfrak{A}\in\mathscr{V}$ is an *absolute retract* in $\mathscr{V}$ iff whenever $\mathfrak{A}\subseteq\mathfrak{C}\in\mathscr{V}$, there exists a homomorphism f retracting $\mathfrak{C}$ onto $\mathfrak{A}$, i.e. $f:\mathfrak{C}\to\mathfrak{A}$ which is the identity on $\mathfrak{A}$. An algebra $\mathfrak{A}\in\mathscr{V}$ is $\mathscr{V}$*-injective* iff whenever $\mathfrak{B}\subseteq\mathfrak{C}\in\mathscr{V}$ and $h:\mathfrak{B}\to\mathfrak{A}$ is a homomorphism there exists a homomorphism $f:\mathfrak{C}\to\mathfrak{A}$ extending h. It is not hard to check (see e.g. [38]) that every $\mathscr{V}$-injective algebra is an absolute retract in $\mathscr{V}$, that every absolute retract in $\mathscr{V}$ is equationally compact, and that the converse statements are false.

The notion of injectivity comes from category theory, but we do not state here the definition within category theory. The interested reader may consult e.g. [2], [5], [12] or [13] and check that the category theorist's definition reduces to ours in the category consisting of all homomorphisms between algebras of a given variety $\mathscr{V}$. The same is true of the notion of essential extension: $\mathfrak{B}$ is an *essential extension* of $\mathfrak{A}$ in this category if and only if $\mathfrak{A}\subseteq\mathfrak{B}$ and every proper congruence on $\mathfrak{B}$ must identify two points of $\mathfrak{A}$. Since this definition does not mention $\mathscr{V}$, we will say simply that $\mathfrak{B}$ is an essential extension of $\mathfrak{A}$.

We now state some simple propositions about subdirect irreducibility (and some related facts about essential extensions) which are essentially well known and will be useful in what follows. Recall that an algebra $\mathfrak{A}$ is *subdirectly irreducible* iff any family of homomorphisms defined on $\mathfrak{A}$ which separates points of $\mathfrak{A}$ must contain a homomorphism which is one-to-one.

DEFINITION 0.1. The algebra $\mathfrak{A}$ is (a, b)*-irreducible* iff $a\neq b$, $a, b\in A$, and $f(a)=f(b)$ whenever f is a homomorphism defined on $\mathfrak{A}$ and f is not one-to-one.

PROPOSITION 0.2. *$\mathfrak{A}$ is subdirectly irreducible if and only if there exist $a, b\in A$, $a\neq b$ such that A is (a, b)-irreducible.*

PROPOSITION 0.3. *Let θ be the smallest congruence on the algebra $\mathfrak{A}$ containing $(c, d)\in A^2$. Then for any $(a, b)\in A^2$, $(a, b)\in\theta$ if and only if there exists a formula $\varphi(\cdot,\cdot,\cdot,\cdot)$ in the first order language of $\mathfrak{A}$ with the following properties:*

(i) *φ is positive;*

(ii) $\vdash\forall yz\,[\exists x\varphi(x, x, y, z)\to y\simeq z]$;

(iii) $\mathfrak{A}\models\varphi\,[c, d, a, b]$.

Proof. It is easy to check that the existence of such a formula φ implies that $(a, b)\in\theta$. Conversely if $(a, b)\in\theta$, then a result of Mal'cev [28] (or see [19, §10, Theorem 3]) states that there exists a sequence of terms $\tau_0, \ldots, \tau_{n-1}$ whose variables are among say $x_0, \ldots, x_m$ such that if φ is

$$\exists z_0\ldots z_n\left[z_0\simeq x_2\wedge z_n\simeq x_3\wedge\bigwedge_{i=0}^{n-1}\exists x_2\ldots x_m\big(z_i\simeq\tau_i\wedge z_{i+1}\simeq\tau_i(\sigma)\big)\right]$$

(where σ is the substitution interchanging x_0 and x_1), then φ satisfies condition (iii) above. Clearly this φ also satisfies conditions (i) and (ii).

COROLLARY 0.4. *The algebra* $\mathfrak{A}$ *is* (a, b)*-irreducible if and only if* $a, b \in A$, $a \neq b$ *and for all* $c, d \in A$ *with* $c \neq d$ *there exists a formula* φ *satisfying (i), (ii) and (iii) of 0.3.*

COROLLARY 0.5. $\mathfrak{B}$ *is an essential extension of* $\mathfrak{A}$ *if and only if* $\mathfrak{A} \subseteq \mathfrak{B}$ *and for each* $c, d \in B$ *with* $c \neq d$, *there exist* $a, b \in A$ *with* $a \neq b$ *and a formula* φ *satisfying (i), (ii) and (iii) of 0.3.*

Any ordinal is the set of smaller ordinals; ordinals are denoted α, β, γ and δ. A cardinal is an ordinal equipotent with no smaller ordinal; infinite cardinals are denoted $\mathfrak{m}$, $\mathfrak{n}$. The cardinality of X (the least ordinal equipotent with X) is denoted $|X|$. $2^{\mathfrak{n}}$ is the cardinality of the set of subsets of $\mathfrak{n}$. The least infinite cardinal is $\aleph_0$, *alias* ω. If A is any set, then $A^{(2)}$ denotes the set of pairs $\{a, b\} \subseteq A$ such that $a \neq b$. We will use the following theorem of combinatorial set theory due to P. Erdős [16] (or see [17, Theorem 4 (i)]). For a model-theoretic proof, see [33].

THEOREM 0.6. (P. Erdős). *If* $|A| > 2^{\mathfrak{m}}$ *and* $A^{(2)} = \bigcup \mathscr{D}$ *where* $|\mathscr{D}| \leqslant \mathfrak{m}$, *then there exists* $D \in \mathscr{D}$ *and* $B \subseteq A$ *with* $|B| > \mathfrak{m}$ *such that* $B^{(2)} \subseteq D$.

COROLLARY 0.7. *Let* $\mathfrak{B}$ *be an essential extension of* $\mathfrak{A}$, $|B| > 2^{\mathfrak{m}}$, *where* $\mathfrak{m} = (\aleph_0 + |A| +$ *the number of operations of* $\mathfrak{A}$). *Then there exist* $a, b \in A$, $a \neq b$, *and a positive formula* $\varphi(\cdot, \cdot, \cdot, \cdot)$ *in the language of* $\mathfrak{A}$ *such that* $\vdash \forall yz [\exists x \varphi(x, x, y, z) \rightarrow y \approx z]$ *and such that*

$$(*) \qquad D(\mathfrak{A}) \cup Eq(\mathfrak{B}) \cup \{\varphi(x_i, x_j, a, b) : i < j < \omega\}$$

is consistent. (Here $Eq(\mathfrak{B})$ is the set of equations holding in $\mathfrak{B}$, and $D(\mathfrak{A})$ is the atomic diagram of $\mathfrak{A}$, i.e. the set of atomic sentences and negations of atomic sentences with constants from $\mathfrak{A}$ which hold in $\mathfrak{A}$.)

Proof. Let Λ be the set of all $\lambda = (\varphi, a, b)$, where φ is a positive formula satisfying $\vdash \forall yz [\exists x \varphi(x, x, y, z) \rightarrow y \approx z]$, $a, b \in A$, and $a \neq b$. Let $\prec$ be a strict linear ordering of B. For each $\lambda = (\varphi, a, b) \in \Lambda$, we define

$$C_\lambda = \{\{c, d\} \in B^{(2)} : c \prec d,\ \mathfrak{B} \models \varphi[c, d, a, b]\}.$$

It follows from Corollary 0.5 that $B^{(2)} = \bigcup \{C_\lambda : \lambda \in \Lambda\}$. Since $|\Lambda| = \mathfrak{m}$, we may apply Theorem 0.6 to see that there exists infinite $C \subseteq B$ and $\lambda = (\varphi, a, b) \in \Lambda$ such that $C^{(2)} \subseteq C_\lambda$. But clearly $(*)$ is consistent for this φ, a and b.

The following lemma was known to many people and was perhaps first stated in [14].

LEMMA 0.8 *Let v be $\mathscr{V}$ variety, and let $\mathfrak{B} \in \mathscr{V}$ be an essentialex tension of $\mathfrak{A}$. Then $\mathfrak{B}$ is an absolute retract in $\mathscr{V}$ if and only if no proper extension of $\mathfrak{B}$ in $\mathscr{V}$ is an essential extension of $\mathfrak{A}$.*

COROLLARY 0.9. *If $\mathfrak{A} \in \mathscr{V}$ has (within isomorphism) only a set of essential extensions which are in $\mathscr{V}$, then some essential extension of $\mathfrak{A}$ is an absolute retract in $\mathscr{V}$.*

Proof. Follows immediately from 0.8, Zorn's Lemma and the fact that the union of a chain of essential extensions of $\mathfrak{U}$ is an essential extension of $\mathfrak{U}$.

The following representation theorem from [36] will be used in the proof of our main Theorem (1.2).

THEOREM 0.10. [36, Corollary 5.8]. *Let K be a class of algebras of the same type closed under the formation of products and retracts. Let $\mathfrak{n} = \aleph_0 +$ the number of operations of algebras of K. Then there exists a set $K_0 \subseteq K$ such that*

(i) $|K_0| \leqslant 2^{\mathfrak{n}}$;

(ii) *each member of K_0 is equationally compact and of power $\leqslant 2^{\mathfrak{n}}$;*

(iii) *for each $\mathfrak{A} \in K$, $\mathfrak{A}$ is equationally compact if and only if $\mathfrak{A}$ is a retract of a product of members of K_0.*

1. Residual smallness

DEFINITION 1.1. A variety $\mathscr{V}$ is *residually small* if and only if $\mathscr{V}$ satisfies any (and hence all) of the eleven conditions of the following theorem.

THEOREM 1.2.[2]) *Let $\mathscr{V}$ be the variety of algebras defined by the set Σ of equations, and let $\mathfrak{n} = (\aleph_0 +$ the number of operations of algebras in $\mathscr{V}$). Then the following eleven conditions are equivalent:*

(i) *there exists a cardinal $\mathfrak{m}$ such that every subdirectly irreducible algebra in $\mathscr{V}$ has power $\leqslant \mathfrak{m}$;*[3])

(ii) *every subdirectly irreducible algebra in $\mathscr{V}$ has power $\leqslant 2^{\mathfrak{n}}$;*

(iii) *there are $\leqslant 2^{2^{\mathfrak{n}}}$ non-isomorphic subdirectly irreducible algebras in $\mathscr{V}$;*

(iv) *there exists a set K such that $\mathscr{V} \subseteq$ **ISP** K;*

(v) *there exists a set $K \subseteq \mathscr{V}$ with $|K| \leqslant 2^{\mathfrak{n}}$ and $|A| \leqslant 2^{\mathfrak{n}}$ for all $\mathfrak{A} \in K$, such that $\mathscr{V} =$ **ISP** K;*

(vi) *each $\mathfrak{A} \in \mathscr{V}$ has (within isomorphism) only a set of essential extensions in $\mathscr{V}$;*

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[2]) Announced in [37].

[3]) Condition (i) of Theorem 1.2 contrasts strongly with the condition that subdirect irreducibility be a first order property relative to the variety $\mathscr{V}$. This latter condition has been investigated by K. Baker [3].

(vii) *if* $\mathfrak{B} \in \mathscr{V}$ *is an essential extension of* $\mathfrak{A}$, *then* $|B| \leqslant 2^{\mathfrak{n}+|A|}$;

(viii) *every algebra* $\mathfrak{A} \in \mathscr{V}$ *is a subalgebra of some equationally compact algebra*;

(ix) *every algebra* $\mathfrak{A} \in \mathscr{V}$ *is a subalgebra of some equationally compact algebra* $\in \mathscr{V}$;

(x) *every algebra* $\mathfrak{A} \in \mathscr{V}$ *is a subalgebra of some absolute retract in* $\mathscr{V}$;

(xi) *for every positive formula* $\varphi(\cdot, \cdot, \cdot, \cdot)$ *in the language of* $\mathscr{V}$ *such that* $\vdash \forall yz\,[\exists x \varphi(x, x, y, z) \rightarrow y \doteq z]$, *there exists a finite number n such that*

$$\Sigma \vdash \forall yz\,[\exists x_1 \ldots x_n \bigwedge_{1 \leqslant i < j \leqslant n} \varphi(x_i, x_j, y, z) \rightarrow y \doteq z].$$

Proof. The proof proceeds *via*

$$\begin{array}{ccc} (\mathrm{i}) \Rightarrow (\mathrm{xi}) \Rightarrow (\mathrm{vii}) \Rightarrow (\mathrm{vi}) \Rightarrow (\mathrm{x}) \Rightarrow (\mathrm{ix}) \Rightarrow (\mathrm{viii}) \Rightarrow (\mathrm{iv}) & \searrow & \\ \Downarrow \qquad\qquad & & (\mathrm{i}). \\ (\mathrm{v}) \Rightarrow (\mathrm{ii}) \quad \Rightarrow (\mathrm{iii}) & \nearrow & \end{array}$$

(i)⇒(xi). Suppose that (xi) fails and that $\mathfrak{m}$ is as in (i). Let U be a set (of variables) of power $\mathfrak{m}^+$ with $y \notin U$ and $z \notin U$, and let $\prec$ be a strict linear ordering on U. The failure of (xi) implies that for some positive φ satisfying $\vdash \forall yz\,[\exists x \varphi(x, x, y, z) \rightarrow y \doteq z]$, the set of formulas

$$(*) \qquad \Sigma \cup \{\varphi(u, v, y, z) : u, v \in U, u \prec v\} \cup \{\neg\, y \doteq z\}$$

is consistent. Thus there exists an algebra $\mathfrak{A} \in \mathscr{V}$ and an assignment $f: U \cup \{y, z\} \rightarrow A$ satisfying the formulas $(*)$ in $\mathfrak{A}$. Let θ be a maximal congruence on $\mathfrak{A}$ separating $f(y)$ and $f(z)$. Thus $\mathfrak{A}/\theta$ is subdirectly irreducible. Now let $u, v \in U$, $u \prec v$. Since $\mathfrak{A} \models \varphi[f(u), f(v), f(y), f(z)]$ and φ is positive, it follows that $\mathfrak{A}/\theta \models \varphi[f(u)/\theta, f(v)/\theta, f(y)/\theta, f(z)/\theta]$. Since $f(y)/\theta \neq f(z)/\theta$, it follows that $f(u)/\theta \neq f(v)/\theta$. Thus the subdirectly irreducible algebra $\mathfrak{A}/\theta$ has power $\geqslant |U| > \mathfrak{m}$, in contradiction to (i).

(xi)⇒(vii) by Corollary 0.7.

(vii)⇒(vi) is clear.

(vi)⇒(x) by Corollary 0.9.

(x)⇒(ix) since every absolute retract in $\mathscr{V}$ is equationally compact (see §0).

(ix)⇒(viii) *a fortiori.*

(ix)⇒(v) and (viii)⇒(iv) by Theorem 0.10.

(v)⇒(ii)⇒(iii)⇒(i) and (iv)⇒(i) are immediate. Q.E.D.

Remarks 1.3. Varieties (and more generally, categories) satisfying condition (vi) of Theorem 1.2 were studied by B. Banaschewski (see Theorem 2.3 and its accompanying remarks below).[4]

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[4]) D. Higgs has given a direct proof that conditions (i) and (vi) of Theorem 1.2 are equivalent [22].

1.4. The varieties of Abelian groups, Boolean algebras, distributive lattices, and semilattices are all residually small, as is essentially well known. Moreover, it is an immediate corollary of [23, Corollary 3.4] that if K is a finite set of finite lattices, then $\mathbf{HSP}K$ has only finitely many subdirect irreducibles (each of which is finite), and hence that $\mathbf{HSP}K$ is residually small. A variety of commutative rings with unit satisfying an additional law of the form $x^n \approx x$ is residually small (see 2.7 below). Any variety of unary algebras is residually small, since any unary algebra may be embedded in a compact topological algebra [40], e.g. in its Stone-Čech compactification.[5])

1.5. The varieties of groups, modular lattices and commutative rings with unit all have arbitrarily large simple algebras, and so are not residually small. There also exist varieties which have only two non-isomorphic simple algebras, and yet are not residually small, as has been shown by R. McKenzie [private communication].

1.6. Notice that the equivalence of conditions (viii) and (ix) of Theorem 1.2 follows immediately from the stronger result of Węglorz [39, Theorem 3.1] that if $\mathfrak{A}$ has an equationally compact extension, then $\mathfrak{A}$ has an equationally compact extension in $\mathbf{HSP}\{\mathfrak{A}\}$. But one cannot deduce Węglorz' result directly from Theorem 1.2, as the next remark shows.

1.7. An algebra $\mathfrak{A}$ may be equationally compact (in fact even a compact topological algebra) and still not be a member of any residually small variety. Such is the case for $\mathfrak{A} = \mathbf{SO}(3)$, the group of all rotations of the 2-sphere. This follows from the fact, essentially due to Hausdorff (see [15] and [4]), that $\mathbf{SO}(3)$ has a subgroup isomorphic to the free group on $\aleph_0$ generators, and the fact that the variety of all groups is not residually small (1.5).

1.8. If $\mathscr{V}$ is residually small and $\mathfrak{A} \in \mathscr{V}$ is written as a subdirect product of subdirectly irreducible algebras $\mathfrak{B}_i \in \mathscr{V}$ $(i \in I)$, then an equationally compact extension of $\mathfrak{A}$, whose existence is asserted in 1.2 (viii), clearly may be constructed by finding an equationally compact extension of each $\mathfrak{B}_i$. One easily checks (by Lemma 0.8) that an (a, b)-irreducible (0.1) algebra $\mathfrak{B}$ is an absolute retract in $\mathscr{V}$ if and only if $\mathfrak{B}$ is $(\mathscr{V}, \subseteq)$-maximal with respect to the property of being (a, b)-irreducible. We will call such algebras $\mathfrak{B}$ *$\mathscr{V}$-maximal irreducible*; their isomorphism types in a given residually small $\mathscr{V}$ form a set, and every subdirectly irreducible algebra has a $\mathscr{V}$-maximal extension, which is equationally compact. In the variety $\mathscr{V}$ of Boolean algebras, semilattices or distributive lattices, the $\mathscr{V}$-maximal irreducibles are the two-element algebras. In the variety $\mathscr{V}$ of Abelian groups, the $\mathscr{V}$-maximal irreducibles are the Prüfer groups Z_{p^∞}.

1.9. It follows from Corollary 0.7 that any $\mathscr{V}$-maximal irreducible algebra has

[5]) D. Higgs has given a direct proof that any variety of unary algebras satisfies condition (ii) of Theorem 1.2 [22]. See Banaschewski [5, 2(5)] for yet a third reason.

power $\leqslant 2^{n}$ (even if $\mathscr{V}$ is not necessarily residually small). According to a general theorem of S. Shelah, in the case of a countable number of operations, a $\mathscr{V}$-maximal irreducible algebra has power $\leqslant \aleph_0$ or exactly $=2^{\aleph_0}$ (see [29]).

1.10. If $\mathscr{V}$ is residually small, then each algebra $\mathfrak{A}\in\mathscr{V}$ has an essentially unique equational compactification, namely, there exists an algebra $\mathfrak{B}$, $\mathfrak{A}\subseteq\mathfrak{B}\in\mathscr{V}$, such that if $\mathfrak{C}$ is equationally compact and $f:\mathfrak{A}\to\mathfrak{C}$ is any homomorphism, then there exists a homomorphism $g:\mathfrak{B}\to\mathfrak{C}$ extending f, and such that there exists no equationally compact algebra $\mathfrak{B}_0$ with $\mathfrak{A}\subseteq\mathfrak{B}_0\subseteq\mathfrak{B}$. Such $\mathfrak{B}$ is unique to within isomorphism fixing the points of $\mathfrak{A}$ [36, §2]. In fact $\mathfrak{B}$ does not depend on $\mathscr{V}$, as may be seen e.g. from the theorem of Węglorz mentioned in 1.6 above. Notice that a maximal essential extension of $\mathfrak{A}$, although equationally compact, may not be a compactification of $\mathfrak{A}$ in the above sense. Thus $\langle Z, x+1\rangle$ is an essential extension of $\langle Z^{+}, x+1\rangle$, where Z (resp. Z^{+}) is the set of all (resp. all positive) integers. But no maximal essential extension of $\langle Z, x+1\rangle$ can be the compactification of $\langle Z^{+}, x+1\rangle$, since this compactification is isomorphic to a disjoint union of $\langle Z^{+}, x+1\rangle$ with another copy of $\langle Z, x+1\rangle$ [36, 2.19]. Notice also that $\mathfrak{A}\in\mathscr{V}$ may have two non-isomorphic maximal essential extensions, as is shown by 2.7 below, although this cannot happen if $\mathscr{V}$ has enough injectives (see §2).

1.11. There exist varieties $\mathscr{V}$ such that no proper variety $\mathscr{V}_0\subseteq\mathscr{V}$ is residually small. Let $\mathscr{V}$ be the variety of algebras

$$\mathfrak{A}=\langle A, \vee, \wedge, 0, 1, a_i\rangle_{i\in\omega},$$

where $\langle A, \vee, \wedge, 0, 1\rangle$ is a lattice with 0 and 1, and the a_i are nullary operations satisfying the laws $a_i\wedge a_j \doteq 0, a_i\vee a_j\doteq 1$ $(i<j<\omega)$. Now suppose that $\mathscr{V}_0\subseteq\mathscr{V}$ is a proper variety, i.e. that there exists $\mathfrak{A}\in\mathscr{V}_0$ with $|A|\geqslant 2$. Clearly no law $a_i\doteq a_j$ $(i\neq j)$ holds in $\mathfrak{A}$, for this law entails $0\doteq 1$ which in turn entails $x\doteq y$. Thus $\mathfrak{A}$ has a subalgebra $\mathfrak{A}_0$ with (infinite) universe $\{0, 1, a_i\}_{i\in\omega}$. Clearly all elementary extensions of $\mathfrak{A}_0$ are subdirectly irreducible, and so $\mathscr{V}_0$ is not residually small. Thus there exist *equationally complete* (i.e. minimal, see e.g. [19, §27]) varieties which are not residually small. But if $\mathscr{V}$ is equationally complete and contains a *primal* algebra [19, §27], then $\mathscr{V}$ is residually small, by e.g. [19, Theorem 27.5] (cf. 2.9 below).

1.12. The upper bound 2^{n} appearing in 1.2 (ii) is best possible. Let A be the Cantor set 2^{ω} of countable sequences of 0's and 1's and let $\mathfrak{A}$ be the unary algebra $\langle A, f, g\rangle$, where

$$f(\langle a_0, a_1, \ldots\rangle)=\langle a_1, a_2, \ldots\rangle$$

and

$$g(\langle a_0, a_1, \ldots\rangle)=\langle a_0, a_0, \ldots\rangle.$$

This $\mathfrak{A}$ is clearly subdirectly irreducible, and the variety of all unary algebras $\langle A, f, g\rangle$ is residually small (1.4). (Two unary operations are necessary, since all subdirectly irreducible algebras with just one unary operation are countable [42].) For any infinite $\mathfrak{n}$ a similar construction yields a subdirectly irreducible algebra of power $2^{\mathfrak{n}}$ having $\mathfrak{n}$ unary operations.

1.13. 1.12 and 1.2 ((i)⇒(ii)) imply that the Hanf number for subdirect irreducibility in varieties is $(2^{\mathfrak{n}})^+$; i.e. that any variety with a subdirectly irreducible algebra of power $>2^{\mathfrak{n}}$ has arbitrarily large subdirectly irreducible algebras, and that $2^{\mathfrak{n}}$ is best possible. By related but subtler arguments, R. McKenzie and S. Shelah have shown that the Hanf number for simplicity in varieties is also $(2^{\mathfrak{n}})^+$ [29]. (Cf. Problem 1.22 below.)

1.14. The upper bound $2^{2^{\mathfrak{n}}}$ appearing in 1.2 (iii) is best possible. Let S be any subset of 2^{ω} and define the unary algebra $\mathfrak{A}_S=\langle A, f, g, 0, h_S\rangle$, where A, f and g are as in 1.12, and

$$O(a) = \langle 0, 0, \ldots\rangle$$
$$h_S(a) = \begin{cases}\langle 1, 1, \ldots\rangle & \text{if } a\in S,\\ \langle 0, 0, \ldots\rangle & \text{otherwise}.\end{cases}$$

Clearly distinct subsets S yield non-isomorphic subdirectly irreducible algebras $\mathfrak{A}_S$. For any infinite $\mathfrak{n}$ a similar construction yields $2^{2^{\mathfrak{n}}}$ non-isomorphic subdirectly irreducible algebras each having $\mathfrak{n}$ unary operations.

1.15. Both upper bounds $2^{\mathfrak{n}}$ appearing in 1.2 (v) are best possible. That the bound $|A|\leqslant 2^{\mathfrak{n}}$ is best possible follows immediately from 1.12. To show that the bound $|K|\leqslant 2^{\mathfrak{n}}$ is best possible it suffices to exhibit $2^{\mathfrak{n}}$ subdirectly irreducible algebras in $\mathscr{V}$ no two of which have a common isomorphic extension. We first do this for $\mathscr{V}$ the variety of unary algebras $\langle A, f_0, f_1, f_2, f_3\rangle$ defined by the law $f_0(x)\approx f_0(y)$. For each subset S of ω define the algebra $\mathfrak{A}_S=\langle \omega, f_0, f_1, f_2, f_{3S}\rangle$, where

$$f_0(n) = 0$$
$$f_1(n) = n+1$$
$$f_2(n) = \begin{cases}0 & \text{if } n = 0\\ n-1 & \text{otherwise},\end{cases}$$
$$f_{3S}(n) = \begin{cases}0 & (n\in S)\\ 1 & (n\notin S).\end{cases}$$

It is easy to check that each $\mathfrak{A}_S$ is subdirectly irreducible and that $\mathfrak{A}_S$ and $\mathfrak{A}_T$ have no common extension when $S\neq T$. A similar construction may be used for arbitrary infinite $\mathfrak{n}$.

1.16. Let $\mathscr{V}$ be the variety defined in 1.15, and suppose that $K_0\subseteq\mathscr{V}$ has the property that every equationally compact algebra in $\mathscr{V}$ is a retract of a product of members of K_0. Extending each $\mathfrak{A}_S$ of 1.15 to a $\mathscr{V}$-maximal irreducible $\mathfrak{B}_S$, which is equationally compact (1.8), we see that each $\mathfrak{A}_S\in\mathbf{ISP}K_0$, and thus as in 1.15 that $|K_0|\geqslant 2^{\aleph_0}$ (or similarly $|K_0|\geqslant 2^{\mathfrak{n}}$ for a variety of algebras with $\mathfrak{n}$ unary operations).

This fact provides a positive answer to Problem 5.9 of [36], which asks whether the upper bound $2^{\mathfrak{n}}$ of Theorem 0.10 (i) above is best possible.

1.17. From 1.15 we see that there exists a residually small variety $\mathscr{V}$ such that for no single algebra $\mathfrak{A}$ is $\mathscr{V} = \mathbf{ISP}\{\mathfrak{A}\}$; i.e. the category of all $\mathscr{V}$-homomorphisms does not have a *cogenerator*. (Such a variety was previously discovered by S. Burris.)

1.18. If $\mathfrak{n} = \aleph_0$, then the upper bound $2^{\mathfrak{n}+|A|}$ of 1.2 (vii) is best possible for any $|A|$. For let $\mathfrak{m}$ be any cardinal, and let X be a set of power $\mathfrak{m}$. Let $\mathfrak{A}$ be the Boolean algebra of finite and cofinite subsets of X, and let $\mathfrak{B}$ be the Boolean algebra of all subsets of X. It is known (and easy to check) that $\mathfrak{B}$ is an essential extension of $\mathfrak{A}$ (see [1] or [7]), $|A| = \mathfrak{m}$ and $|B| = 2^{\mathfrak{m}}$.

Recall that a *quasivariety* [21] is a class of algebras defined by a set of implications $\forall x_1 \dots x_n [(p_1 \simeq q_1 \wedge \cdots \wedge p_k \simeq q_k) \rightarrow (p \simeq q)]$ (such a sentence is called an *identical implication* in [10, p. 148]). Equivalently [21], a quasivariety is a class of similar algebras closed under formation of subalgebras, products, ultraproducts and isomorphic images.

THEOREM 1.19. *If $\mathscr{V}$ is a quasivariety, then the following implications hold among the conditions of Theorem 1.2:*

$$(x) \Rightarrow (ix) \Rightarrow (v) \Rightarrow (vii) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) \Rightarrow (vi).$$

Proof. Similar to the above, except for (v)⇒(vii) and (i)⇒(vi), which are immediate from the following lemma.

LEMMA 1.20. *Suppose that $\mathfrak{B}$ is a subalgebra of a product of algebras each of power $\leqslant \mathfrak{m}$ and that $\mathfrak{B}$ is an essential extension of $\mathfrak{A}$. Then $|B| \leqslant \mathfrak{m}^{|A|^2}$.*

Proof.[6]) Projection to a certain product of only $|A|^2$ of the given factors must separate the points of $\mathfrak{A}$; since $\mathfrak{B}$ is an essential extension of $\mathfrak{A}$, this projection must also separate the points of $\mathfrak{B}$.

Remark 1.21. The implication (vii)⇒(ix) among the conditions of Theorem 1.2 does not hold for quasivarieties. We let $\mathscr{V}$ be the quasivariety of unary algebras $\langle A, f, g \rangle$ defined by the identical implication $(fx \simeq gx \rightarrow y \simeq z)$. $\mathscr{V}$ satisfies condition (vii) by 1.4. Let $\mathfrak{A} \in \mathscr{V}$ be the algebra $\langle \omega, f, g \rangle$, where f and g are the two components of a bijection of ω onto $\omega^2 \backslash \{\langle k, k \rangle : k \in \omega\}$. (This algebra, due to C. Ryll-Nardzewski, was used as an example in [41, Example 21], and later in [36, 3.13].) It may be checked that the equation $fx \simeq gx$ is satisfiable in any extension in $\mathscr{V}$ of $\mathfrak{A}$ which is equationally compact. Thus $\mathscr{V}$ does not satisfy condition (ix).

PROBLEM 1.22. What is the Hanf number for subdirect irreducibility in a quasivariety? (See 1.13 above.)

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[6]) A similar argument is used by D. Higgs in [22].

PROBLEM 1.23. Is every equationally compact algebra a subalgebra of some compact (Hausdorff) topological algebra?

PROBLEM 1.24. Does every residually small variety $\mathscr{V}$ have the property that every algebra in $\mathscr{V}$ is a subalgebra of some compact (Hausdorff) topological algebra?

The last two problems are related to J. Mycielski's question [30, Problem 484] whether every equationally compact algebra is a retract of a compact topological algebra. It is now known that there exists a residually small variety $\mathscr{V}$ such that not every equationally compact algebra in $\mathscr{V}$ is a retract of a compact topological algebra, namely the variety $\mathscr{V}$ of all algebras with two unary operations (see 1.4 and [35]). A negative answer to Problem 1.24 would be stronger than a negative answer to Problem 1.23, which in turn would be stronger than the above mentioned negative solution to the original problem of Mycielski.

PROBLEM 1.25. For which residually small varieties does there exist $n<\omega$ such that 1.2(xi) holds for this n for all φ?

1.26. There exist varieties such as are described in Problem 1.25. For example, if there exists $m<\omega$ such that each subdirectly irreducible algebra in $\mathscr{V}$ has power $\leqslant m$, then $n=m+1$ is as required in Problem 1.25; the proof is very similar to that of 1.2 ((i)$\Rightarrow$(xi)). Thus the varieties of semilattices, distributive lattices and Boolean algebras have the property described in 1.25 with $n=3$.

2. Injective algebras in varieties

We say that the variety $\mathscr{V}$ *has enough injectives* iff every algebra in $\mathscr{V}$ can be embedded in some $\mathscr{V}$-injective. Following Bacsich [2], we say that *injections are transferable* in the variety $\mathscr{V}$ if the following is true: if in the following diagram $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}\in\mathscr{V}$, f is a homomorphism and u is an embedding,

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{u} & \mathfrak{B} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathfrak{C} & \xrightarrow[v]{} & \mathfrak{D} \end{array}$$

then there exists $\mathfrak{D}\in\mathscr{V}$, a homomorphism g and an embedding v such that $v\circ f = g\circ u$. (This is referred to as property (E4) in [5].)

We say the variety $\mathscr{V}$ has the *congruence extension property* iff the following is true: if $\mathfrak{A}\subseteq\mathfrak{B}\in\mathscr{V}$ and θ is a congruence on $\mathfrak{A}$, then there exists a congruence ψ on $\mathfrak{B}$ such that $\theta=\psi\cap A^2$.

We say the variety $\mathscr{V}$ has the *amalgamation property* iff the following is true: whenever u and v are embeddings with the same domain, there exist embeddings i and j such that $i\circ u=j\circ v$.

PROPOSITION 2.1. [2]. *Injections are transferable in $\mathscr{V}$ if and only if $\mathscr{V}$ has the congruence extension property and the amalgamation property.*

The following lemma was known to many people.

LEMMA 2.2. *If injections are transferable in $\mathscr{V}$, then $\mathfrak{A}\in\mathscr{V}$ is $\mathscr{V}$-injective if and only if $\mathfrak{A}$ is an absolute retract in $\mathscr{V}$.*

THEOREM 2.3. *For any variety $\mathscr{V}$, the following conditions are equivalent:*

(i) *$\mathscr{V}$ has enough injectives;*

(ii) *$\mathscr{V}$ is residually small and injections are transferable in $\mathscr{V}$.*

Proof. (i)⇒(ii) by standard arguments from category theory (see e.g. [2], [5], [12] or [13]). (ii)⇒(i) by 1.1, 1.2(x) and 2.2.

Remarks 2.4. Theorem 2.3 was first proved by B. Banaschewski [5], using in 2.3(ii) the equivalent form 1.2(vi) of residual smallness: each $\mathfrak{A}\in\mathscr{V}$ has (within isomorphism) only a set of essential extensions.[7])

2.5. The two conditions of 2.3(ii) are independent. We first note that the variety $\mathscr{V}$ of pseudocomplemented distributive lattices is not residually small [25] (or implicitly [26]), but that injections are transferable in $\mathscr{V}$ [20].[8]) (A distributive lattice $\langle D, \vee, \wedge\rangle$ is pseudocomplemented iff each element $a\in D$ has a pseudocomplement a^*, which is the greatest element b such that $b\wedge a=0$; this property may be expressed by a finite set of lattice equations in the unary operation * [32]. Also see [26] or [25].)

2.6. On the other hand there exist residually small varieties which do not have enough injectives (i.e. in which injections are not transferable). Such are in fact certain varieties of pseudocomplemented distributive lattices (2.5). The lattice of proper subvarieties of the variety of pseudocomplemented distributive lattices is a chain of type $\omega+1$ (whose first two elements are the varieties of Boolean algebras and of Stone algebras, and whose last element is the full variety of pseudocomplemented distributive lattices) [26]. If $\mathscr{V}$ is any variety in this chain other than the first, second, third or last, then $\mathscr{V}$ is residualy small [26] and the amalgamation property fails in $\mathscr{V}$ [20].

2.7. Another example of a residually small variety which does not have enough injectives is provided by B. Banaschewski [5]. We let $\mathscr{V}$ be the variety of commutative rings with unit satisfying the additional law $x^{22}\doteq x$. Clearly no ring in this variety has non-zero nilpotent elements, and hence by a result of G. Birkhoff [9, Lemma 2], the subdirectly irreducible algebras in $\mathscr{V}$ are fields, clearly having $\leqslant 22$ elements. One easily checks that the only such fields satisfying $x^{22}\doteq x$ are GF(2),

[7]) Several variants of Theorem 2.3 (e.g. for hypoinjectivity) have been announced by P. D. Bacsich [private communication].

[8]) The author is indebted to D. Higgs for suggesting the variety needed for 2.5.

GF(4), and GF(8). Thus $\mathscr{V}$ is residually small. Notice that if $\mathscr{V}$ had enough injectives, then GF(8), being $\mathscr{V}$-maximal irreducible and hence an absolute retract in $\mathscr{V}$ by 1.8, would itself be injective in $\mathscr{V}$ by 2.2. But GF(8) is not injective in $\mathscr{V}$ because $GF(2) \subseteq GF(4)$ and the identity embedding of GF(2) in GF(8) cannot be extended to GF(4), since GF(8) has no subfield of four elements.

2.8. Another example of a residually small variety which does not have enough injectives is as follows. Let $\mathscr{V}$ be any variety of lattices other than the variety of distributive lattices or the trivial variety defined by the law $x \approx y$. (Such varieties exist which are residually small, by 1.4) Then a theorem of A. Day [13] says that the only $\mathscr{V}$-injective is the one-element lattice. (In fact the variety $\mathscr{V}$ of modular lattices does not have the amalgamation property [24].)

2.9. A. Day has recently proved that every variety generated by a primal algebra has enough injectives [14].

If $\mathscr{V}$ has enough injectives and $\mathfrak{A} \in \mathscr{V}$, then by 2.3 and 1.2, $\mathfrak{A}$ has only a set of essential extensions in $\mathscr{V}$, and by 0.9 and 2.2, there exist maximal essential extensions $\mathfrak{B}$ of $\mathfrak{A}$, and such $\mathfrak{B}$ must be $\mathscr{V}$-injective. Any two such algebras $\mathfrak{B}$ are isomorphic over $\mathfrak{A}$, and any such $\mathfrak{B}$ is called an *injective envelope* (or *hull*) of $\mathfrak{A}$. (In particular the injective envelope of a subdirectly irreducible $\mathfrak{A} \in \mathscr{V}$ is any $\mathscr{V}$-maximal irreducible extension of $\mathfrak{A}$.) Equivalently, an injective envelope of $\mathfrak{A}$ is any $\subseteq$-minimal $\mathscr{V}$-injective $\supseteq \mathfrak{A}$. (Consult [2], [5], [12], [13] or [14] for these and related facts.) The following theorem was proved in [36, Corollary 2.12], and is included here because these two proofs seem simpler and more direct.

THEOREM 2.10. *If $\mathfrak{B}$ is the $\mathscr{V}$-injective envelope of $\mathfrak{A}$, then $|B| \leqslant 2^{\mathfrak{n}+|A|}$ ($\mathfrak{n} = \aleph_0 +$ the number of operations of $\mathfrak{A}$).*

First proof. Suppose $|B| > 2^{\mathfrak{n}+|A|}$; since $\mathfrak{B}$ is an essential extension of $\mathfrak{A}$, it follows from Corollary 0.7 that

$$(*) \qquad D(\mathfrak{A}) \cup Eq(\mathfrak{B}) \cup \{\varphi(x_\alpha, x_\beta, a, b) : \alpha < \beta < |B|^+\}$$

is consistent, for some $a \neq b \in A$ and positive formula φ such that $\vdash \forall yz[\exists x \varphi(x, x, y, z) \rightarrow y \approx z]$. Let $\mathfrak{C}$ be a model of $(*)$ and let θ be a maximal congruence on $\mathfrak{C}$ separating a and b. Thus $\mathfrak{C}/\theta$ is an essential extension of $\mathfrak{A}$ with $|C/\theta| > |B|$ and $\mathfrak{C}/\theta \in \mathscr{V}$. Since $\mathfrak{B}$ is $\mathscr{V}$-injective, there exists a homomorphism $f : \mathfrak{C}/\theta \rightarrow \mathfrak{B}$ extending the identity function on $\mathfrak{A}$. The fact that $\mathfrak{C}/\theta$ is an essential extension of $\mathfrak{A}$ implies then that f is one-to-one, which is a contradiction.

Second proof. Since $\mathfrak{B}$ is injective, and hence equationally compact, it follows from Theorem 0.10 that $\mathfrak{B}$ is a subalgebra of a product of algebras each of power $\leqslant 2^{\mathfrak{n}}$. Since $\mathfrak{B}$ is an essential extension of $\mathfrak{A}$, the result now follows immediately from Lemma 1.20. Q.E.D.

Remark 2.11. The upper bound on $|B|$ in 2.10 is best possible, as the case of Boolean algebras shows (cf. 1.18 above).

PROBLEM 2.12. Let $\mathscr{V}$ be the variety defined by the set Σ of equations. Does there exist a necessary and sufficient (syntactic) condition on Σ for $\mathscr{V}$ to have enough injectives (e.g. a condition similar to 1.2(xi) or to 3.12(vii) below)?

3. Pure representations

In this section we deal with structures which may have both operations and relations (as mentioned in §0).

DEFINITION 3.1. A *pure representation* of a structure $\mathfrak{A}$ is a family of surjective homomorphisms $f_i:\mathfrak{A}\to\mathfrak{B}_i$ $(i\in I)$ such that the associated homomorphism $f:\mathfrak{A}\to \to\prod_{i\in I}\mathfrak{B}_i$ is a pure embedding. ($\prod$ denotes product.)

As in [36], by an *$(\exists, \wedge)$-formula* we mean a first order formula whose only logical connectives are $\exists$, $\wedge$ (and $\approx$).

LEMMA 3.2. *The family of homomorphisms $f_i:\mathfrak{A}\to\mathfrak{B}$ $(i\in I)$ is a pure representation of $\mathfrak{A}$ if and only if the following holds: if φ is an $(\exists, \wedge)$-formula in the language of $\mathfrak{A}$, $a_i, \ldots, a_n\in A$ and $\mathfrak{A}\models\neg\varphi[a_1, \ldots, a_n]$, then there exists $i\in I$ such that $\mathfrak{B}_i\models\neg\varphi[f_i(a_1), \ldots, f_i(a_n)]$.*

DEFINITION 3.3. The structure $\mathfrak{A}$ is *pure-irreducible* iff the following holds: if the family of homomorphisms f_i $(i\in I)$ is a pure representation of $\mathfrak{A}$, then for some $i\in I, f_i$ is an isomorphism.

LEMMA 3.4. *$\mathfrak{A}$ is pure-irreducible if and only if there exists an $(\exists, \wedge)$-formula φ and $a_1, \ldots, a_n\in A$ such that $\mathfrak{A}\models\neg\varphi[a_1, \ldots, a_n]$ and such that $\mathfrak{B}\models\varphi[f(a_1), \ldots, f(a_n)]$ whenever $f:\mathfrak{A}\to\mathfrak{B}$ is a homomorphism which is not one-to-one.*

3.5. Any subdirectly irreducible algebra is pure-irreducible, but the converse is false. If $\mathfrak{A}$, for example, is a three-element lattice, then $\mathfrak{A}$ is pure-irreducible, since the middle element has a complement in any proper homomorphic image; but $\mathfrak{A}$ is not subdirectly irreducible.

The following theorem is a counterpart for pure embeddings to G. Birkhoff's subdirect representation theorem of 1944.

THEOREM 3.6. *Any structure $\mathfrak{A}$ has a pure representation $f_i:\mathfrak{A}\to\mathfrak{B}_i$ $(i\in I)$ with each $\mathfrak{B}_i$ pure-irreducible.*

Proof. For each $(\exists, \wedge)$-formula φ and $a_i, \ldots, a_n \in A$ such that $\mathfrak{A} \models \neg \varphi [a_1, \ldots, a_n]$, let θ be a maximal congruence on $\mathfrak{A}$ such that $\mathfrak{A}/\theta \models \neg \varphi [a_1/\theta, \ldots, a_n/\theta]$. Clearly the family of homomorphisms $\mathfrak{A} \to \mathfrak{A}/\theta$, with θ ranging over all possible θ defined as above, is a pure representation of $\mathfrak{A}$ via pure-irreducible structures.

We turn now to our counterpart for pure embeddings of Theorem 1.2 (see Theorem 3.12 below). The following notion extends the notion of pure-essential extension in the theory of Abelian groups (see [18] and references given there).

DEFINITION 3.7. $\mathfrak{B}$ is a *pure-essential* extension of $\mathfrak{A}$ iff $\mathfrak{B}$ is a pure extension of $\mathfrak{A}$, and for any proper congruence θ on $\mathfrak{B}$, the natural homormophism $\mathfrak{A} \to \mathfrak{B}/\theta$ is not a pure embedding.

LEMMA 3.8. *Let $\mathfrak{B}$ be a pure-essential extension of $\mathfrak{A}$, $|B| > 2^{\mathfrak{m}}$, where $\mathfrak{m} = \aleph_0 + |A|$ + the number of operations and relations of $\mathfrak{A}$). Then there exists an $(\exists, \wedge)$-formula φ in the language of $\mathfrak{A}$ having v_0 and v_1 as its only free variables and having constants from $\mathfrak{A}$, such that $\mathfrak{A} \models \neg \exists x \varphi(x, x)$ and such that*

$$(*) \qquad D(\mathfrak{A}) \cup NE(\mathfrak{A}) \cup \{\varphi(x_i, x_j) : i < j < \omega\}$$

is consistent. (Here $D(\mathfrak{A})$ is the diagram of $\mathfrak{A}$ and $NE(\mathfrak{A})$ is the set of negations of those $(\exists, \wedge)$-sentence with constants from $\mathfrak{A}$ which fail to be satisfied in $\mathfrak{A}$.)

The proof is similar to that of Corollary 0.7 and thus is omitted.

COROLLARY 3.9. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be as in* 3.8 *and let φ be as supplied by* 3.8. *Then*

$$Th(\mathfrak{A}_A) \cup \{\varphi(x_i, x_j) : i < j < \omega\}$$

is consistent, where $Th(\mathfrak{A}_A)$ is the set of all sentences with constants from $\mathfrak{A}$ which are true in $\mathfrak{A}$.

Proof. Follows immediately from Lemma 3.8 and the fact (due to C. Ryll-Nardzewski – see [38]) that if $\mathfrak{C}$ is a pure extension of $\mathfrak{A}$, then there is a homomorphism $g: \mathfrak{C} \to \mathfrak{D}$ such that $g \restriction \mathfrak{A}$ is an elementary embedding. (In fact one may take $\mathfrak{D}$ an ultrapower of $\mathfrak{A}$, with $g \restriction \mathfrak{A}$ the natural embedding of $\mathfrak{A}$ into $\mathfrak{D}$.)

LEMMA 3.10 *Let $\mathfrak{B}$ be a pure-essential extension of $\mathfrak{A}$. Then $\mathfrak{B}$ is atomic-compact if and only if no proper pure extension of $\mathfrak{B}$ is a pure-essential extension of $\mathfrak{A}$.*

The proof is similar to that of Lemma 0.8 and thus is omitted.

COROLLARY 3.11. *If $\mathfrak{A}$ has (within isomorphism) only a set of pure-essential extensions, then some pure-essential extension of $\mathfrak{A}$ is atomic-compact.*

The proof is similar to that of Corollary 0.9 and thus is omitted 9).

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9) The present proof of 3.11 is due to B. Banaschewski and E. Nelson, who also suggested Lemma 3.10. They pointed out a mistake in a previous proof.

THEOREM 3.12. *Let $\mathscr{K}$ be an elementary class of structures (i.e. the class of models of a set Γ of first order sentences), and let* $\mathfrak{n} = (\aleph_0 +$ *the number of operations and relations of structures in* $\mathscr{K}$*). Then conditions* (iv)–(vii) *below are equivalent. If in addition $\mathscr{K}$ is a variety, then all seven conditions are equivalent.*

(i) *There exists a cardinal $\mathfrak{m}$ such that every pure-irreducible structure in $\mathscr{K}$ has power $\leqslant \mathfrak{m}$;*

(ii) *every pure-irreducible structure in $\mathscr{K}$ has power $\leqslant 2^{\mathfrak{n}}$;*

(iii) *there exists a set $K \subseteq \mathscr{K}$ with $|K| \leqslant 2^{\mathfrak{n}}$ and $|A| \leqslant 2^{\mathfrak{n}}$ for all $\mathfrak{A} \in K$, such that every structure in $\mathscr{K}$ is isomorphic to a pure substructure of some product of members of K;*

(iv) *each $\mathfrak{A} \in \mathscr{K}$ has (within isomorphism) only a set of pure-essential extensions;*

(v) *if $\mathfrak{B}$ is any pure-essential extension of $\mathfrak{A} \in \mathscr{K}$, then $|B| \leqslant 2^{\mathfrak{n}+|A|}$;*

(vi) *every structure $\mathfrak{A} \in \mathscr{K}$ is a pure substructure of some atomic-compact structure;*

(vii) *for every $(\exists, \wedge)$-formula φ in the language of $\mathscr{K}$, there exists a finite number n such that*

$$\Gamma \models \forall u_2 \dots u_m [\exists x_1 \dots x_n \bigwedge_{1 \leqslant i < j \leqslant n} \varphi(x_i, x_j, u_2, \dots, u_m) \to \exists x_0 \varphi(x_0, x_0, u_2, \dots, u_m)].$$

Proof. In case $\mathscr{K}$ is a variety, the proof proceeds *via* (i)$\Rightarrow$(vii)$\Rightarrow$(v)$\Rightarrow$(iv)$\Rightarrow$(vi)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i).

(i)$\Rightarrow$(vii). If (vii) is false, then the set of formulas

$$(*) \qquad \Gamma \cup \{\neg \exists x_0 \varphi(x_0, x_0, a_2, \dots, a_m)\} \cup \{\varphi(x_\alpha, x_\beta, a_2, \dots, a_m) : \alpha < \beta < \mathfrak{m}^+\}$$

is consistent. If $\mathfrak{A}$ is a model of $(*)$ and θ is a maximal congruence on $\mathfrak{A}$ subject to $\mathfrak{A}/\theta \models \neg \exists x_0 \varphi(x_0, x_0, a_2/\theta, \dots, a_m/\theta)$, then by 3.4, $\mathfrak{A}/\theta$ is a pure-irreducible model of Γ of power $> \mathfrak{m}$, in contradiction to (i).

(vii)$\Rightarrow$(v) by Corollary 3.9.
(v)$\Rightarrow$(iv) is clear.
(iv)$\Rightarrow$(vi) by Corollary 3.11.
(vi)$\Rightarrow$(iii) by [36, Corollary 5.8].
(iii)$\Rightarrow$(ii)$\Rightarrow$(i) is easy.

To complete the proof, we note that if $\mathscr{K}$ is not necessarily a variety, then we may prove (vi)$\Rightarrow$(vii) as follows. Assume that (vii) is false for some φ. Thus every finite subset of

$$(**) \qquad \begin{cases} \{\neg \exists x_0 \varphi(x_0, x_0, b_2, \dots, b_m)\} \\ \cup \{\exists x_1 \dots x_n \bigwedge_{1 \leqslant i < j \leqslant n} \varphi(x_i, x_j, b_2, \dots, b_m) : n < \omega\} \end{cases}$$

has a model in $\mathscr{K}$. Since $\mathscr{K}$ is elementary, there is a model $\mathfrak{A} \in \mathscr{K}$ of $(**)$. By [36, Theorem 3.8(ii)$\Rightarrow$(v)], this $\mathfrak{A}$ is not a pure substructure of any atomic-compact structure. Q.E.D.

Remarks 3.13. If $\mathscr{K}$ is a variety, then the equivalence of conditions (iv) and (vi) of Theorem 3.12 follows from Proposition 5 of [5], together with the remarks in §3 of [5]. (One applies Banaschewski's more general version of our Theorem 2.3 above; one need only check that Banaschewski's condition (E4) is satisfied for the class of pure embeddings, i.e. that 'pure injections are transferable,' in the terminology of §2 above. This fact follows fairly directly from the ultrapower construction of Ryll-Nardzewski mentioned in the proof of Corollary 3.9 above.) Similar and related remarks occur in [2].

3.14. Both the variety of Abelian groups and the variety of Boolean algebras satisfy the equivalent conditions of Theorem 3.12. J. Łoś in effect proved [27] that the pure-irreducible Abelian groups are precisely the groups Z_{p^n} (p prime, $n=1, 2, \ldots, \infty$). The only pure-irreducible Boolean algebra is the two-element Boolean algebra, as follows from Stone's representation theorem and the fact that every embedding of Boolean algebras is a pure embedding [36, 2.20] [2, §4].

3.15. It has been proved by B. Banaschewski [6] that the equivalent conditions of Theorem 3.12 are satisfied for the variety $\mathscr{V}_G$ of permutation representations of an arbitrary but fixed group G. (Each member of $\mathscr{V}_G$ is a unary algebra of the form $\langle A, \bar{g}\rangle_{g\in G}$ subject to the laws

$$1(x) \doteq x \quad (1 = \text{unit of } G)$$
$$\bar{g}(\bar{h}(x)) \doteq (\overline{gh})(x) \quad (g, h \in G).$$

Members of $\mathscr{V}_G$ are also called G-sets.) J. Mycielski has pointed out that from Theorems 1 and 5 of [31] one may immediately deduce the stronger fact that every member of $\mathscr{V}_G$ has an equationally compact elementary extension. We remark that it is not difficult to directly verify the syntactic condition (vii) of Theorem 3.12 for Γ taken as the equational theory Σ_G of $\mathscr{V}_G$, as follows. If φ is any $(\exists, \wedge)$-formula, then φ is logically equivalent to $\exists x_1 \ldots \exists x_n(\alpha_1 \wedge \ldots \wedge \alpha_m)$, where each α_i is an equation. If no equation α_i involves both x_n and x_k $(k \neq n)$, then φ is logically equivalent to the conjunction of a sentence and a formula of the form $\exists x_1 \ldots \exists x_{n-1}\psi$, where ψ is a conjunction of equations. The alternative is that some α_i is Σ_G-equivalent to $x_n \doteq \bar{g}_i(x_k)$ $(k \neq n)$. If we replace each occurrence of x_n in each α_j by $\bar{g}(x_k)$, we see that φ is Σ_G-equivalent to $\exists x_1 \ldots \exists x_{n-1}\psi$, where ψ is a conjunction of equations. Thus by induction we see that every $(\exists, \wedge)$-formula is Σ_G-equivalent to the conjunction of a sentence and an open $(\exists, \wedge)$-formula. Thus we need to verify 3.12 (vii) only for open $(\exists, \wedge)$-formulas, i.e. for conjunctions $\alpha_1 \wedge \cdots \wedge \alpha_s$ of equations. The reader may easily check that if no α_i is Σ_G-equivalent to $x_0 \doteq \bar{g}(x_1)$, then 3.12(vii) holds with $n=2$. But if such an equation does occur, then 3.12(vii) holds with $n=3$, because

$$\Sigma_G \vdash \forall x_1 x_2 x_3 \left[(x_1 \doteq \bar{g}(x_2) \wedge x_1 \doteq \bar{g}(x_3)) \rightarrow x_2 \doteq x_3\right].$$

The problem remains of making some other direct applications of 3.12(vii) (or of 1.2(xi) above).

3.16. The variety of all algebras with two unary operations is residually small, but does not satisfy the equivalent conditions of Theorem 3.12. It is not hard to check, for example, that the algebra $\mathfrak{A}$ described in 1.21 above is not a pure subalgebra of any equationally compact algebra [41, Example 21].

3.17. Another residually small variety not satisfying the equivalent conditions of Theorem 3.12 is the variety of distributive lattices. We present an example due to R. McKenzie to show that there exist pure-irreducible distributive lattices of arbitrarily large infinite power. Given infinite $\mathfrak{m}$ we first define a partially ordered set $(P, \leqslant)$ as follows: $P=\{x_\alpha, y_\alpha, u_\alpha, v_\alpha : \alpha<\mathfrak{m}\}$, $y_\alpha<x_\alpha$, $y_{\alpha+1}<x_\alpha$, $v_\alpha<u_\alpha$, $v_{\alpha+1}<u_\alpha$. Then define L as the set of all subsets $J\subseteq P$ with the following five properties:

(a) J is an ideal, i.e. if $x\in J$ and $y\leqslant x$ then $y\in J$;

(b) $J_1=\{\alpha : x_\alpha\in J\}$ is closed, i.e. if $K\subseteq J_1$ and $\bigcup K\in\mathfrak{m}$, then $\bigcup K\in J_1$;

(c) $J_2=\{\alpha : u_\alpha\in J\}$ is closed;

(d) if $x_0\in J$, then there exists $\beta\in\mathfrak{m}$ such that $u_\alpha\in J$ whenever $\beta<\alpha<\mathfrak{m}$;

(e) if $u_0\in J$, then there exists $\beta\in\mathfrak{m}$ such that $x_\alpha\in J$ whenever $\beta<\alpha<\mathfrak{m}$.

One easily checks that L is closed under finite union and intersection, and thus that $\langle L, \cup, \cap\rangle$ is a distributive lattice. Now we define $A, B\in L$ as $A=\{y_0\}$ and $B=\{v_0\}$. We first claim that $\langle L, \cup, \cap\rangle\models\neg\varphi[\varnothing, P, A, B]$, where φ is $\exists XY(X\cap Y\doteq x_0 \wedge X\cup Y\doteq x_1 \wedge X\cap x_2\doteq x_2 \wedge Y\cap x_3\doteq x_3)$. For suppose that our claim is false. Since $y_0\in A\subseteq X$ and Y is an ideal, we cannot have $x_0\in Y$. Thus $x_0\in X$ and similarly $u_0\in Y$. By (d) there exists α such that $u_\alpha\in X$, and thus we may let β be the smallest member of $\{\alpha : u_\alpha\in X\}$. By (c) β cannot be a limit ordinal, and thus $\beta=\gamma+1$ for some γ. Thus $v_{\gamma+1}\leqslant u_{\gamma+1}\in X$ and $v_{\gamma+1}\leqslant u_\gamma\in Y$, and thus $v_{\gamma+1}\in X\cap Y$. This contradiction establishes the above claim. We next claim that if $f:\langle L, \cup, \cap\rangle\to\langle M, \vee, \wedge\rangle$ is any homomorphism with $|M|<2^{\mathfrak{m}}$, then $\langle M, \vee, \wedge\rangle\models\varphi[f(\varnothing), f(P), f(A), f(B)]$. Notice first that there are $2^{\mathfrak{m}}$ subsets of $K=\{y_{\alpha+1} : \alpha<\mathfrak{m}\}$ (each of which satisfies conditions (a)–(e)). Thus there exist $J, \bar{J}\subseteq K$, $J\neq\bar{J}$ with $f(J)=f(\bar{J})$. Without loss of generality we assume that $y_{\gamma+1}\in J$, $y_{\gamma+1}\notin\bar{J}$. And so $f(\{y_{\gamma+1}\})=f(\{y_{\gamma+1}\}\cap J)=f(\{y_{\gamma+1}\}\cap\bar{J})=f(\varnothing)$. Likewise there exists $\delta<\mathfrak{m}$ such that $f(\{v_{\delta+1}\})=f(\varnothing)$. Now we let

$$X=\{x_\alpha : \alpha\leqslant\gamma\}\cup\{y_\alpha : \alpha\leqslant\gamma+1\}\cup\{u_\alpha : \alpha>\delta\}\cup\{v_\alpha : \alpha>\delta\}$$

and

$$Y=\{u_\alpha : \alpha\leqslant\delta\}\cup\{v_\alpha : \alpha\leqslant\delta+1\}\cup\{x_\alpha : \alpha>\gamma\}\cup\{y_\alpha : \alpha>\gamma\}.$$

It is not hard to check that $X, Y\in L$, $X\supseteq A$, $Y\supseteq B$ and $X\cup Y=P$. One also easily checks that $X\cap Y=\{y_{\gamma+1}, v_{\delta+1}\}$ and so $f(X\cap Y)=f(\{y_{\gamma+1}\}\cup\{v_{\delta+1}\})=f(\{y_{\gamma+1}\})\vee \vee f(\{v_{\delta+1}\})=f(\varnothing)\vee f(\varnothing)=f(\varnothing)$. And so we have checked the second claim above. Thus finally we see that if θ is a maximal congruence on $\langle L, \cup, \cap\rangle$ subject to $\langle L, \cup,$

$\cap\rangle/\theta \models \varphi[\varnothing/\theta, P/\theta, A/\theta, B/\theta]$, then $\langle L, \cup, \cap\rangle/\theta$ is pure-irreducible and has power $\geqslant 2^{\mathfrak{m}}$.

3.18. Notice that in contrast to the situation described in Remark 1.7 above, there is a local version of Theorem 3.12. Namely one may apply the theorem in case $\mathscr{K}$ is the class of structures elementarily equivalent to a given structure $\mathfrak{A}$. The resulting theorem is the same as [36, Theorem 3.8(iii)$\Leftrightarrow$(v)], with the addition of conditions 3.12(iv) and 3.12(v).

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Added in proof, December 31, 1971

J. A. Kalman has recently proved that the variety of *distributive quasillattices* is residually small [Fund. Math. *71* (1971), 161–163].

W. Nemitz and T. Whaley have proved, *inter alia*, that the variety of *implicative semilattices* is not residually small, but that any finite set of finite implicative semilattices generates a residually small variety [Pacific J. Math. *37* (1971), 759–769]. (Cf. the similar facts for lattices mentioned in 1.4-5 above.)

See[7] for the fact that every non-trivial lattice has arbitrarily large essential extensions which are lattices (and thus that all injectives in the variety of all lattices are trivial).

A detailed presentation of Simpson's model-theoretic proof [33] of Erdős' Theorem 0.6 above appears in *Model Theory for Infinitary Logic*, by H. J. Keisler, North-Holland, 1971, pp. 75–77.

E. Fisher and P. D. Bacsich have proved [private communication] that conditions (x) and (vi) of Theorem 1.19 are equivalent for quasivarieties $\mathscr{V}$ if essential extensions are understood as relative to (the category) $\mathscr{V}$. Thus in Example 1.21, the algebra $\mathfrak{A}$ has arbitrarily large extensions which are essential relative to $\mathscr{V}$.

It is implicit in 2.5 and 2.6 above that varieties of pseudocomplemented distributive lattices other than the first three do not have enough injectives. This was originally proved by A. Day, who also proved that the first three do have enough injectives; see G. Grätzer and H. Lakser, *The structure of pseudocomplemented distributive lattices.* III: *injectives and absolute subretracts* (to appear). The existence of enough injective Stone algebras was proved by R. Balbes and G. Grätzer [Duke Math. J. *38* (1971), 339–347].

We take this opportunity to mention some results in the literature related to results of [36]. The existence of a maximal compact representation of a topological group (i.e. [36, Theorem 6.1] for groups) was proved by E. M. Alfsen and P. Holm [Math. Scand. *10* (1962), 127–136]. The existence of enough injective semilattices, mentioned in [36, 2.14], was also proved by A. Horn and N. Kimura [Alg. Univ. *1* (1971), 26–38]. The existence of enough injective distributive lattices, mentioned in [36, 2.14], is also implicit in R. Balbes [Pacific J. Math. *21* (1967), 405–420].