Implication Algebras are 3-Permutable and 3-Distributive

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In this note we give some examples to a question of G. Grätzer [3] about Mal'cev-type conditions. In Universal Algebra there are several results of the following form.

THEOREM 1. *For any equational class* $\mathfrak{A}$ *the statements in each of the following pairs are equivalent to each other.*

Permutability ([3], [6], *and* [7])

(Pa) *The congruence relations of every algebra of* $\mathfrak{A}$ *are n-permutable (of* Θ*-type* $n-1$*).*

(Pb) *There exist* $(n+1)$*-ary algebraic operations* $\bar{p}_0, \ldots, \bar{p}_n$ *of* $\mathfrak{A}$ *satisfying the following identities:*

$$\bar{p}_0(x_0, \ldots, x_n) = x_0$$
$$\bar{p}_{i-1}(x_0, x_0, x_2, x_2, \ldots) = \bar{p}_i(x_0, x_0, x_2, x_2, \ldots) \quad (i \text{ even})$$
$$\bar{p}_{i-1}(x_0, x_1, x_1, x_3, x_3, \ldots) = \bar{p}_i(x_0, x_1, x_1, x_3, x_3, \ldots) \quad (i \text{ odd})$$
$$\bar{p}_n(x_0, \ldots, x_n) = x_n.$$

Distributivity (B. Jónsson [5]*)*

(Da) *The congruence lattice of every algebra of* $\mathfrak{A}$ *is distributive.*

(Db) *There exist a natural number n and ternary algebraic operations* $\bar{p}_0, \ldots, \bar{p}_n$ *of* $\mathfrak{A}$ *satisfying the following identities:*

$$\bar{p}_i(x_0, x_1, x_0) = x_0 \quad (1 \leqslant i \leqslant n)$$
$$\bar{p}_0(x_0, x_1, x_2) = x_0$$
$$\bar{p}_{i-1}(x_0, x_0, x_2) = \bar{p}_i(x_0, x_0, x_2) \quad (i \text{ odd})$$
$$\bar{p}_{i-1}(x_0, x_2, x_2) = \bar{p}_i(x_0, x_2, x_2) \quad (i \text{ even})$$
$$\bar{p}_n(x_0, x_1, x_2) = x_2.$$

Modularity (A. Day [2]*)*

(Ma) *The congruence lattice of every algebra in* $\mathfrak{A}$ *is modular.*

(Mb) *There exist a natural number n and* 4*-ary algebraic operations* $\bar{p}_0, \ldots, \bar{p}_n$ *of* $\mathfrak{A}$ *satisfying the following identities:*

$$\bar{p}_i(x_0, x_1, x_1, x_0) = x_0 \quad (1 \leqslant i \leqslant n)$$
$$\bar{p}_0(x_0, x_1, x_2, x_3) = x_0$$
$$\bar{p}_{i-1}(x_0, x_0, x_2, x_2) = \bar{p}_i(x_0, x_0, x_2, x_2) \quad (i \text{ odd})$$
$$\bar{p}_{i-1}(x_0, x_1, x_1, x_3) = \bar{p}_i(x_0, x_1, x_1, x_3) \quad (i \text{ even})$$
$$\bar{p}_n(x_0, x_1, x_2, x_3) = x_3.$$

For a general theory of this type of theorem see R. Wille [8].

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We define an equational class to be *n-permutable* for some natural number n if there exist $(n+1)$-ary algebraic operations $\bar{p}_0, \ldots, \bar{p}_n$ of $\mathfrak{A}$ which satisfy the identities of Theorem 1 (Pb). Analogously we call an equational class $\mathfrak{A}$ *n-distributive* (*n-modular*) for some natural number n if there exist ternary (4-ary) operations of $\mathfrak{A}$ satisfying the identities of (Db) ((Mb)). It is known that an n-permutable (n-distributive, n-modular) equational class is m-permutable (m-distributive, m-modular) for every natural number m greater than n.

In [3] G. Grätzer asks for examples of equational classes which show that n-permutability and $(n+1)$-permutability are not equivalent and poses the same question for n-modularity, n-distributivity and other results of this form. The following theorem gives an answer to this question for $n=2$.

THEOREM 2. *The equational class of all implication algebras is*

(P) *3-permutable, but not 2-permutable.*
(M) *3-modular, but not 2-modular.*
(D) *3-distributive, but not 2-distributive.*

Remark: E. T. Schmidt has shown that for every natural number $n \geqslant 2$ there exists an $(n+1)$-permutable equational class, which is not n-permutable (preprint, Bonn 1970).

B. Jónsson gave in [5] an example of a 3-distributive equational class, which is not 2-distributive.

The referee shortened my proof of (D) by giving another counter-example, which I shall use in the following proof.

For the following definition and properties of implication algebras see J. C. Abbott [1].

An *implication algebra* is a pair $\langle I, \cdot \rangle$ consisting of a carrier set I closed under a binary operation $\cdot$ (we write ab instead of $a \cdot b$) satisfying the identities

(I1) $(ab)\,a = a$
(I2) $(ab)\,b = (ba)\,a$
(I3) $a(bc) = b(ac)$.

From the definition follows the existence of a unique element 1 with the properties $aa=1$, $a1=1$, $1a=a$ for every $a \in I$. Furthermore every implication algebra $\langle I, \cdot \rangle$ determines a partially ordered set $\langle I, \leqslant \rangle$ with greatest element 1 by: $a \leqslant b$ iff $ab=1$. With respect to the partial ordering I is a join semi-lattice, where $(ab)\,b$ is the least upper bound for a and b. Every principal filter of this semi-lattice is a boolean algebra. Conversely, every join semi-lattice, in which every principal filter is a boolean algebra, determines an implication algebra under $ab = (a \vee b)'_b$, where $(a \vee b)'_b$ is the complement of $a \vee b$ in the principal filter $[b)$. In particular any boolean algebra with 0 deleted is an implication algebra.

Proof of theorem 2:

(P) 3-*permutable:*

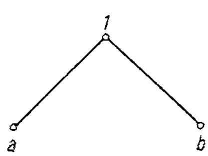
If we define algebraic operations $\bar{p}(x, y, z, u) = (zy)\, x$ and $\bar{q}(x, y, z, u) = (yz)\, u$ we get:

$\bar{p}(x, y, y, z) = (yy)\, x = 1x = x,$
$\bar{p}(x, x, z, z) = (zx)\, x = (xz)\, z$ by (I2),
$\bar{q}(x, x, z, z) = (xz)\, z$ and
$\bar{q}(x, y, y, z) = (yy)\, z = z$

satisfying the identities of condition (Pb) of theorem 1 for $n = 3$.

not permutable:

We consider the implication algebra



Then $\Theta = \{\{a, 1\}, \{b\}\}$ and $\Phi = \{\{b, 1\}, \{a\}\}$ are congruences, if we denote congruences by the partitions they induce. We get $(a, b) \in \Theta \circ \Phi$ and $(a, b) \notin \Phi \circ \Theta$, while the condition that $\Theta_1 \circ \Theta_2 = \Theta_2 \circ \Theta_1$ holds for every pair of congruences is equivalent to 2-permutability.

(M) 3-*modular:*

By properties of congruences one can show that 3-modularity follows from 3-permutability (see Jónsson [4], theorem 1.2.). Another way to show this is by using results of theorem 1. If $\bar{p}$, $\bar{q}$ are 4-ary algebraic operations of an equational class with the properties $\bar{p}(x, y, y, z) = x$, $\bar{q}(x, y, y, z) = z$, $\bar{p}(x, x, y, y) = \bar{q}(x, x, y, y)$ then the class is 3-permutable.
If we define 4-ary algebraic operations $\bar{r}$, $\bar{s}$ by

$\bar{r}(x, y, z, u) = \bar{p}(x, \bar{p}(x, y, z, u), \bar{q}(x, y, z, u), u)$

and

$\bar{s}(x, y, z, u) = \bar{q}(x, \bar{q}(u, z, y, x), \bar{p}(u, z, y, x), u)$

then $\bar{r}$ and $\bar{s}$ satisfy the identities for 3-modularity. In particular, for implication algebras we get the operations

$\bar{r}(x, y, z, u) = \{(zy)[((yz)\,u)\,x]\}\,x$

and

$\bar{s}(x, y, z, u) = \{(yz)\,[((zy)\,x)\,u]\}\,u.$

not 2-modular:

In [2, theorem 2] Day has shown that an equational class is 2-modular if and only if it is permutable. For the class of implication algebras we have shown that it is not permutable, it follows that it cannot be 2-modular.

(D) *3-distributive:*

We consider the ternary algebraic operations

$\bar{p}(x, y, z) = (y(zx))\,x$ and $\bar{q}(x, y, z) = (xy)\,z.$

Then we get the following identities

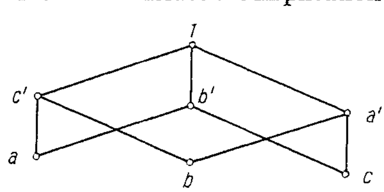
$\bar{p}(x, x, z) = (x(zx))\,x = (z(xx))\,x = x,$
$\bar{p}(x, y, x) = (y1)\,x = x,$
$\bar{p}(x, z, z) = (z(zx))\,x = (zx)\,x = (xz)\,z$

because $z(zx) = ((zx)\,z)\,(zx) = zx$ by (I1),

$\bar{q}(x, z, z) = (xz)\,z,$
$\bar{q}(x, x, z) = (xx)\,z = z,$
$\bar{q}(x, y, x) = (xy)\,x = x.$

not 2-distributive:

Let $\Theta(a, b)$ be the least congruence relation collapsing a and b and $[a]\,\Theta\,(a, b)$ the congruence class of $\Theta(a, b)$ containing a. For any equational class $\mathfrak{A}$ condition (Db) of theorem 1 for $n=2$ holds if and only if for every algebra $A \in \mathfrak{A}$ and all $a, b, c \in A$ $[a]\,\Theta(a, b) \cap [b]\,\Theta(b, c) \cap [c]\,\Theta(a, c) \neq \emptyset$ (see Wille [8], theorem 6.6.). Now we consider the implication algebra



and the congruences $\Theta(a, b)$, $\Theta(b, c)$, $\Theta(a, c)$. Then it is easily seen (see the definition of $\cdot$ on page 80) that $[a]\,\Theta(a, b) = \{a, b, c'\}$, $[b]\,\Theta(b, c) = \{b, c, a'\}$ and $[c]\,\Theta(a, c) = \{a, c, b'\}$, which completes the proof.

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