The Category of Semilattices[1])

Alfred Horn and Naoki Kimura

1. Introduction and Preliminaries

In this paper, we will determine the injective and projective semilattices. Moreover explicit constructions will be given for the injective hull and projective cover (when it exists) of a semilattice. Thus the category of semilattices is seen to be unusually amenable and a good source of illustrations for the concepts of category theory.

A *semilattice* is an algebra $\langle A,\cdot\rangle$ with a binary operation which is associative, commutative and idempotent. Any partially ordered set in which any two elements have a greatest lower bound (meet) $x\wedge y$ is a semilattice under $\wedge$. Dually, a partially ordered set in which least upper bounds $x\vee y$ exist is a semilattice under $\vee$. If A is a semilattice then A has a canonical ordering defined by $x\leqslant y$ if and only if $xy=x$. In this ordering, $x\wedge y$ exists and is equal to xy. This ordering will always be used unless otherwise stated.

For future reference we name some particular semilattices. The two element semilattice $\{0, 1\}$ in which $0\cdot 1=0$ is denoted by $\underset{\sim}{2}$. If X is a set, $S(X)$ denotes the semilattice of all subsets of X under *intersection*. $S(X)$ is of course a direct product of copies of $\underset{\sim}{2}$. Subalgebras of $S(X)$ are called *set semilattices*. They are of course families of sets closed under finite intersections. $S_\omega(X)$ denotes the semilattice under *union* of the family of all nonempty finite subsets of X.

The category of semilattices is denoted by $\mathscr{S}$. An $\mathscr{S}$ morphism is a homomorphism of semilattices. If f is an $\mathscr{S}$ morphism, then $x\leqslant y$ implies $f(x)\leqslant f(y)$. If f is one-to-one, then $f(x)\leqslant f(y)$ implies $x\leqslant y$.

A semilattice in which $x\vee y$ always exists (in the canonical ordering) is called a *lattice*. A lattice in which $x\cdot(y\vee z)=xy\vee xz$ is called a *distributive* lattice. An equivalent form of the distributive law is the dual form $x\vee yz=(x\vee y)\cdot(x\vee z)$. An element x of a semilattice is called *meet irreducible* if $x=yz$ implies $x=y$ or $x=z$. x is called *super meet irreducible* (*s.m.i.* for short) if $x\geqslant yz$ implies $x\geqslant y$ or $x\geqslant z$. Clearly s.m.i. elements are meet irreducible. If A is a distributive lattice, then every meet irreducible element is s.m.i. If P is a partially ordered set and $x\in P$, let x_P denote the principal ideal $\{y\in P: y\leqslant x\}$ and let x^P be the principal filter $\{y\in P: y\geqslant x\}$. A *filter* in a semilattice A is a subset F such that (1) if $y\geqslant x\in F$ then $y\in F$, and (2) if $x\in F$ and $y\in F$, then $xy\in F$. The smallest and largest elements of a semilattice (if they exist) are denoted by 0 and 1

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[1]) This research was supported in part by National Science Foundation Grants GP-9044 and GP-7429.

Presented by G. Grätzer. Received March 25, 1970. Accepted for publication June 24, 1970.

respectively. If A is a semilattice let $0*A=A$ if A has a 0, while if A has no 0, let $0*A$ denote the result of adding to A an element 0 such that $0\cdot x=x\cdot 0=0$ for all $x\in A$. It is obvious that A is a subalgebra of the semilattice $0*A$. Similarly, we let $A*1$ denote the result of adding a 1 to A in case A does not already have a 1. A partially ordered set is called *discrete* if $x\leqslant y$ only when $x=y$.

A morphism $f:A\to B$ is called a *monomorphism*, or an *embedding* if for any two morphisms $g_1, g_2: C\to A$, $fg_1=fg_2$ implies $g_1=g_2$. f is called an *epimorphism* if $g_1 f=g_2 f$ implies $g_1=g_2$ for any morphisms $g_1, g_2: B\to C$.

THEOREM 1.1. *Let $f:A\to B$ be an $\mathscr{S}$ morphism. Then*

(1) *f is a monomorphism if and only if f is one-to-one*

(2) *f is an epimorphism if and only if f is onto.*

Proof. The sufficiency is obvious in (1) and (2). Suppose f is not one-to-one. There exist distinct $a, b\in A$ such that $f(a)=f(b)$. Let C be any semilattice and define $g_1, g_2: C\to A$ by $g_1(x)=a$ and $g_2(x)=b$ for all x. Then $fg_1=fg_2$ but $g_1\neq g_2$. This proves (1).

Suppose f is not onto. Let $C=f(A)$. Then C is a proper subalgebra of B. Now a function $g:B\to 2$ is a homomorphism if and only if $\{x:g(x)=1\}$ is a filter. Therefore we need only find distinct filters F_1, F_2 in B such that $F_1\cap C=F_2\cap C$. There exists an $x\in B-C$. Let F_1 be the principal filter x^B, and let $F_2=\{y\in B\text{: for some } z\in C, y\geqslant z>x\}$. Clearly F_2 is a filter and $F_1\cap C=F_2\cap C$. But $F_1\neq F_2$ since $x\in F_1-F_2$.

2. Injective Semilattices

There is a very simple representation theorem for semilattices.

THEOREM 2.1 *Every semilattice A is isomorphic with a set semilattice. Specifically, there exists a monomorphism $g:A\to S(A)$.*

Proof. Define g by $g(x)=x^A$. It is obvious that $g(xy)=g(x)\cap g(y)$ and g is one-to-one.

As an immediate consequence, we have the following corollary.

COROLLARY 2.2. *The only non-trivial subdirectly irreducible semilattice is $\underset{\sim}{2}$.*

DEFINITION 2.3. A is called a *retract* of B if there exist homomorphisms $f:B\to A$ and $g:A\to B$ such that $fg=I_A$.

DEFINITION 2.4. A semilattice A is called *injective* (more exactly $\mathscr{S}$ injective) if for every $\mathscr{S}$ monomorphism $f:B\to C$ and every homomorphism $g:B\to A$, there exists a homomorphism $h:C\to A$ such that $hf=g$.

DEFINITION 2.5. A is called an *absolute subretract* if for every $\mathscr{S}$ monomorphism $g: A \to B$ there exists a homomorphism $f: B \to A$ such that $fg = I_A$. In other words, A is a retract of every semilattice in which it can be embedded.

When dealing with arbitrary equational classes of algebras, it is more natural to define monomorphisms (or epimorphisms) to be homomorphisms which are one-to-one (or onto). In our general remarks on equational classes, we shall always use this convention. The following facts are easy to prove in any equational class of algebras. A direct product of injective algebras is injective. A retract of an injective algebra is injective. Every injective algebra is an absolute subretract. We shall see that the converse holds for $\mathscr{S}$. Also it is obvious that every factor of a direct product A of semilattices is a retract of A. Therefore a direct product of semilattices is injective if and only if each of them is injective.

We shall now determine the injective semilattices. First some definitions.

DEFINITION 2.6. A semilattice A is called *complete* if it is complete in its canonical ordering, that is, every subset S of A has a least upper bound $\bigvee(S)$ and a greatest lower bound $\bigwedge(S)$. A sufficient condition for the completeness of A is that A has a 1 and $\bigwedge(S)$ exists for every nonempty $S \subseteq A$.

DEFINITION 2.7. A complete semilattice A is called $(2, \infty)$ *distributive* if for any family $\langle a_i : i \in I \rangle$ of elements of A and any $a \in A$, we have $a \cdot \bigvee_{i \in I} a_i = \bigvee_{i \in I} aa_i$. If A is $(2, \infty)$ distributive, then

$$\bigvee_{i \in I} a_i \bigvee_{j \in J} b_j = \bigvee_{\substack{i \in I \\ j \in J}} a_i b_j$$

for any families $\langle a_i : i \in I \rangle$ and $\langle b_j : j \in J \rangle$.

THEOREM 2.8. *If A is a semilattice then A is injective if and only if A is complete and* $(2, \infty)$ *ditributive.*

Proof. Suppose A is injective. Then A is an absolute subretract, and so the monomorphism $g: A \to S(A)$ of Theorem 2.1 has a left inverse $f: S(A) \to A$. Let $\langle a_i : i \in I \rangle$ be any family of elements of A. It is easy to check that $f(\bigcup_{i \in I} g(a_i))$ is the least upper bound of the family, and $f(\bigcap_{i \in I} g(a_i))$ is its greatest lower bound. Also, for any $a \in A$,

$$\bigvee_{i \in I} aa_i = f\Big(\bigcup_{i \in I} g(aa_i)\Big) = f\Big(\bigcup_{i \in I} (g(a) \cap g(a_i))\Big) = f\Big(g(a) \cap \bigcup_{i \in I} g(a_i)\Big)$$
$$= fg(a) \cdot f\Big(\bigcup_{i \in I} g(a_i)\Big) = a \cdot \bigvee_{i \in I} a_i .$$

This proves the necessity of the conditions.

Conversely suppose A is complete and $(2, \infty)$ distributive. Let $f: B \to C$ be an $\mathscr{S}$ monomorphism and $g: B \to A$ be a homomorphism. Define $h: C \to A$ as follows:

$$h(x) = \bigvee \{g(z) : z \in B \quad \text{and} \quad f(z) \leqslant x\}.$$

If $x_1, x_2 \in C$, then $h(x_1 x_2) \leqslant h(x_1)\, h(x_2)$ since h clearly preserves order. Also by $(2, \infty)$ distributivity, $h(x_1)\, h(x_2) = \vee(T)$, where

$$T = \{g(z_1) \cdot g(z_2) : z_1, z_2 \in B, f(z_1) \leqslant x_1, f(z_2) \leqslant x_2\}.$$

But $T \subseteq \{g(z) : z \in B, f(z) \leqslant x_1 x_2\}$. Therefore $h(x_1)\, h(x_2) \leqslant h(x_1 x_2)$, and h is a homomorphism. Finally, for any element z_0 of B, we have

$$\begin{aligned} h(f(z_0)) &= \vee\{g(z) : z \in B, f(z) \leqslant f(z_0)\} \\ &= \vee\{g(z) : z \in B, z \leqslant z_0\} = g(z_0), \end{aligned}$$

since f is a monomorphism. The proof is complete.

COROLLARY 2.9. *A finite semilattice is injective if and only if it is a distributive lattice.*

An immediate consequence of 2.8 is the following.

THEOREM 2.10. *$\mathscr{S}$ is injectively complete, that is, for every semilattice A there exists a monomorphism f of A into an injective semilattice.*

Proof. By Theorem 2.1 there exists a monomorphism of A into $S(A)$, which is certainly injective by Theorem 2.8.

Now it is clear that in an injectively complete category, every absolute subretract is injective. Therefore a semilattice is injective if and only if it is an absolute subretract.

Injective semilattices exist in great profusion. It is well known that the class of $(2, \infty)$ distributive complete lattices coincides with the class of complete Heyting algebras. Thus examples of injective semilattices are finite distributive lattices and the set semilattice of all open sets of a topologic space.

There is another interesting consequence of the fact that $\mathscr{S}$ is injectively complete. Namely, not only do free products (coproducts) exist in $\mathscr{S}$ but also free products with amalgamated subalgebras. See the (not quite accurate) discussion in [2, p. 109].

3. Injective Hulls

From the general theory of categories we have the following definitions.

DEFINITION 3.1. A monomorphism $f : A \to B$ is called *essential* if for every morphism $g : B \to C$ we have: if gf is a monomorphism then g is a monomorphism.

DEFINITION 3.2. If A is a semilattice then an *injective hull* of A is an essential monomorphism of A into an injective semilattice B. We shall also refer to B itself as an injective hull of A.

It is more convenient here to use the language of extensions rather than monomorphisms. We say B is an *extension* of A if A is a subalgebra of B. B is an *essential* extension of A if the identity monomorphism of A into B is essential.

The following facts can be proved in any injectively complete equational class of algebras with finitary operations. See for example the treatment in [1, p. 91]. If B is an essential extension of A and C is an essential extension of B, then C is an essential extension of A. If C is an essential extension of A then C is an essential extension of any intermediate algebra. An algebra A is injective if and only if it has no proper extension which is essential. An extension B of A is an injective hull of A if and only if B is a maximal essential extension of A (that is, no extension of B is an essential extension of A). Also B is an injective hull of A if and only if B is a monimal injective extension of A (that is, no intermediate algebra is injective). Every injective extension of A contains an injective hull of A. If B and B' are injective hulls of A, then B and B' are isomorphic over A.

In order to construct the injective hull of a semilattice, we need the following definitions.

DEFINITION 3.3. A family $\langle a_i : i \in I \rangle$ of elements of a semilattice A is said to be *distributive* if (1) $\bigvee_{i\in I} a_i$ exists, and (2) for any $a \in A$, $\bigvee_{i \in I} aa_i$ exists and is equal to $a \cdot \bigvee_{i\in I} a_i$.

LEMMA 3.4. *If $\langle a_i : i \in I \rangle$ is distributive, then $\langle aa_i : i \in I \rangle$ is distributive for any $a \in A$. If $\{a_{ij} : j \in I_i\}$ is distributive for each $i \in I$, $a_i = \bigvee_{j\in I_i} a_{ij}$ and $\{a_i : i \in I\}$ is distributive, then $\{a_{ij} : i \in I, j \in I_i\}$ is distributive.*

DEFINITION 3.5. An $\mathscr{S}$ monomorphism $f: A \to B$ is called a *regular* embedding if

(1) If $a_i \in A$ for all $i \in I$ and $\bigwedge_{i\in I} a_i$ exists, then $\bigwedge_{i \in I} f(a_i)$ exists and is equal to $f(\bigwedge_{i\in I} a_i)$.

(2) If $\{a_i : i \in I\}$ is distributive, then $\bigvee_{i\in I} f(a_i)$ exists and is equal to $f(\bigvee_{i\in I} a_i)$.

DEFINITION 3.6. A nonempty subset J of a semilattice A is called a *d-ideal* provided:

(1) J is hereditary, that is, if $x \in J$ and $y \leqslant x$, then $y \in J$,

(2) If $\langle a_i : i \in I \rangle$ is distributive, and $a_i \in J$ for all $i \in I$, then $\bigvee_{i\in I} a_i \in J$.

LEMMA 3.7. *Let B be the set of all d-ideals of a semilattice A with an 0. Then B is closed under arbitrary intersections and has a largest member. If $\mathscr{C}(S)$ denotes the smallest d-ideal containing S, then for any nonempty hereditary subset S of A, we have*

$$\mathscr{C}(S) = \{x : x = \bigvee_{i \in I} x_i \text{ for some family } \langle a_i : i \in I \rangle \text{ contained in } S\}. \tag{1}$$

Proof. The first part of the lemma is obvious. Let J denote the right hand side of (1). If $J' \in B$ and $J' \supseteq S$ then $J' \supseteq J$. It remains only to show that J is a d-ideal. Suppose $y \leqslant x \in J$. Then $x = \bigvee_{i \in I} x_i$, where $\langle x_i : i \in I \rangle$ is distributive and $x_i \in S$ for all i. Therefore $y = xy = \bigvee_{i \in I} x_i y$ and $\langle x_i y : i \in I \rangle$ is distributive by Lemma 3.4. Also $x_i y \in S$ for all i, since S is hereditary. Thus $y \in J$ and we have shown that J is hereditary. Using Lemma 3.4, it is easily seen that J is closed under joins of distributive families. Thus J is a d-ideal and the proof is complete.

THEOREM 3.8. *Let A be a semilattice with* 0 *and let B be the set of all d-ideals of A. Let $f : A \to B$ be defined by $f(x) = x_A$. Then the set semilattice B is the injective hull of A and f is a regular embedding.*

Proof. By Lemma 3.7, B is a complete set semilattice, the join of a family $\langle J_i : i \in I \rangle$ of d-ideals being $\mathscr{C}(\bigcup_{i \in I} J_i)$. To prove that B is $(2, \infty)$ distributive, suppose $J \in B$ and $J \in B$ for all $i \in I$. If x is any member of $J \cap \bigvee_{i \in I} J_i$, then $x \in J$ and $x \in \mathscr{C}(\bigcup_{i \in I} J_i)$. By Lemma 3.7. $x = \bigvee_{k \in K} x_k$, where $\langle x_k : k \in K \rangle$ is distributive and $x_k \in \bigcup_{i \in I} J_i$ for all k. Therefore $x_k \in J \cap \bigcup_{i \in I} J_i = \bigcup_{i \in I} (J \cap J_i)$ and so $x \in \mathscr{C}(\bigcup_{i \in I} (J \cap J_i)) = \bigvee_{i \in I} (J \cap J_i)$. The converse inclusion $\bigvee_{i \in I} (J \cap J_i) \subseteq J \cap \bigvee_{i \in I} J_i$ is obvious. Thus B is injective. It is obvious that f is a monomorphism and $f(\bigwedge_{i \in I} a_i) = \bigcap_{i \in I} f(a_i)$ whenever $\bigwedge_{i \in I} a_i$ exists. Suppose $\langle a_i : i \in I \rangle$ is distributive, and $a = \bigvee_{i \in I} a_i$. Then $f(a) \supseteq f(a_i)$ for all i. Also if $J \in B$ and $J \supseteq f(a_i)$ for all i, then $a_i \in J$ for all i, which implies $a \in J$ and therefore $f(a) \subseteq J$. This proves that f is a regular embedding. It remains to prove f is essential. Let $g : B \to C$ be an $\mathscr{S}$ morphism such that gf is a monomorphism. We first prove:

(i) If $a \in A$, $J \in B$, and $g(f(a)) = g(J)$, then $a = \bigvee(J)$.

If $x \in J$, then $g(f(ax)) = g(f(a)) \cdot g(f(x)) = g(J) \cdot g(f(x)) = g(f(x))$ since $f(x) \subseteq J$. Therefore $ax = x$, or $x \leqslant a$ for all $x \in J$. Suppose $x \leqslant b$ for all $x \in J$. Then $J \subseteq f(b)$ and therefore

$$g(f(ab)) = g(f(a)) \cdot g(f(b)) = g(J) \cdot g(f(b)) = g(J) = g(f(a)).$$

Hence $ab = a$ and so $a \leqslant b$. Thus $a = \bigvee(J)$. Next we show:

(ii) If $a \in A$, $J \in B$, and $g(f(a)) = g(J)$, then $J = f(a)$.

If $x \in A$, then

$$g(f(xa)) = g(f(x)) \cdot g(f(a)) = g(f(x)) \cdot g(J) = g(f(x) \cap J).$$

Therefore by (*i*), $xa = \bigvee(f(x) \cap J)$. Arrange the members of J in a sequence $\langle x_k : k \in K \rangle$. Then by (i), $a = \bigvee_{k \in K} x_k$. It is easy to check that

$$J \cap f(x) = \{x x_k : k \in K\}.$$

Therefore $xa = \bigvee(f(x) \cap J) = \bigvee_{k \in K} x x_k$. This shows that $\langle x_k : k \in K \rangle$ is distributive. Since f is a regular embedding, it follows that $f(a) = \bigvee_{k \in K} f(x_k)$. But it is obvious that $J = \bigvee_{k \in K} f(x_k)$. This proves (ii).

Now we are ready to prove that g is a monomorphism. Suppose $g(J_1)=g(J_2)$, where $J_1, J_2 \in B$. If $a \in J_1$, then

$$g(f(a)) = g(f(a))\cdot g(J_1) = g(f(a))\cdot g(J_2) = g(f(a) \cap J_2).$$

By (ii), it follows that $f(a)=f(a) \cap J_2$. Thus $f(a) \subseteq J_2$ for all $a \in J_1$. This proves $J_1 \subseteq J_2$ and similarly $J_2 \subseteq J_1$. The proof is complete.

THEOREM 3.9. *If A is any semilattice, then $0 * A$ is an essential extension of A and therefore the injective hull of A is the same as that of $0 * A$.*

Proof. We may assume A has no 0. Suppose $g: 0 * A \to C$ is any $\mathscr{S}$ morphism such that $g \mid A$ is a monomorphism but $g(0)=g(a)$ for some $a \in A$. There exists $b \in A$ such that $a \nleqslant b$, so that $a \neq ab$. But $g(0)=g(0\cdot b)=g(ab)$, so that $g(a)=g(ab)$. Since $g \mid A$ is a monomorphism, we have a contradiction.

Theorems 3.8 and 3.9 give an explicit construction for the injective hull of any semilattice. If A has no 0, we may describe its injective hull as the set semilattice of all d-ideals of A together with the empty set. As an illustration of this construction, it is easy to see that if A is a set semilattice containing all one-element subsets of a set X, then the injective hull of A is $S(X)$.

We close this section with the following remark.

THEOREM 3.10. *If A and B are semilattices with 0 and A' and B' are there spective injective hulls of A and B, then the injective hull of $A \times B$ is $A' \times B'$.*

Proof. We may assume A' and B' are extensions of A and B. Then $A' \times B'$ is an injective extension of $A \times B$. Suppose $g: A' \times B' \to C$ is an $\mathscr{S}$ morphism such that $g \mid (A \times B)$ is a monomorphism. Suppose (x_1, y_1) and (x_2, y_2) are in $A' \times B'$ and $g(x_1, y_1) = g(x_2, y_2)$ but $(x_1, y_1) \neq (x_2, y_2)$. Without loss of generality, suppose $x_1 \neq x_2$, and say $x_1 \nleqslant x_2$. Since every d-ideal of A is a union of principal ideals, every element x of A' is the least upper bound of all elements of A which are $\leqslant x$. Therefore exists $a \in A$ such that $a \leqslant x_1$ but $a \nleqslant x_2$. Also there exist $b_1, b_2 \in B$ such that $b_1 \leqslant y_1$ and $b_2 \leqslant y_2$. Let $b=b_1 b_2$. Let $h: A' \to C$ be defined by $h(x)=g(x, b)$. Then $h \mid A$ is a monomorphism. Therefore h is itself a monomorphism. But

$$\begin{aligned} h(a) = g(a, b) &= g((a, b)\cdot(x_1, y_1)) = g((a, b)\cdot(x_2, y_2)) \\ &= g(ax_2, b) = h(ax_2). \end{aligned}$$

This implies that $a=ax_2$, or $a \leqslant x_2$, which is a contradiction. This completes the proof.

If both A and B have no 0, the injective hull of $A \times B$ is $(A'-\{0\}) \times (B'-\{0\}) \cup \cup \{(0, 0)\}$, and there is a corresponding result in the case where exactly one of A, B has a 0. We omit the proofs which are quite similar to that of Theorem 3.10.

4. Free Semilattice Over a Partially Ordered Set

If A is a semilattice and X is a subset of A then X is said to freely generate A if X generates A and any function $f:X\to B$, $B\in\mathscr{S}$, can be extended to a homomorphism $g:A\to B$. We shall need the following generalization of this concept.

THEOREM 4.1. *Let P be any partially ordered set. There exists a semilattice P^* containing P such that P generates P^* and every order preserving function $f:P\to B$, $B\in\mathscr{S}$, can be extended to a homomorphism $f^*:P^*\to B$. The set of s.m.i. elements of P^* coincides with P. P^* is unique up to isomorphism over P.*

Proof. Let P^* be the set of all nonempty finite discrete subsets of P. If $S_1, S_2\in P^*$ define $S_1\leqslant S_2$ if every member of S_2 is $\geqslant$ a member of S_1. This is clearly a partial ordering and $S_1\wedge S_2$ is the set of minimal elements of $S_1\cup S_2$. Hence $\langle P^*, \wedge\rangle$ is a semilattice. Let $h:P\to P^*$ be defined by $h(x)=\{x\}$. Then h is an order embedding, that is, $x\leqslant y$ if and only if $h(x)\leqslant h(y)$. Now suppose $f:P\to B$ is an order preserving function, where $B\in\mathscr{S}$. Define $f^*:P^*\to B$ as follows: $f^*(S)=\bigwedge(f(S))$. For any $S_1, S_2\in P^*$,

$$f^*(S_1)\cdot f^*(S_2)=\bigwedge(f(S_1))\cdot\bigwedge(f(S_2))=\bigwedge(f(S_1\cup S_2))$$
$$\leqslant\bigwedge(f(S_1\wedge S_2))=f^*(S_1\wedge S_2),$$

since $S_1\cup S_2\supseteq S_1\wedge S_2$. Since f^* clearly preserves order we have $f^*(S_1\wedge S_2)\leqslant f^*(S_1)\times\times f^*(S_2)$, and therefore f^* is a homomorphism. Also for any $x\in P$, $f^*h(x)=f^*(\{x\})=f(x)$.

If $x\in P$, $S_1\in P^*$, $S_2\in P^*$ and $\{x\}\geqslant S_1\wedge S_2$, then x is $\geqslant$ a member of $S_1\cup S_2$. Therefore $\{x\}\geqslant S_1$ or $\{x\}\geqslant S_2$, and so $\{x\}$ is s.m.i. The uniqueness of P^* follows in the usual way from the fact that $h(P)$ generates P^* since $S=\bigwedge_{x\in S}\{x\}$ for any $S\in P^*$.

Note that if P is already a semilattice, then P^* may not coincide with P. For example if $P=\{0, a, b\}$, where $0<a$, $0<b$ and a and b are incomparable, then P^* has four members $\{0\}$, $\{a\}$, $\{b\}$ and $\{a, b\}=\{a\}\cdot\{b\}$.

THEOREM 4.2. *Let A be a semilattice which is generated by the set M of its s.m.i. elements. Then $A=M^*$.*

Proof. Let $f:M^*\to A$ be defined by $f(S)=\bigwedge(S)$. Since M generates A, f is onto. f is a homomorphism because $\bigwedge(S_1\wedge S_2)=\bigwedge(S_1\cup S_2)$ for any $S_1, S_2\in M^*$. Suppose $f(S_1)=f(S_2)$ and $x\in S_1$. Then $x\geqslant\bigwedge(S_1)=\bigwedge(S_2)$. Since x is s.m.i., x must be $\geqslant$ some member of S_2. Thus $S_1\geqslant S_2$ and similarly $S_2\leqslant S_1$. This proves that f is one-to-one and is thus an isomorphism.

THEOREM 4.3. *Every free semilattice is isomorphic to $S_\omega(X)$ for some X.*

Proof. A free semilattice is of the form X^*, where X is a discrete partially ordered set. In this case X^* consists of all nonempty finite subsets of X and $S_1 \wedge S_2 = S_1 \cup S_2$ for all $S_1, S_2 \in X^*$.

COROLLARY 4.4. *Every free semilattice is isomorphic with the set semilattice of all proper cofinite subsets of a set X.*

Proof. The canonical ordering of X^* is exactly the opposite of the inclusion relation. If we map subsets of X into their complements this will also reverse the inclusion relation.

5. Projective Semilattices

DEFINITION 5.1. A semilattice A is called projective if for every $\mathcal{S}$ epimorphism $f: C \to B$ and every homomorphism $g: A \to B$ there exists a homomorphism $h: A \to C$ such that $fh = g$.

For any equation class of algebras, the following statements are easy to prove. Every free algebra is projective. A retract of a projective algebra is projective. An algebra A is projective if and only if it is an absolute quotient retract, that is, every epimorphism onto A has a right inverse. An algebra is projective if and only if it is a retract of a free algebra.

DEFINITION 5.2. A *ring of sets* is a family R of sets such that if $A, B \in R$, then $A \cup B \in R$ and $A \cap B \in R$.

The following theorem describes the projective semilattices.

THEOREM 5.3. *For any semilattice A, the following are equivalent:*

(1) *A is projective,*

(2) (a) *for all $x \in A$, x^A is finite, and*

(b) *If $x, y, z \in A$ and $x \geqslant yz$, then either $x \geqslant y$, or $x \geqslant z$, or $x = x_1 x_2$ for some x_1, x_2 such that $x_1 \geqslant y$ and $x_2 \geqslant z$,*

(3) (a) *For all $x \in A$, x^A is finite, and*

(b) *$A * 1$ is a distributive lattice,*

(4) *$A * 1$ is isomorphic with $\langle R, \cup \rangle$, where R is a ring of finite sets,*

(5) *If M is the set of s.m.i. elements of A, then M generates A and for each $x \in M$, x^M is finite.*

(6) *$A = P^*$ for some partially ordered set P such that x^P is finite for all $x \in P$.*

Proof. (1) → (2). If A is projective, then A is a retract of a free semilattice. Therefore for some X there exist homomorphisms $f: S_\omega(X) \to A$ and $g: A \to S_\omega(X)$ such that $fg = J_A$. Since the canonical order of $S_\omega(X)$ is opposite to the order by inclusion, $g(x^A)$ is a set of subsets of $g(x)$. But g is necessarily one-to-one and $g(x)$ is finite. Therefore x^A must be finite. Now suppose $x \geqslant y \cdot z$. Then $g(x) \geqslant g(y) \cdot g(z)$, and so

$g(x)\subseteq g(y)\cup g(z)$. Therefore

$$g(x)=(g(x)\cap g(y))\cup(g(x)\cap g(z)).$$

If $g(x)\cap g(y)$ is empty, then $g(x)\subseteq g(z)$ and therefore $x=fg(x)\geqslant fg(z)=z$. Similarly if $g(x)\cap g(z)$ is empty, then $x\geqslant y$. Otherwise, let $x_1=f(g(x)\cap g(y))$ and $x_2=f(g(x)\cap \cap g(z))$. Then $x=fg(x)=x_1\cdot x_2$, and $x_1\geqslant fg(y)=y$, $x_2\geqslant fg(z)=z$.

(2)→(3). Let $x, y\in A*1$. Then $\{z\in A*1:z\geqslant x \text{ and } z\geqslant y\}$ is finite, nonempty and closed under $\cdot$. Therefore $x\vee y$ exists, and $A*1$ is a lattice. Next we show that $A*1$ satisfies the dual form of the distributive law, that is,

$$(x\vee y)(x\vee z)=x\vee yz.$$

This holds if $x=1$, $y=1$ or $z=1$. Since $x\vee yz\geqslant yz$, we have $x\vee yz=x_1\cdot x_2$ where $x_1\geqslant y$ and $x_2\geqslant z$ (x_1 or x_2 may be 1). Since $x_1\geqslant x$ and $x_2\geqslant x$, we have $x\vee yz=x_1\cdot x_2\geqslant \geqslant(x\vee y)\cdot(x\vee z)$. The reverse inequality $x\vee yz\leqslant(x\vee y)\cdot(x\vee z)$ is obvious and this completes the proof.

(3)→(4). We first show that every element of $A*1$ is a finite product of meet irreducible elemets. If some element x is not so expressible, then $x=y\cdot z$ where $y>x$, $z>x$, and either y or z is not so expressible. Continuing, we obtain an infinite chain of elements $>x$, which contradicts (3) (a). Let M be the set of all meet irreducible elements of $A*1$. Define $f:A*1\to S_\omega(M)$ by $f(x)=\{y\in M:y\geqslant x\}$. By the first remark, f is one-to-one. Now $f(x_1x_2)=f(x_1)\cup f(x_2)$ for any $x_1, x_2\in A$, since every element of M is s.m.i. in $A*1$ by (3) (b). Also $f(x_1\vee x_2)=f(x_1)\cap f(x_2)$, obviously. Thus $A*1$ is isomorphic with $f(A*1)$, which is a ring of finite sets.

(4)→(3) is obvious.

(3)→(5). In the proof of (3)→(4), we saw that every element of $A*1$ is a product of s.m.i. elements of $A*1$. But every element of A which is s.m.i. in $A*1$ is s.m.i. in A. It is now obvious that (5) holds.

(5)→(6) is an immediate consequence of Theorem 4.2.

(6)→(1) There exists an epimorphism $f:F\to P^*$, where F is a free semilattice. For each $x\in P$, select an element $g(x)$ such that $fg(x)=x$. Let $h:P\to F$ be defined by

$$h(x)=\wedge\{g(y):y\geqslant x\}.$$

Then h is order preserving and $fh(x)=x$ for all $x\in P$. Extend h to a homomorphism $h^*:P^*\to F$. Since P generates P^*, we have $fh^*=I_{P^*}$. Thus P^* is a retract of F and P^* is projective.

COROLLARY 5.4. *If A is a finite semilattice, then A is projective if and only if $A*1$ is a distributive lattice.*

Proof. This follows immediately from (3) of Theorem 5.3.

DEFINITION 5.5. *If A is a semilattice, let $\bar{A}=0*A*1$.*

THEOREM 5.6. *If A is a projective semilattice, then $\bar{A}$ is injective.*

Proof. Let X be a set of the same cardinal as A, and let $F=S_\omega(X)$. Then $\bar{F}$ is the semilattice, under union, of all finite subsets of X together with X itself. $\bar{F}$ is certainly complete, and if $a_i \in \bar{F}$ for all $i \in I$, then in the canonical ordering, $\bigwedge_{i\in I} a_i = \bigcap_{i\in I} a_i$. Therefore for any $a \in \bar{F}$, we have

$$a \cdot \bigvee_{i\in I} a_i = a \cup \bigcap_{i\in I} a_i = \bigcap_{i\in I} (a \cup a_i) = \bigvee_{i\in I} aa_i.$$

By Theorem 2.8, $\bar{F}$ is injective. Since F is free and A is projective, there exist homomorphsims $f: F \to A$ and $g: A \to F$ such that $fg = I_A$. Extend f to a homomorphism $\bar{f}: \bar{F} \to \bar{A}$ by defining $\bar{f}(0)=0$ and $\bar{f}(1)=1$. This definition is consistent since if F already has a 0, then so does A because f is onto. Extend g to a homomorphism $\bar{g}: \bar{A} \to \bar{F}$ by defining $\bar{g}(0)=0$ if A has no 0, and $\bar{g}(1)=1$ if A has no 1. Clearly $\bar{f}\bar{g}=I_{\bar{A}}$. Thus $\bar{A}$ is a retract of $\bar{F}$ and so $\bar{A}$ is injective.

THEOREM 5.7. *A semilattice A is a homomorphic image of an injective semilattice if and only if A has a 0 and a 1.*

Proof. The necessity is obvious. Suppose A has a 0 and a 1. There exists an epimorphism $f: F \to A$, where F is a free semilattice. f can be extended to an epimorphism $\bar{f}: \bar{F} \to \bar{A}$ by defining $\bar{f}(0)=0$ and $\bar{f}(1)=1$. By Theorem 5.6, $\bar{F}$ is injective and the proof is complete.

We close this section with the following example.

THEOREM 5.8. *There exists a projective semilattice A such that $A \times A$ is not projective.*

Proof Let A be the three-element semilattice $\{0, a, b\}$, where $a\cdot 0 = b \cdot 0 = a \cdot b = 0$. Then A is projective by Theorem 5.8(3), since $A*1$ is the four element Boolean algebra. However $(A \times A)*1$ is not distributive since

$$(0, a)\cdot((a, 0) \vee (b, 0)) = (0, a)\cdot 1 = (0, a),$$

while

$$(0, a)\cdot(a, 0) \vee (0, a)\cdot(b, 0) = (0, 0).$$

Therefore by Theorem 5.8(3), $A \times A$ is not projective.

6. Projective Covers

The notion of projective cover is obtained by dualizing the concept of injective hull.

DEFINITION 6.1. An epimorphism $f: B \to A$ is called *tight* if for every morphism $g: C \to B$ we have: if fg is an epimorphism then g is an epimorphism.

It is easily seen that an epimorphism $f: B \to A$ is tight if and only if for every proper subalgebra C of $B, f(C) \neq A$.

DEFINITION 6.2. A projective cover of a semilattice A is a tight epimorphism from a projective algebra B onto A. We shall also refer to B as a projective cover of A. The following theorem gives the uniqueness of projective covers.

THEOREM 6.3. *If $f: B \to A$ and $f': B' \to A$ are projective covers of A, then there exists an isomorphism $h: B' \to B$ such that $f' = fh$.*

Proof. Since B is projective, there exists a homomorphism $g: B \to B'$ such that $f = f'g$. g is an epimorphism because f' is tight. B' being an absolute quotient retract, there exists a monomorphism $h: B' \to B$ such that $gh = I_{B'}$. Now $fh = f'gh = f'$, and f is tight. Therefore h is onto and h is an isomorphism.

In contrast to the situation for injective hulls, not every semilattice has a projective cover even though every semilattice is a homomorphic image of a free semilattice. The following theorem describes the situation.

THEOREM 6.4. *Let A be a semilattice and let M be the set of meet irreducible elements of A. Then A has a projective cover if and only if M generates A and for each $x \in M$, x^M is finite (compare with Theorem 5.3(5)). In this case, the projective cover of A is M^*.*

Proof. Suppose $f: B \to A$ is a projective cover. Let N be the set of s.m.i. elements of B. To prove that M generates A, we need only to show $f(N) \subseteq M$. Let $x \in N$ and let C be the subalgebra of B which is generated by $N - \{x\}$.

If $f(x) \in f(C)$, then $f(N) \subseteq f(C)$ and so $f(C) = A$. Since f is tight, $C = B$ and so $x \in C$. Therefore $x = x_1 \cdot \cdots \cdot x_n$, where $x_i \in N$ and $x_i \neq x$ for all i. This contradicts the fact that x is meet irreducible. Hence $f(x) \notin f(C)$. Now suppose $f(x) \notin M$. Then there exist u, v such that $f(x) = f(u) \cdot f(v), f(u) > f(x)$ and $f(v) > f(x)$. But then $uv \notin C$. Since N generates B, we have

$$uv = x \cdot x_1 \cdot \cdots \cdot x_n,$$

Where $x_i \in N - \{x\}$. Hence $x \geqslant uv$ and therefore $x \geqslant u$ or $x \geqslant v$. This implies $f(x) \geqslant f(u)$ or $f(x) \geqslant f(v)$, which is a contradiction. This proves that M generates A.

Next we shall show that for each $x \in M$, x^A is finite. Suppose x^A is infinite for some $x \in M$. For each $a \in A$, choose an element $g(a)$ such that $fg(a) = a$. Let $T = \{g(a): a > x\}$. By Theorem 5.3(2), only finitely many elements of T are $\geqslant g(x)$. Since T is infinite, there exists an $a_0 \in A$ such that $a_0 > x$ and $g(a_0) \ngeqslant g(x)$. Let $S = \{g(a): a \neq x\} \cup \{g(a_0) \cdot g(x)\}$. Then $f(S) = A$ since $f(g(a_0) \cdot g(x)) = a_0 \cdot x = x$. Therefore S generates B because f is tight. Hence

$$g(x) = g(a_1) \cdot \cdots \cdot g(a_n)$$

where $a_i \neq x$ for $i = 1, \ldots, n$, because $g(a_0) \cdot g(x) < g(x)$. Applying f we have

$$x = a_1 \cdot \cdots \cdot a_n,$$

which contradicts the fact that x is meet irreducible. This proves the necessity of the conditions.

Now assume M generates A and x^M is finite for all $x \in M$. By Theorem 4.1 there exists a homomorphism $f: M^* \to A$ such that $f(\{x\}) = x$ for all $x \in M$. f is onto because M generates A. By Theorem 5.3(6), M^* is projective. Since $\{\{x\}: x \in M\}$ generates M^*, f will be tight if we prove: if $S \in M^*$ and $f(S) = x$, $x \in M$, then $S = \{x\}$. Let $S = \{x_1, D, x_n\}$. Then $S = \{x_1\} \cdot \cdots \cdot \{x_n\}$ and so $f(S) = x = x_1 \cdot \cdots \cdot x_n$. Therefore $x = x_i$ for some i and $x_i \geqslant x$ for all i. Since S is discrete, we must have $S = \{x\}$.

As an illustration, let X be a set of four elements and let A be the set semilattice consisting of all subsets of X with cardinal $\neq 3$. Then the projective cover of A is $S(Y)$, where Y is a set of cardinal 6, while the injective hull of A is $S(X)$.

For a final example, let A be the partially ordered set $\{a_i : i = 0, 1, 2, \ldots\} \cup \{b_i : i = 0, 1, 2, \ldots\}$, where $a_i < a_{i+1}$ and $a_i < b_i$ for all i. Then $\langle A, \wedge \rangle$ is a semilattice in which $a_i = b_i \wedge b_{i+1}$ and b_i is meet irreducible for all i. Since $\{b_i : i = 0, 1, \ldots\}$ is discrete, the projective cover of A is the free semilattice with $\aleph_0$ generators. This example shows that if x is an element of a semilattice A which has a projective cover, x^A need not be finite.

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Addendum. Theorems 2.8 and 3.8 were obtained independently by G. Bruns and H. Lakser in their paper Injective hulls of semilattices, Canadian Mathematical Bulletin, vol. 13, 1970, pp. 115–118. The problem of characterizing the subalgebras of free semilattices has been solved by K. Baker (to appear).

University of California,
Los Angeles, California.
University of Arkansas,
Fayetville, Arkansas,
U.S.A.