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Inequalities for the Gamma and Polygamma Functions

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1 Introduction

The classical gamma function

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, \mathrm{d}t, \tag{1}$$

$(0 < x \in \mathbb{R})$, is one of the most important functions in Analysis and its applications. It was introduced in 1729 by L. EULER in a letter to C. GOLDBACH as an infinite product, from which the integral representation (1) can be derived. A detailed description of the history and the development of the gamma function is given in [10]. An interesting stochastic approach can be found in [15].

Another important special function is the logarithmic derivative of Γ,

$$\psi(x) = \Gamma'(x)/\Gamma(x) = -C + \sum_{n=0}^{\infty}\Big(\frac{1}{1+n} - \frac{1}{x+n}\Big)$$

$(0 < x \in \mathbb{R};\ C =$ Euler's constant), which is known in literature as psi or digamma function. ψ and its derivatives are called polygamma functions. In the recent past, several authors published remarkable properties of these functions. In particular, many interesting inequalities can be found in the literature; see [3] and the references therein.

It is the main purpose of this paper to present new inequalities for the gamma and polygamma functions. In Section 2 we prove that the function $x \mapsto \log\big(\Gamma(x+1)\big)/(x\,\log(x))$ is strictly increasing on $(0, \infty)$, which extends a recent result of G. D. ANDERSON and S.-L. QIU. In Section 3 we provide sharp upper and lower bounds for the difference $\psi^{(n)}(x+1) - \psi^{(n)}(x+s)$. And, in Section 4 we use some properties of ψ and ψ' in order to obtain estimates for Euler's constant, which refine those given by R. M. YOUNG and others.

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2 A monotonicity property of the gamma function

In 1989, G. D. ANDERSON et al. [6] conjectured that the function $x \mapsto \log\big(\Gamma(x/2+1)\big)/\big(x\ \log(x)\big)$ is strictly increasing on $[2,\infty)$. This conjecture was recently proved by G. D. ANDERSON and S.-L. QIU [5]. Actually, they proved a bit more, namely, that even the function

$$f(x) = \log\big(\Gamma(x+1)\big)/\big(x\ \log(x)\big)$$

is strictly increasing on $(1,\infty)$. Moreover, they used this result to show that the sequence $n \mapsto \Omega_n^{1/(n\log(n))}$ $(n=2,3,\dots)$ is strictly decreasing. Here, $\Omega_n = \pi^{n/2}/\Gamma(n/2+1)$ denotes the n-dimensional volume of the unit ball in $\mathbb{R}^n$.

The monotonicity proof given in [5] is quite long and complicated, so that the following short and simple proof of an extension might be of interest. In order to show that f is strictly increasing not only on $(1,\infty)$ but even on $(0,\infty)$, we make use of the following elementary lemma which modifies slightly a result given in [7].

Lemma. *Let $u \in C^1(0,\infty)$ with $u(1)=0$ and $v \in C^1(0,\infty)$ such that $v<0$ on $(0,1)$, $v>0$ on $(1,\infty)$ and $v'>0$ on $(0,\infty)$. If u'/v' is strictly increasing on $(0,\infty)$, then u/v is also strictly increasing on $(0,\infty)$.*

We are now in a position to establish our first result.

Theorem 1. *The function $f(x) = \log\big(\Gamma(x+1)\big)/\big(x\ \log(x)\big)$ is strictly increasing on $(0,\infty)$.*

Proof. We define for $x>0$:

$$u(x) = \tfrac{1}{x}\log\big(\Gamma(x+1)\big) \quad \text{and} \quad v(x) = \log(x)\,.$$

Moreover, let

$$w(x) = x^2\Big(\tfrac{u'(x)}{v'(x)}\Big)' = x^2\,\psi'(x+1) - x\,\psi(x+1) + \log\big(\Gamma(x+1)\big)\,.$$

Using the integral representations

$$\psi'(z) = \int_0^\infty e^{-zt}\frac{t}{1-e^{-t}}\,\mathrm{d}t, \quad \psi''(z) = -\int_0^\infty e^{-zt}\frac{t^2}{1-e^{-t}}\,\mathrm{d}t$$

and

$$\frac{1}{z} = \int_0^\infty e^{-zt}\,\mathrm{d}t$$

$(z>0)$, (see [1], p. 260), and the convolution theorem for Laplace transforms, we obtain for $x>0$:

$$\frac{1}{x^2}\,w'(x) = \frac{1}{x}\,\psi'(x+1) + \psi''(x+1) = \int_0^\infty e^{-xt}\,h(t)\,\mathrm{d}t,$$

where

$$h(t) = \int_0^t \Big(\frac{s}{e^s-1} - \frac{t}{e^t-1}\Big)\,\mathrm{d}s\,.$$

Since $x \mapsto x/(e^x - 1)$ is strictly decreasing on $(0, \infty)$, we get $h(t) > 0$ $(t > 0)$, and, hence, $w'(x) > 0$ and $w(x) > w(0) = 0$ $(x > 0)$. This implies that u'/v' is strictly increasing on $(0, \infty)$. From the Lemma we conclude that $f = u/v$ is also strictly increasing on $(0, \infty)$. □

Remarks.

1) From Theorem 1 we obtain

$$\left(\frac{y\log(y)}{x\log(x)}\right)^{\alpha} < \frac{\log\big(\Gamma(y+1)\big)}{\log\big(\Gamma(x+1)\big)} \tag{2}$$

$(1 < x < y)$ with $\alpha = 1$. If we write inequality (2) as

$$\alpha < \frac{\log\big(\log(\Gamma(y+1))\big) - \log\big(\log(\Gamma(x+1))\big)}{\log\big(y\ \log(y)\big) - \log\big(x\ \log(x)\big)}, \tag{3}$$

and let y tend to ∞, then the ratio on the right-hand side of (3) tends to 1. This implies that the best possible constant in (2) is given by $\alpha = 1$.

2) In [5] the authors conjecture that $f(x) = \log\big(\Gamma(x+1)\big)/\big(x\ \log(x)\big)$ is concave on $(1, \infty)$. Computer experiments suggest that f is even concave on $(0, \infty)$.

3) Recently, P. J. GRABNER et al. [16] proved that the related function $g(x) = \log\big(\Gamma(x+1)\big)/x$ is increasing and concave on $(0, \infty)$. Moreover, they presented several inequalities for g and used their results to obtain bounds for the permanents of $0 - 1$ matrices.

3 Inequalities for the polygamma functions

Many authors studied inequalities for the difference

$$D_s(x) = \log\big(\Gamma(x+1)\big) - \log\big(\Gamma(x+s)\big)$$

$(x > 0;\ s \in (0, 1))$; (see [2, 8, 9, 11, 12, 14, 17-20, 23-26, 28]). Inequalities for $D_s(x)$ have a remarkable application: in [22] it is shown that they can be used to obtain estimates for ultraspherical polynomials.

The results presented in this section have been inspired by an interesting paper of I. B. LAZAREVIĆ and A. LUPAS [21], who published in 1974 the following theorem.

Proposition. *If $s \in (0, 1)$ is a real number, then we have for all real numbers $x > 0$:*

$$(1-s)\ \log(x + s/2) < D_s(x) < (1-s)\ \log\Big(x + \big(\Gamma(s)\big)^{1/(s-1)}\Big), \tag{4}$$

where the constants $s/2$ and $\big(\Gamma(s)\big)^{1/(s-1)}$ are best possible.

It is natural to ask whether this result can be extended to the derivatives of $x \mapsto D_s(x)$. This means, we are looking for sharp bounds for the difference $\psi^{(n)}(x+1) - \psi^{(n)}(x+s)$ $(0 \le n \in \mathbb{Z})$. Our next theorem provides such bounds. The following companion of double-inequality (4) holds.

Theorem 2. *Let $n \geq 0$ be an integer and let $s \in (0,1)$ be a real number. Then we have for all real numbers $x > 0$:*

$$\frac{n!(1-s)}{[x+\alpha_n(s)]^{n+1}} < (-1)^n[\psi^{(n)}(x+1) - \psi^{(n)}(x+s)] < \frac{n!(1-s)}{[x+\beta_n(s)]^{n+1}}, \tag{5}$$

with the best possible constants

$$\alpha_n(s) = \left(\frac{n!(1-s)}{(-1)^n[\psi^{(n)}(1) - \psi^{(n)}(s)]}\right)^{1/(n+1)} \quad \text{and} \quad \beta_n(s) = s/2. \tag{6}$$

Proof. Let $s \in (0,1)$ be a (fixed) real number. We denote by f_n the function

$$f_n(x) = \left[\frac{\Delta_n(x)}{n!(1-s)}\right]^{-1/(n+1)} - x,$$

where

$$\Delta_n(x) = (-1)^n[\psi^{(n)}(x+1) - \psi^{(n)}(x+s)].$$

We shall prove that

$$\lim_{x\to\infty} f_n(x) = s/2 \tag{7}$$

and that f_n is strictly decreasing on $[0,\infty)$. This implies

$$s/2 < f_n(x) < f_n(0)$$

($x > 0$), which is equivalent to double-inequality (5) with $\alpha_n(s)$ and $\beta_n(s)$ given in (6). Moreover, we conclude that these constants are best possible.

From the asymptotic formula

$$\psi(x) = \log(x) - \tfrac{1}{2x} - \tfrac{1}{12x^2} + O(x^{-4})$$

($x \to \infty$) (see [1], p. 259 or [13], p. 824), we get

$$\psi(x+1) - \psi(x+s) = \tfrac{1-s}{x} - \tfrac{s(1-s)}{2(x+s)(x+1)} + O(x^{-3}). \tag{8}$$

This leads to

$$f_0(x) = \frac{\frac{1}{2}sx^2(x+s)^{-1}(x+1)^{-1} + O(x^{-1})}{1+O(x^{-1})}$$

which implies (7) for $n = 0$. Let $n \geq 1$; from

$$\psi^{(n)}(x) = (-1)^{n-1}\left[(n-1)!x^{-n} + \tfrac{1}{2}n!x^{-n-1} + \tfrac{1}{12}(n+1)!x^{-n-2} + O(x^{-n-3})\right]$$

($x \to \infty$) (see [1], p. 260), we obtain

$$\begin{aligned} x^{n+1}\frac{\Delta_n(x)}{n!(1-s)} &= \frac{1+\frac{1}{2}(n-1)(1+s)\frac{1}{x} + O(x^{-2})}{1+n(1+s)\frac{1}{x} + O(x^{-2})} \\ &\quad + \frac{\frac{1}{2}(n+1)\frac{1}{x} + O(x^{-2})}{1+(n+1)(1+s)\frac{1}{x} + O(x^{-2})} + O(x^{-2}). \end{aligned} \tag{9}$$

This implies

$$f_n(x) = \frac{\left(1 - \frac{s}{2}(n+1)\frac{1}{x} + O(x^{-2})\right)^{-1/(n+1)} - 1}{1/x}. \tag{10}$$

From (10) we conclude $\lim_{x\to\infty} f_n(x) = s/2$, which proves (7) for $n \geq 1$. It remains to establish that

$$\left(\Delta_n(x)\right)^{1+1/(n+1)} f_n'(x) = \frac{1}{n+1}[n!(1-s)]^{1/(n+1)} \Delta_{n+1}(x) - \left(\Delta_n(x)\right)^{1+1/(n+1)} < 0\,. \tag{11}$$

To prove (11) for $x > 0$ it suffices to show that the function

$$g_n(x) = -\log\bigl(n!(1-s)\bigr) + (n+1)\log(n+1) - (n+1)\log\bigl(\Delta_{n+1}(x)\bigr) + (n+2)\log\bigl(\Delta_n(x)\bigr)$$

is positive on $(0, \infty)$. From (8) and (9) we get for $n \geq 0$:

$$\lim_{x\to\infty} x^{n+1}\Delta_n(x) = n!(1-s)\,, \tag{12}$$

which implies $\lim_{x\to\infty} g_n(x) = 0$. Therefore, it is enough to establish that g_n is strictly decreasing on $(0, \infty)$. The inequality $g_n'(x) < 0$ is equivalent to

$$(n+2)\bigl(\Delta_{n+1}(x)\bigr)^2 > (n+1)\Delta_{n+2}(x)\,\Delta_n(x)\,. \tag{13}$$

We set

$$u(t) = \frac{e^{-ts} - e^{-t}}{1 - e^{-t}}$$

($t > 0$) and make use of the integral representation

$$\psi(z) = -C + \int_0^\infty \frac{e^{-t} - e^{-tz}}{1 - e^{-t}}\,\mathrm{d}t$$

($z > 0$; $C =$ Euler's constant) (see [1], p. 259). Then we get

$$\begin{aligned}\bigl(\Delta_{n+1}(x)\bigr)^2 &= \Bigl(\int_0^\infty e^{-tx}\,t^{n+1}\,u(t)\,\mathrm{d}t\Bigr)^2 \\ &= \int_0^\infty e^{-tx}\bigl((t^{n+1}u(t)) * (t^{n+1}\,u(t))\bigr)\,\mathrm{d}t\,,\end{aligned}$$

where $*$ denotes Laplace convolution. Moreover, we obtain

$$\begin{aligned}\Delta_{n+2}(x)\,\Delta_n(x) &= \int_0^\infty e^{-tx}\,t^{n+2}\,u(t)\,\mathrm{d}t \int_0^\infty e^{-tx}\,t^n u(t)\,\mathrm{d}t \\ &= \int_0^\infty e^{-tx}\bigl((t^{n+2}u(t)) * (t^n\,u(t))\bigr)\,\mathrm{d}t\,.\end{aligned}$$

Thus, to prove (13) it suffices to show that the following inequality holds for $t > 0$:

$$\begin{aligned}&(n+2)\bigl((t^{n+1}u(t)) * (t^{n+1}u(t))\bigr) - (n+1)\bigl((t^{n+2}u(t)) * (t^n u(t))\bigr) \\ &\quad = \int_0^t u(t-x)\,u(x)\,(t-x)^n\,x^{n+1}[t(n+2) - (2n+3)x]\,\mathrm{d}x > 0\,.\end{aligned} \tag{14}$$

We denote the integral in (14) by $I(t)$ and we set $P_a(y) = u\big(a(1-y)\big)\,u\big(a(1+y)\big)$. Next, we change the variable, $x = \frac{t}{2}(1+y)$, and take into account that $y \mapsto P_{t/2}(y)(1-y^2)^n y$ is an odd function. Then we get

$$\begin{aligned} I(t) &= (t/2)^{2n+3} \int_{-1}^{1} P_{t/2}(y)\,(1-y^2)^n[1-2(n+1)y-(2n+3)y^2]\,\mathrm{d}y \\ &= 2\,(t/2)^{2n+3} \int_0^1 P_{t/2}(y)\,(1-y^2)^n[1-(2n+3)y^2]\,\mathrm{d}y\,. \end{aligned}$$

We shall prove that $y \mapsto P_a(y)$ $(a>0)$ is strictly decreasing on $(0,1)$. We set $c=(2n+3)^{-1/2}$; then we obtain

$$\begin{aligned} I(t)\,4^{n+1}\,t^{-(2n+3)} &= \int_0^c P_{t/2}(y)\,(1-y^2)^n[1-(y/c)^2]\,\mathrm{d}y \\ &\quad + \int_c^1 P_{t/2}(y)\,(1-y^2)^n[1-(y/c)^2]\,\mathrm{d}y \\ &> P_{t/2}(c)\Big[\int_0^c (1-y^2)^n[1-(y/c)^2]\,\mathrm{d}y \\ &\quad + \int_c^1 (1-y^2)^n[1-(y/c)^2]\,\mathrm{d}y\Big] \\ &= P_{t/2}(c)\int_0^1 (1-y^2)^n[1-(2n+3)y^2]\,\mathrm{d}y \\ &= P_{t/2}(c)\big[y(1-y^2)^{n+1}\big]_0^1 = 0\,. \end{aligned}$$

It remains to prove that

$$P_a'(y) < 0 \tag{15}$$

$\big(y \in (0,1)\big)$. We set

$$Q_a(x) = \log\big(u(ax)\big)\,;$$

then we have

$$P_a'(y) = P_a(y)[-Q_a'(1-y)+Q_a'(1+y)]\,.$$

Hence, to establish (15) it suffices to show that $x \mapsto Q_a(x)$ is strictly concave on $(0,\infty)$. Elementary calculations reveal that the inequality

$$Q_a''(x) = \big(a/u(ax)\big)^2\big[u(ax)u''(ax) - \big(u'(ax)\big)^2\big] < 0$$

is equivalent to

$$0 < b^2z^2 - z^{1+b} - 2(b^2-1)z - z^{1-b} + b^2 = R_b(z), \tag{16}$$

say, where $z>1$ and $b \in (0,1)$. From

$$R_b(1) = R_b'(1) = R_b''(1) = 0$$

and

$$R_b'''(z) = b(1-b^2)z^{-b-2}(z^{2b}-1) > 0$$

we conclude the validity of inequality (16). This completes the proof of Theorem 2. $\sqcap$

Remark. Let $n \geq 0$ be an integer and let

$$\Delta_n(x,s) = \psi^{(n)}(x+1) - \psi^{(n)}(x+s).$$

Inequality (13) implies that the following converse of the Cauchy-Schwarz inequality holds for all real numbers $x > 0$ and $s \in (0,1)$:

$$\tfrac{n+1}{n+2}\,\Delta_n(x,s)\,\Delta_{n+2}(x,s) < \big(\Delta_{n+1}(x,s)\big)^2. \qquad (17)$$

From $\psi^{(n)}(x+1)-\psi^{(n)}(x) = (-1)^n n! x^{-n-1}$ (see [1], p. 260) and (12) we conclude that the ratio $\big(\Delta_{n+1}(x,s)\big)^2/[\Delta_n(x,s)\,\Delta_{n+2}(x,s)]$ tends to $(n+1)/(n+2)$ if $s \to 0$ or if $x \to \infty$. Hence, the constant factor $(n+1)/(n+2)$ in (17) cannot be replaced by a larger number.

4 Inequalities for Euler's constant

Euler's constant $C = 0.57721\ldots$ – "the third mysterious number of real analysis" [29], p. 187 – is defined by the well-known limit relation

$$C = \lim_{n\to\infty} d_n\,,$$

where

$$d_n = \sum_{k=1}^{n} \tfrac{1}{k} - \log(n)$$

$(n = 1, 2, \ldots)$. There is a close connection between the difference $d_n - C$ and the psi function. Indeed, using the recurrence formula $\psi(x+1) = \psi(x) + \frac{1}{x}$ $(x > 0)$ and $\psi(1) = -C$ (see [1], p. 258), we get $d_n - C = \psi(n+1) - \log(n)$.

Several bounds for $d_n - C$ are given in the literature. In 1971, S. R. TIMS and J. A. TYRRELL [27] used analytical methods to establish

$$\tfrac{1}{2(n+1)} < d_n - C < \tfrac{1}{2(n-1)} \qquad (18)$$

$(n = 2, 3, \ldots)$, which leads to the asymptotic formula $d_n - C \sim \frac{1}{2n}$. In 1991, R. M. YOUNG [29] presented an elegant geometrical proof for the double-inequality

$$\tfrac{1}{2(n+1)} < d_n - C < \tfrac{1}{2n} \qquad (19)$$

$(n = 1, 2, \ldots)$, which provides a slight improvement of the right-hand side of (18). Recently, G. D. ANDERSON et al. [4] proved that the function $h(x) = x\big(\log(x) - \psi(x)\big)$ is strictly decreasing on $(0, \infty)$ with $\lim_{x\to\infty} h(x) = \frac{1}{2}$. This leads to

$$\tfrac{1-C}{n} \leq d_n - C < \tfrac{1}{2n} \qquad (20)$$

$(n = 1, 2, \ldots)$, which sharpens the lower bound given in (19) if $n \leq 5$.

In view of these inequalities it is natural to ask: what is the smallest number a and what is the largest number b such that the inequalities

$$\tfrac{1}{2(n+a)} \leq d_n - C \leq \tfrac{1}{2(n+b)}$$

are valid for all integers $n \geq 1$? The answer to this question provides refinements of the bounds given in (18), (19) and (20).

Theorem 3. *For all integers $n \geq 1$, we have*

$$\frac{1}{2(n+a)} \leq d_n - C < \frac{1}{2(n+b)}, \tag{21}$$

with the best possible constants

$$a = \frac{1}{2(1-C)} - 1 = 0.1826\ldots \quad \text{and} \quad b = \frac{1}{6}.$$

Proof. Since

$$d_n - C = \psi(n+1) - \log(n),$$

double-inequality (21) can be written as

$$b < \frac{1}{2}\,\frac{1}{\psi(n+1)-\log(n)} - n \leq a. \tag{22}$$

In order to prove (22) we define for positive real x:

$$f(x) = \frac{1}{2}\,\frac{1}{\psi(x+1)-\log(x)} - x.$$

Differentiation yields

$$\begin{aligned} f'(x)[\psi(x+1) - \log(x)]^2 &= \frac{1}{2}\left(\frac{1}{x} - \psi'(x+1)\right) - \left(\psi(x+1) - \log(x)\right)^2 \\ &= \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x^2} - \psi'(x)\right) - \left(\psi(x) + \frac{1}{x} - \log(x)\right)^2. \end{aligned}$$

Using the inequalities

$$\log(x) - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x)$$

and

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x)$$

($x > 0$) (see [15]), we obtain for $x \geq 2.4$:

$$f'(x)[\psi(x+1) - \log(x)]^2 < \frac{1}{144x^5}(2.4 - x) \leq 0. \tag{23}$$

From (23) and $f(1) = 0.182\ldots$, $f(2) = 0.177\ldots$, $f(3) = 0.174\ldots$, we conclude that the sequence $f(n) = \frac{1}{2(d_n - C)} - n$ ($n = 1, 2, \ldots$) is strictly decreasing. This leads to

$$\lim_{k\to\infty} f(k) < f(n) \leq f(1) = \frac{1}{2(1-C)} - 1$$

($n = 1, 2, \ldots$). It remains to prove that

$$\lim_{k\to\infty} f(k) = \frac{1}{6}. \tag{24}$$

From the representation

$$\psi(x) = \log(x) - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4})$$

($x \to \infty$), we get

$$f(x) = \left(\frac{1}{6} + O(x^{-2})\right)/\left(1 + O(x^{-1})\right),$$

which implies (24). This completes the proof of Theorem 3. □

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