Abh. Math. Sem. Univ. Hamburg 59, 191–224 (1989)

Jacobi Forms of Higher Degree

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Introduction

Let $\boldsymbol{H}_{n+j}$ denote Siegel's upper halfplane of degree $n+j$ $(n, j \in \mathbb{N})$ and let $\Gamma_{n+j} := Sp(n+j, \mathbb{Z})$. We consider a Siegel modular form $F \in [\Gamma_{n+j}, k]$ of weight k, i.e. a holomorphic function $F\colon \boldsymbol{H}_{n+j} \to \mathbb{C}$ satisfying the functional equation

$$F(M\langle Z\rangle) = \det(CZ+D)^k \cdot F(Z) \text{ for every } M \in \Gamma_{n+j}$$

and having a Fourier expansion of the form

$$F(Z) = \sum_{\substack{T = T^t \geqq 0 \\ T \text{ half integer}}} c(T) \cdot e^{2\pi i \sigma(TZ)}.$$

For all that we use the usual notation $M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}$ for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{n+j}$ with $A, B, C, D \in \mathbb{Z}^{(m+j, n+j)}$; σ denotes the tracefunction and T^t the transposed matrix to T. Writing

$$Z = \begin{pmatrix} Z_1 & W^t \\ W & Z_2 \end{pmatrix}$$

with $Z_1 \in \mathbb{C}^{(n,n)}$, $Z_2 \in \mathbb{C}^{(j,j)}$ and $W \in \mathbb{C}^{(j,n)}$ the partial Fourier expansion of F with respect to the variable Z_2 is given by

$$F(Z) = F(Z_1, W, Z_2) = \sum_{\substack{\mathcal{M} \geqq 0 \\ \text{half integer}}} \Phi_{\mathcal{M}}(Z_1, W) \cdot e^{2\pi i \sigma(\mathcal{M} Z_2)},$$

where

$$\Phi_{\mathcal{M}}(Z_1, W) := \sum_{T_1, R} c\left(\begin{pmatrix} T_1 & \frac{1}{2} R \\ \frac{1}{2} R^t & \mathcal{M} \end{pmatrix}\right) \cdot e^{2\pi i \sigma(T_1 Z_1)} \cdot e^{2\pi i \sigma(R W)}.$$

The above formula is well known as Fourier-Jacobi expansion of the Siegel modular form F. Now the functions $\Phi_{\mathcal{M}}$ inherit certain functional equations from F:

$$\text{Let } \tilde{M} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & E & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & E \end{pmatrix} \in \Gamma_{n+j} \text{ with } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n.$$

Then $F(\tilde{M}\langle Z\rangle) = \det(\tilde{C}Z + \tilde{D})^k \cdot F(Z)$ implies:

$$\Phi_{\mathcal{M}}(M\langle Z_1\rangle, W(CZ_1 + D)^{-1}) = \det(CZ_1 + D)^k \cdot e^{2\pi i\sigma(\mathcal{M}W(CZ_1+D)^{-1}CW^t)} \cdot \Phi_{\mathcal{M}}(Z_1, W) \tag{1}$$

for every $M \in \Gamma_n$. The same argument for

$$\tilde{M} = \begin{pmatrix} E & 0 & 0 & \mu^t \\ \lambda & E & \mu & \varkappa \\ 0 & 0 & E & -\lambda^t \\ 0 & 0 & 0 & E \end{pmatrix} \in \Gamma_{n+j}$$

with $\lambda, \mu \in \mathbb{Z}^{(j,n)}, \varkappa \in \mathbb{Z}^{(j,j)}$ satisfying $(\varkappa + \mu\lambda^t) = (\varkappa + \mu\lambda^t)^t$ shows:

$$\Phi_{\mathcal{M}}(Z_1, W + \lambda Z_1 + \mu)\, e^{-2\pi i\sigma(\mathcal{M}(\lambda Z_1\lambda^t + 2\lambda W^t + (\varkappa + \mu\lambda^t)))} \cdot \Phi_{\mathcal{M}}(Z_1, W) \tag{2}$$

for every such triple $[(\lambda, \mu), \varkappa]$.

The attempt of this paper is to give a starting point for a systematic investigation of holomorphic functions $\Phi : \boldsymbol{H}_n \times \mathbb{C}^{(j,n)} \to \mathbb{C}$ satisfying the functional equations (1) and (2) as well as a certain condition on their Fourier expansion. EICHLER and ZAGIER have recently developped a theory of these functions (called Jacobi Forms) in the special case $n = 1, j = 1$ (see [2]). As far as I know the only papers investigating higher dimensional cases are GRITSENKO [4], MURASE [8], SHIMURA [10] and YAMAZAKI [14]. Nevertheless no satisfactory general theory seems to exist, so this paper may be viewed as a first serious attempt to build up such a theory in the spirit of Eichler and Zagier.

In 1. we give the precise definition of Jacobi Forms and discuss some basic concepts of the theory; the main result of this section will be the finite dimensionality of the space of Jacobi Forms. In 2. we shall construct Jacobi Forms by means of Eisenstein Series. Analogously to the theory of Siegel modular forms we shall first consider ordinary Eisenstein Series before we generalize our results to Eisenstein Series of Klingen's type. Furthermore we shall develop some technical tools, for example the notion of Petersson scalar product. In 3. we investigate various topics concerning theta series: First we shall prove a result of Shimura, which states an isomorphism between the space of Jacobi Forms and a certain space of vector valued Siegel modular forms of half integral weight. Then using even unimodular lattices we shall construct Jacobi Forms on the modular group Γ_n by means of theta series and discuss some of the related problems, especially the theory of singular Jacobi Forms. 4. is devoted to applications of our theory: First we shall investigate (non) surjectiveness properties of the Fourier Jacobi expansion of Siegel modular forms. Then we shall be concerned with Siegel's Hauptsatz for Jacobi forms and as a corollary shall obtain a stability theorem for Poincaré Square Series.

This paper is a short version of the author's 1988 thesis—also some slight modifications and corrections have been carried out.

1. Jacobi Forms

We consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,j)} := \{[(\lambda, \mu), \varkappa] \mid \lambda, \mu \in \mathbb{R}^{(j,n)}, \varkappa \in \mathbb{R}^{(j,j)}, (\varkappa + \mu\lambda^t) \text{ symmetric}\}$$

which is a group with the following composition law:

$$[(\lambda, \mu), \varkappa] \circ [(\lambda', \mu'), \varkappa'] := [(\lambda + \lambda', \mu + \mu'), \varkappa + \varkappa' + \lambda\mu'^t - \mu\lambda'^t].$$

The mapping

$$[(\lambda, \mu), \varkappa] \to \begin{pmatrix} E & 0 & 0 & \mu^t \\ \lambda & E & \mu & \varkappa \\ 0 & 0 & E & -\lambda^t \\ 0 & 0 & 0 & E \end{pmatrix}$$

defines an imbedding of $H_{\mathbb{R}}^{(n,j)}$ into $Sp\,(n + j, \mathbb{R})$. Now the group $Sp\,(n, \mathbb{R})$ acts on $H_{\mathbb{R}}^{(n,j)}$ by multiplication from the left:

$$[(\lambda, \mu), \varkappa] \cdot M := [(\lambda, \mu) \cdot M, \varkappa] \quad .$$

So we can define the Jacobi group $G_{\mathbb{R}}^{(n,j)} := Sp\,(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,j)}$ with the associated composition law:

$$(M, \zeta) \cdot (M', \zeta') := (MM', \zeta M' \circ \zeta')$$

i.e.

$$(M, [(\lambda, \mu), \varkappa]) \cdot (M', [\lambda', (\mu'), \varkappa'])$$

$$:= (MM', [(\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'), \varkappa + \varkappa' + \tilde{\lambda}\mu'^t - \tilde{\mu}\lambda'^t])$$

where $(\tilde{\lambda}, \tilde{\mu}) := (\lambda, \mu) \cdot M'$.

It is easy to verify that $G_{\mathbb{R}}^{(n,j)}$ acts on $H_n \times \mathbb{C}_{\mathbb{R}}^{(j,n)}$ as a group of automorphisms. The action is given by:

$$(M, [(\lambda, \mu), \varkappa]) \cdot (Z, W) := (M\langle Z\rangle, (W + \lambda Z + \mu)(CZ + D)^{-1}).$$

Our next aim is to define a factor of automorphy.

Let E be a finite dimensional $\mathbb{C}$-vectorspace. We shall consider holomorphic mappings $\Phi: H_n \times \mathbb{C}^{(j,n)} \to E$ and denote the $\mathbb{C}$-vectorspace of all such mappings with $\mathcal{O}(H_n \times \mathbb{C}^{(j,n)}, E)$.

1.1. Definition. Let $\varrho: Gl\,(n, \mathbb{C}) \to Gl\,(E)$ be a rational representation of $Gl\,(n, \mathbb{C})$ on E. For $\Phi \in \mathcal{O}(H_n \times \mathbb{C}^{(j,n)}, E)$, $M \in Sp\,(n, \mathbb{R})$, $\zeta = [(\lambda, \mu), \varkappa] \in H^{(n,j)}$ and $\mathcal{M} \in \mathbb{R}^{(j,j)}$ with $\mathcal{M} \geqq 0$ symmetric and half integer, i.e. $2\mathcal{M}_{ij}, \mathcal{M}_{ii} \in \mathbb{Z}$, we define:

$$(\Phi\,|_{\varrho,\mathcal{M}}\, M)(Z, W)$$

$$:= \varrho(CZ + D)^{-1} \cdot e^{-2\pi i\sigma(\mathcal{M}\cdot W(CZ+D)^{-1}CW^t)} \cdot \Phi(M\langle Z\rangle, W(CZ + D)^{-1})$$

$$(\Phi\,|_{\mathcal{M}}\,\zeta)(Z, W) := e^{2\pi i\sigma(\mathcal{M}\cdot(\lambda Z\lambda^t + 2\lambda W^t + (\varkappa + \mu\lambda^t)))} \cdot \Phi(Z, W + \lambda Z + \mu).$$

Of special interest is the case $E = \mathbb{C}$ and $\varrho(N) := (\det N)^k$ for $N \in Gl(n, \mathbb{C})$ and some fixed $k \in \mathbb{Z}$.

1.2. Lemma. For $M, M' \in Sp(n, \mathbb{R})$ and $\zeta, \zeta' \in H_{\mathbb{R}}^{(n,j)}$ we have the following relations:

$$\Phi \,|M|\, M' = \Phi \,|\, (MM'), \tag{1}$$

$$\Phi \,|\zeta|\, \zeta' = \Phi \,|\, (\zeta \circ \zeta'), \tag{2}$$

$$\Phi \,|\zeta|\, M = \Phi \,|M|\, (\zeta M). \tag{3}$$

Corollary: $G_{\mathbb{R}}^{(n,j)}$ acts on $\mathcal{O}(H_n \times \mathbb{C}^{(j,n)}, E)$ by

$$\Phi \to \Phi|_{\varrho,\mathcal{M}} (M, \zeta) := \Phi|_{\varrho,\mathcal{M}} M|_{\mathcal{M}} \zeta .$$

Proof. Straightforward computation.

Remark. An element $(M, \zeta) \in G_{\mathbb{R}}^{(n,j)}$ acts trivially von $\mathcal{O}(H_n \times \mathbb{C}^{(j,n)}, E)$ if and only if $M = E$ and $\zeta = [(0, 0), \varkappa]$ such that $\sigma(\mathcal{M} \cdot \varkappa) \in \mathbb{Z}$. The set $\mathcal{N}_{\mathcal{M}}$ of all such elements is a normal subgroup of $G_{\mathbb{R}}^{(n,j)}$ and passing to the quotient $G_{\mathbb{R}}^{(n,j)}/\mathcal{N}_{\mathcal{M}}$ we obtain a faithful representation of $G_{\mathbb{R}}^{(n,j)}/\mathcal{N}_{\mathcal{M}}$ on $\mathcal{O}(H_n \times \mathbb{C}^{(j,n)}, E)$.

Let

$$H_{\mathbb{Z}}^{(n,j)} := \{[(\lambda, \mu), \varkappa] \in H_{\mathbb{R}}^{(n,j)} \mid \lambda, \mu \in \mathbb{Z}^{(j,n)}, \varkappa \in \mathbb{Z}^{(j,j)}\}$$

and $G_{\mathbb{Z}}^{(n,j)} := Sp(n, \mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,j)} \subset G_{\mathbb{R}}^{(n,j)}$. Analogously we define $H_{\mathbb{Q}}^{(n,j)}$ and $G_{\mathbb{Q}}^{(n,j)}$. We now give the precise definition of Jacobi Forms:

1.3. Definition. Let ϱ and $\mathcal{M}$ be like in Definition 1.1.

A (vector valued) Jacobi Form of index $\mathcal{M}$ with respect to ϱ on a subgroup $\Gamma \subset Sp(n, \mathbb{Z})$ of finite index is a holomorphic mapping $\Phi \in \mathcal{O}(H_n \times \mathbb{C}^{(j,n)}, E)$ satisfying:

1. $\Phi|_{\varrho,\mathcal{M}} M = \Phi$ for every $M \in \Gamma$;
2. $\Phi|_{\mathcal{M}} \zeta = \Phi$ for every $\zeta \in H_{\mathbb{Z}}^{(n,j)}$;
3. for each $M \in Sp(n, \mathbb{Z})$ the function $\Phi|_{\varrho,\mathcal{M}} M$ has a Fourierexpansion of the following form:

$$(\Phi|_{\varrho,\mathcal{M}} M)(Z, W) = \sum_{\substack{T = T^t \geqq 0 \\ T \text{ half integer}}} \; \sum_{R \in \mathbb{Z}^{(n,j)}} c(T, R) \cdot e^{\frac{2\pi i}{\lambda_\Gamma} \sigma(TZ)} \cdot e^{2\pi i \sigma(RW)}$$

with a suitable $\lambda_\Gamma \in \mathbb{Z}$ and $c(T, R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} R^t & \mathcal{M} \end{pmatrix} \geqq 0.$

We denote the vectorspace of all Jacobi Forms of index $\mathcal{M}$ with respect to ϱ on Γ by $J_{\varrho,\mathcal{M}}(\Gamma)$. In the special case $E = \mathbb{C}$, $\varrho(N) = (\det N)^k$, $k \in \mathbb{Z}$, we write

$J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\varrho,\mathcal{M}}(\Gamma)$ and call k the weight of the corresponding Jacobi Forms. Obviously we have:

1.4. Lemma. Let $\Phi \in J_{\varrho,\mathcal{M}}(\Gamma)$ be a Jacobi Form on Γ. Then $\Phi|_{\varrho,\mathcal{M}} M$ is a Jacobi Form on the conjugate group $M^{-1}\Gamma M$ for each $M \in Sp(n, \mathbb{Z})$, i.e. $\Phi|_{\varrho,\mathcal{M}} M \in J_{\varrho,\mathcal{M}}(M^{-1}\Gamma M)$.

The following simple observation is sometimes useful in order to verify the condition on the Fourier expansion of a Jacobi Form: A symmetric matrix $S \in \mathbb{R}^{(n,n)}$ is semipositive if and only if $S + \varepsilon S_1 \geqq 0$ for every $\varepsilon > 0$ and some fixed semipositive symmetric $S_1 \in \mathbb{R}^{(n,n)}$. For example $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} R^t & \mathcal{M} \end{pmatrix} \geqq 0$ if and only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} R^t & \mathcal{M} + \varepsilon E \end{pmatrix} \geqq 0$ for every $\varepsilon > 0$. Now $\mathcal{M} + \varepsilon E$ is invertible, so we can write:

$$\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} R^t & \mathcal{M} + \varepsilon E \end{pmatrix} = \begin{pmatrix} E & \frac{1}{2} R(\mathcal{M} + \varepsilon E)^{-1} \\ 0 & E \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\lambda_\Gamma} T - \frac{1}{4} R(\mathcal{M} + \varepsilon E)^{-1} R^t & 0 \\ 0 & \mathcal{M} + \varepsilon E \end{pmatrix} \begin{pmatrix} E & \frac{1}{2} R(\mathcal{M} + \varepsilon E)^{-1} \\ 0 & E \end{pmatrix}^t$$

and we obtain the following criterion: $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} R^t & \mathcal{M} \end{pmatrix} \geqq 0$ if and only if $T \geqq 0$, $\mathcal{M} \geqq 0$ and $4T - \lambda_\Gamma R(\mathcal{M} + \varepsilon E)^{-1} R^t \geqq 0$ for every $\varepsilon > 0$.

This criterion is sometimes sufficient for our purposes (for example in order to prove 1.5., 1.6.), nevertheless we would like to have a better insight in what $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} R^t & \mathcal{M} \end{pmatrix} \geqq 0$ means for T, R. Therefore we choose $\tilde{U} \in Gl(j, \mathbb{Z})$ such that $\tilde{U}^t \mathcal{M} \tilde{U} = \begin{pmatrix} \tilde{\mathcal{M}} & 0 \\ 0 & 0 \end{pmatrix}$ with $\tilde{\mathcal{M}} \in \mathbb{R}^{(l,l)}$, $l := \operatorname{rank}(\mathcal{M})$, $\det \tilde{\mathcal{M}} \neq 0$. We write

$\tilde{U} = (U, V)$ with $U \in \mathbb{Z}^{(j,l)}$, $V \in \mathbb{Z}^{(j,j-l)}$, i.e. $\tilde{\mathcal{M}} = U^t \mathcal{M} U$. Now

$$\begin{pmatrix} E & 0 \\ 0 & \tilde{U}^t \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} R^t & \mathcal{M} \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \tilde{U} \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R\tilde{U} \\ \frac{1}{2} \tilde{U}^t R^t & \tilde{U}^t \mathcal{M} \tilde{U} \end{pmatrix}$$

$$= \begin{bmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} RU & \frac{1}{2} RV \\ \frac{1}{2} U^t R^t & \tilde{\mathcal{M}} & 0 \\ \frac{1}{2} V^t R^t & 0 & 0 \end{bmatrix}$$

So $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} R^t & \mathcal{M} \end{pmatrix} \geqq 0$ if and only if $T \geqq 0$, $\mathcal{M} \geqq 0, \frac{1}{2} RV = 0$ and

$$4T - \lambda_\Gamma \tilde{R} \tilde{\mathcal{M}}^{-1} \tilde{R}^t \geqq 0,$$

where $\tilde{R} := RU$ and $\tilde{\mathcal{M}} := U^t \mathcal{M} U$. Using this criterion, the third condition in the definition of Jacobi Forms becomes $c(T, R) \neq 0$ only if $\frac{1}{2} RV = 0$ and $4T - \lambda_\Gamma \tilde{R}^t \tilde{\mathcal{M}}^{-1} \tilde{R} \geqq 0$.

If a Jacobian Form satisfies the stronger condition $c(T, R) \neq 0$ only if $\frac{1}{2} RV = 0$ and $4T - \lambda_\Gamma \tilde{R} \tilde{\mathcal{M}}^{-1} \tilde{R}^t > 0$ it is called cusp form. We now state:

1.5. Theorem. *Let $\Phi \in J_{\varrho,\mathcal{M}}(\Gamma)$ be a Jacobi Form and let $\zeta \in H_{\mathbb{Q}}^{(n,j)}$, i.e. $\zeta = [(\lambda, \mu), \varkappa]$ with $\lambda, \mu \in \mathbb{Q}^{(j,n)}$, $\varkappa \in \mathbb{Q}^{(j,j)}$. Then the function*

$$f(Z) := e^{2\pi i \sigma(\mathcal{M} \cdot \lambda Z \lambda^t)} \cdot \Phi(Z, \lambda Z + \mu)$$

is a vector valued Siegel modular form with respect to ϱ on some subgroup Γ' of finite index in $Sp\,(n, \mathbb{Z})$ depending only on Γ and ζ.

Proof. We have

$$f(Z) = e^{-2\pi i \sigma(\mathcal{M} \cdot (\varkappa + \mu \lambda^t))} \cdot (\Phi|_{\mathcal{M}}\, \zeta)(Z, 0),$$

so it suffices to prove that $\Phi_\zeta(Z) := (\Phi|_{\mathcal{M}}\, \zeta)(Z, 0)$ is a Siegel modular form. First we shall show the functional equation. For any $M \in \Gamma$ we have

$$\begin{aligned} (\Phi_\zeta|_\varrho M)(Z) &= \varrho(CZ + D)^{-1} \cdot \Phi\zeta(M\langle Z\rangle) = \varrho(CZ + D)^{-1} \cdot (\Phi|_{\mathcal{M}}\, \zeta)(M\langle Z\rangle, 0) \\ &= (\Phi|_{\mathcal{M}}\, \zeta|_{\varrho,\mathcal{M}}\, M)(Z, 0) = (\Phi|_{\varrho,\mathcal{M}}\, M|_{\mathcal{M}}(\zeta M))(Z, 0) \\ &= (\Phi|_{\mathcal{M}}\, (\zeta M))(Z, 0) = \Phi_{\zeta\mathcal{M}}(Z). \end{aligned}$$

Now suppose $\zeta M = \zeta'' \circ \zeta' \circ \zeta$ with $\zeta'' \in \mathcal{N}_{\mathcal{M}}$, i.e. $\zeta'' = [(0,0), \varkappa'']$ with $\sigma(\mathcal{M} \cdot \varkappa'') \in \mathbb{Z}$ and $\zeta' \in H_{\mathbb{Z}}^{(n,j)}$. Then

$$(\Phi_\zeta|_\varrho M)(Z) = \Phi_{\zeta\mathcal{M}}(Z) = \Phi_{\zeta''\circ\zeta'\circ\zeta}(Z) = (\Phi|_{\mathcal{M}}\, \zeta''|_{\mathcal{M}}\, \zeta'|_{\mathcal{M}}\, \zeta)(Z,0)$$
$$= (\Phi|_{\mathcal{M}}\, \zeta)(Z,0) = \Phi_\zeta(Z).$$

So Φ_ζ transforms like a Siegel modular form for each M in

$$A_{\Gamma,\zeta} := \{M \in \Gamma \mid \exists\, \zeta'' \in \mathcal{N}_{\mathcal{M}}, \zeta' \in H_{\mathbb{Z}}^{(n,j)}, \text{ such that } \zeta M = \zeta'' \circ \zeta' \circ \zeta\} \subset \Gamma_n.$$

$\zeta M = \zeta'' \circ \zeta' \circ \zeta$ means

$$[(\lambda,\mu)\cdot M, \varkappa] = [(0,0),\varkappa''] \circ [(\lambda',\mu'),\varkappa'] \circ [(\lambda,\mu),\varkappa]$$
$$= [(\lambda+\lambda', \mu+\mu'), \varkappa+\varkappa'+\varkappa''+\lambda'\mu^t-\mu'\lambda^t]$$

and so $M \in A_{\Gamma,\zeta}$ if and only if

$$(\lambda,\mu)\cdot(M-E) = (\lambda',\mu') \equiv 0 \pmod{\mathbb{Z}^{(j,2n)}}, \tag{1}$$

$$\sigma(\mathcal{M}\cdot(\lambda'\mu^t - \mu'\lambda^t - \mu'\lambda'')) \in \mathbb{Z} \ (\text{set } \varkappa' = -\mu'\lambda'^t). \tag{2}$$

Now it is easy to see that $A_{\Gamma,\zeta}$ contains some congruence subgroup $\Gamma' = \Gamma[l] \subset Sp(n,\mathbb{Z})$ and $f|_\varrho M = f$ holds for every $M \in \Gamma'$. So in order to prove $f \in [\Gamma', \varrho]$ it remains to check tde cusp conditions for f. Since this verification may be done exactly like in the one variable case (compare Eichler/Zagier [2], proof of Theorem 1.3.), we omit further details. The reader who is willing to do the slight modifications necessary should recall our preceeding discussion on cusp conditions for Jacobi Forms. □

For $n \geqq 2$ the well known Köcher-principle implies that any holomorphic mapping $f: H_n \to E$ transforming like a Siegel modular form on some subgroup $\Gamma \subset Sp(n,\mathbb{Z})$ of finite index, automatically satisfies the cusp conditions. This together with Theorem 1.5. implies the following Köcher-principle for Jacobi Forms:

1.6. Lemma (Köcher-principle for Jacobi Forms). *Let $n \geqq 2$ and $\Gamma \subset Sp(n,\mathbb{Z})$ be a subgroup of finite index. Then any function $\Phi \in \mathcal{O}(H_n \times \mathbb{C}^{(j,n)}, E)$ satisfying $\Phi|_{\varrho,\mathcal{M}} M = \Phi$ for every $M \in \Gamma$ and $\Phi|_{\mathcal{M}}\, \zeta = \Phi$ for every $\zeta \in H_{\mathbb{Z}}^{(n,j)}$ is a Jacobi Form in $J_{\varrho,\mathcal{M}}(\Gamma)$, i.e. the third condition in the definition of Jacobi Forms can be omitted.*

Proof. It is easy to see that the condition on the Fourier expansion of a Jacobi Form Φ is equivalent to the condition that the functions

$$f_{\lambda,\mu}(Z) := e^{2\pi i\sigma(\mathcal{M}\cdot\lambda Z\lambda^t)} \cdot \Phi(Z, \lambda Z + \mu)$$

satisfy the cusp conditions for Siegel modular forms for every pair $\lambda, \mu \in \mathbb{Q}^{(j,n)}$. □

1.7. Corollary. *Let* $E = \mathbb{C}$ *and* $\varrho(N) := (\det N)^k$, $k \in \mathbb{Z}$. *Then the Jacobi Forms form a bigraded ring*:

$$J_{*,*}(\Gamma) := \bigoplus_{k,\mathcal{M}} J_{k,\mathcal{M}}(\Gamma).$$

Proof. The product of two Jacobi Forms $\Phi_1 \in J_{k,\mathcal{M}}(\Gamma)$, $\Phi_2 \in J_{\tilde{k},\tilde{\mathcal{M}}}(\Gamma)$ obviously transforms like a Jacobi Form of weight $k + \tilde{k}$ and index $\mathcal{M} + \tilde{\mathcal{M}}$. The cusp condition is clearly satisfied for $\Phi(Z, W) := \Phi_1(Z, W) \cdot \Phi_2(T, W)$ since

$$f_{\lambda,\mu}(Z) = (f_1)_{\lambda,\mu}(Z) \cdot (f_2)_{\lambda,\mu}(Z). \quad \square$$

Let $\Phi \in J_{\varrho,\mathcal{M}}(\Gamma)$ be a Jacobi Form. For fixed $Z_0 \in \boldsymbol{H}_n$ we consider the function $g(W) := \Phi(Z_0, W)$. Now each pair $\lambda, \mu \in \mathbb{Z}^{(j,n)}$ occurs in some $\zeta = [(\lambda, \mu), \varkappa] \in H_{\mathbb{Z}}^{(n,j)}$ and $(\varkappa + \mu\lambda^t)$ symmetric implies $\sigma(\mathcal{M} \cdot (\varkappa + \mu\lambda^t)) \in \mathbb{Z}$, so g satisfies the functional equation

$$g(W + \lambda Z_0 + \mu) = e^{-2\pi i\sigma(\mathcal{M}\cdot(\lambda Z_0\lambda^t + 2\lambda W^t))} \cdot g(W)$$

for every $\lambda, \mu \in \mathbb{Z}^{(j,n)}$. Thus g is a $(p := \dim_C E)$-tuple of theta-functions with respect to the lattice

$$\mathbb{Z}^{(j,n)} \cdot Z_0 + \mathbb{Z}^{(j,n)} \subset \mathbb{C}^{(j,n)}.$$

The space of all such theta-functions is finite-dimensional and its dimension D is independent of $Z_0 \in \boldsymbol{H}_n$. For dimensionformulae see Lemma 3.1. The main result of this section is:

1.8. Theorem.

$$\dim_C J_{\varrho,\mathcal{M}}(\Gamma) < \infty.$$

Proof. Let $\theta_1^{Z_0}, \ldots, \theta_D^{Z_0}$ be a basis of the space of theta-functions considered above. We say that a D-tuple $(W_1, \ldots, W_D)$ of points $W_i \in \mathbb{C}^{(j,n)}$ is Z_0-generic, if $\det(\theta_i^{Z_0}(W_j)) \neq 0$. This property is independent of the choice of the basis $\theta_1^{Z_0}, \ldots, \theta_D^{Z_0}$ and by induction on D one easily proves that generic D-tuples always exist. The set of generic D-tuples is open in $\mathbb{C}^{(j,n)} \times \ldots \times \mathbb{C}^{(j,n)}$ (D-times), therefore we may choose D pairs $(\lambda^{(i)}, \mu^{(i)}) \in \mathbb{Q}^{(j,n)}$, such that the points $W_i := \lambda^{(i)} Z_0 + \mu^{(i)}$, $i = 1, \ldots, D$ define a Z_0-generic D-tuple. By Theorem 1.5. the functions

$$f_i(Z) := e^{2\pi i\sigma(\mathcal{M}\cdot\lambda^{(i)} Z \lambda^{(i)^t})} \cdot \Phi(Z, \lambda^{(i)} Z + \mu^{(i)})$$

are (vector valued) Siegel modular forms on subgroups $\Gamma_i \subset Sp(n, \mathbb{Z})$ of finite index. Consequently we obtain a map

$$j: J_{\varrho,\mathcal{M}}(\Gamma) \to \bigoplus_{i=1}^{D} [\Gamma_i, \varrho],$$

$$\Phi \to (f_1, \ldots, f_D).$$

By choice of $\lambda^{(i)}, \mu^{(i)}$ this map is injective, so

$$\dim_C J_{\varrho,\mathcal{M}}(\Gamma) \leqq \sum_{i=1}^{D} \dim_C [\Gamma_i, \varrho] < \infty. \quad \square$$

Corollary. $\dim_C J_{\varrho,\mathcal{M}}(\Gamma)=0$ *if* ϱ *is irreducible and not polynomial. Especially* $\dim_C J_{k,\mathcal{M}}(\Gamma)=0$ *for* $k<0$, *i.e. there are no nonvanishing Jacobi Forms of negative weight.*

1.9. Definition. For $\Phi \in J_{\varrho,\mathcal{M}}(\Gamma)$ *we define* $\mathcal{S}(\Phi)\in \mathcal{O}(\boldsymbol{H}_{n-1}\times \mathbb{C}^{(j,n-1)}, E)$ *by*

$$\mathcal{S}(\Phi)(Z, W) := \lim_{t\to\infty} \Phi\left(\begin{pmatrix} Z & 0 \\ 0 & it\end{pmatrix}, (W, 0)\right)$$

and call $\mathcal{S}$ the Siegel operator.

We claim that the above limit always exists: Since Φ is a Jacobi Form it admits a Fourier expansion converging uniformly on sets of the form

$$\{(Z, W) \mid \operatorname{Im} Z \geqq Y_0 > 0,\ W\in K \subset \mathbb{C}^{(j,n)} \text{ compact}\}.$$

So we are allowed to compute the limit termwise and obtain an expression for $\mathcal{S}$ in terms of Fourier coefficients. Explicitely the Fourier expansion of $\mathcal{S}(\Phi)$ is given by

$$\mathcal{S}(\Phi) = \sum_{T=T^t\in\mathbb{Q}^{(n-1,n-1)}} \ \sum_{R\in\mathbb{Z}^{(n-1,j)}} \tilde{c}(T, R)\cdot e^{\frac{2\pi i}{\lambda_\Gamma}\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)},$$

where the Fourier coefficients $\tilde{c}(T, R)$ are related to the Fourier coefficients $c(T, R)$ of Φ by means of the formula $\tilde{c}(T, R) = c\left(\begin{pmatrix} T & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix} R \\ 0\end{pmatrix}\right)$.

For any $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(n-1,\mathbb{R})$ with $A, B, C, D\in\mathbb{R}^{(n-1,n-1)}$ we set

$$\tilde{M}=\begin{pmatrix} A & 0 & B & 0\\ 0 & 1 & 0 & 0\\ C & 0 & D & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\in Sp(n,\mathbb{R}).$$

For a subgroup $\Gamma\subset\Gamma_n$ of finite index we define

$$\mathcal{S}(\Gamma) := \{M\in\Gamma_{n-1} \mid \tilde{M}\in\Gamma\}$$

which is a subgroup of finite index in Γ_{n-1}.

Finally if $\varrho: Gl(n,\mathbb{C})\to Gl(E)$ is a rational representation of $Gl(n,\mathbb{C})$ on E, we denote by $\mathcal{S}(\varrho)$ the rational representation of $Gl(n-1,\mathbb{C})$ on E defined by

$$\mathcal{S}(\varrho)(N) := \varrho\left(\begin{pmatrix} N & 0\\ 0 & 1\end{pmatrix}\right).$$

1.10. Theorem. *Let* $\Phi\in J_{\varrho,\mathcal{M}}(\Gamma)$ *be a Jacobi Form. Then* $\mathcal{S}(\Phi)$ *is also a Jacobi Form in* $J_{\mathcal{S}(\varrho),\mathcal{M}}(\mathcal{S}(\Gamma))$, *i.e. the Siegel operator defines a linear mapping*

$$\mathcal{S}: J_{\varrho,\mathcal{M}}(\Gamma)\to J_{\mathcal{S}(\varrho),\mathcal{M}}(\mathcal{S}(\Gamma)).$$

Proof. Straightforward computation analogous to the case of Siegel modular forms.

Corollary: *The Siegel operator defines a linear mapping*

$$\mathscr{S}: J_{k,\mathscr{M}}(\Gamma_n) \to J_{k,\mathscr{M}}(\Gamma_{n-1}).$$

Proof. $\mathscr{S}(\Gamma_n) = \Gamma_{n-1}$ *and* $\mathscr{S}(\varrho)(N) = (\det N)^k$, *if* $\varrho(\tilde{N}) = (\det \tilde{N})^k$. $\square$

Remark. If Φ is a cusp form, then $\mathscr{S}(\Phi) \equiv 0$, since $\tilde{c}(T, R) = c\left(\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} R \\ 0 \end{pmatrix}\right) = 0$ for every T, R and $\Phi \in J^{\text{cusp}}_{\varrho,\mathscr{M}}(\Gamma)$. But in contrast to the theory of Siegel modular forms, we do not have $J^{\text{cusp}}_{\varrho,\mathscr{M}}(\Gamma) = \text{kernel}\,(\mathscr{S})$.

2. Eisenstein Series

Our investigations are analogous to the theory of Siegel modular forms. First we shall consider ordinary Eisenstein Series, later on we shall generalize our results to Eisenstein Series of Klingen's type. We restrict ourselves to the special case $E = \mathbb{C}$ and $\varrho(N) := (\det N)^k$ with $k \in \mathbb{N}$ even and define the ordinary Eisenstein Series by

$$E^{(n)}_{k,\mathscr{M}}(Z, W) := \sum_{\gamma \in G_\infty \backslash G^{(n,j)}_{\mathbb{Z}}} (1|_{k,\mathscr{M}}\, \gamma)(Z, W),$$

where

$$\begin{aligned} G_\infty &:= \{\gamma \in G^{(n,j)}_{\mathbb{Z}} \mid 1|_{k,\mathscr{M}}\, \gamma = 1\} \\ &= \{(M, [(\lambda, \mu), \varkappa]) \in G^{(n,j)}_{\mathbb{Z}} \mid M \in \Gamma_{n,0}, \lambda \in \ker(\mathscr{M})\} \end{aligned}$$

with $\Gamma_{n,0} := \left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp\,(n, \mathbb{Z}) \mid C = 0\right\}$ and $\ker(\mathscr{M}) := \{\lambda \in \mathbb{R}^{(j,n)} \mid \mathscr{M} \cdot \lambda = 0\}$. Let $\mathscr{R}$ be a complete system of representatives of the cosets $\Gamma_{n,0} \backslash \Gamma_n$ and Λ be a complete system of representatives of the cosets $\mathbb{Z}^{(j,n)}/(\ker(\mathscr{M}) \cap \mathbb{Z}^{(j,n)})$, then we obtain a complete system of representatives of the cosets $G_\infty \backslash G^{(n,j)}_{\mathbb{Z}}$ by

$$\tilde{\mathscr{R}} := \left\{(M, \zeta) \in G^{(n,j)}_{\mathbb{Z}} \mid M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathscr{R}, \zeta = [(\lambda A, \lambda B), 0], \lambda \in \Lambda\right\}.$$

Therefore we have explicitely:

$$\begin{aligned} E^{(n)}_{k,\mathscr{M}}(Z, W) = \sum_{M \in \mathscr{R}} &\det(CZ + D)^{-k} \cdot e^{-2\pi i \sigma(\mathscr{M} \cdot W(CZ+D)^{-1}CW^t)} \\ &\cdot \sum_{\lambda \in \Lambda} e^{2\pi i \sigma(\mathscr{M} \cdot (M\langle Z\rangle \lambda^t + 2\lambda(CZ+D)^{-t}W^t))}. \end{aligned}$$

It is clear, that $E^{(n)}_{k,\mathscr{M}}$ formally transforms like a Jacobi Form in $J_{k,\mathscr{M}}(\Gamma_n)$, so the problem is to show convergence and to check the cusp condition.

2.1. Theorem. *For* $k > n + \operatorname{rank}(\mathcal{M}) + 1$ *even the Eisenstein Series* $E_{k,\mathcal{M}}^{(n)}$ *converges normally on* $\boldsymbol{H}_n \times \mathbb{C}^{(j,n)}$ *and defines a nonvanishing Jacobi Form in* $J_{k,\mathcal{M}}(\Gamma_n)$. *The convergence is uniform on vertical strips of the form*

$$W_n(\delta) := \{(Z, W) \in \boldsymbol{H}_n \times \mathbb{C}^{(j,n)} \mid Z = X + iY,$$

$$Y \geqq \delta E, \sigma(X^2) \leqq \delta^{-1}, \sigma(W\overline{W}') \leqq \delta^{-1}\}.$$

Proof. Let $l := \operatorname{rank}(\mathcal{M})$. We choose a matrix $\tilde{U} = (U, V) \in Gl(j, \mathbb{Z})$ like in 1.4., such that $\tilde{U}'\mathcal{M}\tilde{U} = \begin{pmatrix} \tilde{\mathcal{M}} & 0 \\ 0 & 0 \end{pmatrix}$ with $\tilde{\mathcal{M}} \in \mathbb{R}^{(l,l)}$, $\det \tilde{\mathcal{M}} \neq 0$. The decomposition

$$\mathbb{Z}^{(j,n)} \cong \tilde{U} \cdot \mathbb{Z}^{(j,n)} \cong U \cdot \mathbb{Z}^{(l,n)} \oplus V \cdot \mathbb{Z}^{(j-l,n)}$$

shows $\ker(\mathcal{M}) \cap \mathbb{Z}^{(j,n)} = V \cdot \mathbb{Z}^{(j-l,n)}$, so we may choose $\Lambda = U \cdot \mathbb{Z}^{(l,n)}$. Analogously we may split each $W \in \mathbb{C}^{(j,n)}$ into two components according to the decomposition

$$\mathbb{C}^{(j,n)} \cong \tilde{U} \cdot \mathbb{C}^{(j,n)} \cong U \cdot \mathbb{C}^{(l,n)} \oplus V \cdot \mathbb{C}^{(j-l,n)}.$$

That is W may be written in a unique manner as $W = UW_1 + VW_2$ with $W_1 \in \mathbb{C}^{(l,n)}$ and $W_2 \in \mathbb{C}^{(j-l,n)}$. Doing so we obtain the identity

$$E_{k,\mathcal{M}}^{(n)}(Z, W) = E_{k,\tilde{\mathcal{M}}}^{(n)}(Z, W_1).$$

Especially $E_{k,\mathcal{M}}^{(n)}(Z, W)$ does not depend on W_2, i.e. is constant along $\ker(\mathcal{M})$. So in order to prove convergence, we may assume without loss of generality $\mathcal{M} > 0$, resp. $l = j$. In this case the series $E_{k,\mathcal{M}}^{(n)}(Z, W)$ may be obtained as a subseries of the $\mathcal{M}$-th Fourier Jacobi coefficient of the Siegelian Eisenstein Series $E_k^{(n+l)}$ of degree $n + l$ (Compare Böcherer [1], formula 13). Using his kind of argument we may conclude that $E_{k,\mathcal{M}}^{(n)}$ inherits convergence from $E_k^{(n+l)}$, which is well known to converge for $k > n + l + 1$ even. The argument also yields the statement about the uniform convergence in vertical strips.

Next we have to verify the cusp condition. By the Köcher principle there is nothing to prove for $n \geqq 2$. In the remaining case $n = 1$ we have to show that certain Fourier coefficients do vanish. This may be done exactly like in Eichler/Zagier [2], Chapter I, 2. by using the Poisson summation formula and deforming the path of integration to infinity. Finally $E_{k,\mathcal{M}}^{(n)}$ does not vanish identically, since

$$\lim_{t \to \infty} E_{k,\mathcal{M}}^{(n)}(itE, 0) = 1. \quad \square$$

Before we shall consider more general types of Eisenstein Series, we shall introduce the notion of Petersson scalar product. Its investigation will be a good preparation for our treatment of Eisenstein Series of Klingen's type.

For $Z \in \boldsymbol{H}_n$, $W \in \mathbb{C}^{(j,n)}$ let $Z = X + iY$ and $W = \alpha + i\beta$ be the decompositions into real and imaginary parts. We define a volume element dV on $\boldsymbol{H}_n \times \mathbb{C}^{(j,n)}$ by

$$dV := (\det Y)^{-(n+j+1)} \cdot dX \wedge dY \wedge d\alpha \wedge d\beta,$$

where $dX := \wedge\, dx_{\mu,\nu}$ $(\mu \leqq \nu)$, etc. A glance at dV shows its $G_{\mathbb{R}}^{(n,j)}$-invariance. The form $(\det Y)^{-(n+1)} \cdot dX \wedge dY$ is the usual $Sp(n, \mathbb{R})$-invariant volume form on H_n, while $(\det Y)^{-j} \cdot d\alpha \wedge d\beta$ is the translation-invariant volume form on $\mathbb{C}^{(j,n)}$ normalized to $\operatorname{vol}(\mathbb{C}^{(j,n)}/(\mathbb{Z}^{(j,n)} \cdot Z + \mathbb{Z}^{(j,n)})) = 1$.

Let $\Phi, \Psi \in J_{\varrho,\mathcal{M}}(\Gamma)$ be Jacobi Forms with respect to some rational representation $\varrho: Gl(n, \mathbb{C}) \to Gl(E)$. Without loss of generality we may assume ϱ to be irreducible and polynomial (compare corollary of 1.8.). Then we choose a hermitean metric on E invariant under the restriction of ϱ to the unitary group $U(n) \subset Gl(n, \mathbb{C})$, i.e.

$$\langle \varrho(U) \cdot v, \varrho(U) \cdot w \rangle = \langle v, w \rangle \text{ for every } U \in U(n).$$

For irreducible ϱ such an inner product is unique up to multiplication with a scalar factor. For any real symmetric positive definite matrix Y let $Y^{\frac{1}{2}}$ denote some positive symmetric squareroot of Y. Then the expression

$$e^{-4\pi\sigma(\mathcal{M}\cdot\beta Y^{-1}\beta^t)} \cdot \langle \varrho(Y^{\frac{1}{2}})\, \Phi(Z, W), \varrho(Y^{\frac{1}{2}})\, \Psi(Z, W) \rangle$$

is $\Gamma \ltimes H_{\mathbb{Z}}^{(n,j)}$-invariant, which may be checked by straightforward computation. Therefore we define the Petersson scalar product of Φ and Ψ in the following way:

2.2. Definition (Petersson scalar product)

$$(\Phi, \Psi)_\Gamma := \int_{\Gamma \ltimes H_{\mathbb{Z}}^{(n,j)} \backslash H_n \times \mathbb{C}^{(j,n)}} e^{-4\pi\sigma(\mathcal{M}\cdot\beta Y^{-1}\beta^t)} \cdot \langle \varrho(Y^{\frac{1}{2}})\, \Phi(Z, W), \varrho(Y^{\frac{1}{2}})\, \Psi(Z, W) \rangle \, dV.$$

A Jacobi Form Φ is called squareintegrable if $\|\Phi\|^2 := (\Phi, \Phi)_\Gamma < \infty$.

If $\Gamma_0 \subset \Gamma_1$ is a subgroup of finite index we have $(\Phi, \Psi)_{\Gamma_0} = [\Gamma_1 : \Gamma_0] \cdot (\Phi, \Psi)_{\Gamma_1}$. For $(M, \zeta) \in G_{\mathbb{Z}}^{(n,j)}$ let $(Z^*, W^*) := (M, \zeta) \cdot (Z, W)$. Then the $U(n)$-invariance of the inner product on E yields

$$\begin{aligned} &e^{-4\pi\sigma(\mathcal{M}\cdot\beta^* Y^{*-1}\beta^{*t})} \cdot \langle \varrho(Y^{*\frac{1}{2}})\, \Phi(Z^*, W^*), \varrho(Y^{*\frac{1}{2}})\, \Phi(Z^*, W^*) \rangle \\ &\quad = e^{-4\pi\sigma(\mathcal{M}\cdot\beta Y^{-1}\beta^t)} \cdot \langle \varrho(Y^{\frac{1}{2}})\, (\Phi|_{\varrho,\mathcal{M}} M)(Z, W), \varrho(Y^{\frac{1}{2}})\, (\Phi|_{\varrho,\mathcal{M}} M)(Z, W) \rangle. \end{aligned}$$

So if $\Gamma \subset Sp(n, \mathbb{Z})$ is a normal subgroup, i.e. $\Gamma \ltimes H_{\mathbb{Z}}^{(n,j)} \subset G_{\mathbb{Z}}^{(n,j)}$ is also normal, then $G_{\mathbb{Z}}^{(n,j)}$ acts on $\Gamma \times H_{\mathbb{Z}}^{(n,j)} \setminus H_n \times \mathbb{C}^{(j,n)}$ and we obtain:

$$(\Phi, \Phi)_\Gamma = (\Phi|_{\varrho,\mathcal{M}} M, \Phi|_{\varrho,\mathcal{M}} M)_\Gamma \text{ for every } M \in Sp(n, \mathbb{Z}).$$

In general we can find some normal subgroup $\Gamma_0 \subset \Gamma \subset Sp(n, \mathbb{Z})$ of finite index, which is also a normal subgroup of finite index in $M^{-1}\, \Gamma M$. Then

$$\begin{aligned} (\Phi, \Phi)_\Gamma &= \frac{1}{[\Gamma : \Gamma_0]} (\Phi, \Phi)_{\Gamma_0} \\ &= \frac{1}{[M^{-1}\, \Gamma M : \Gamma_0]} (\Phi \mid M, \Phi \mid M)_{\Gamma_0} = (\Phi \mid M, \Phi \mid M)_{M^{-1}\Gamma M}. \end{aligned}$$

So Φ is squareintegrable if and only if $\Phi|_{\varrho,\mathcal{M}} M$ is squareintegrable for every $M \in Sp(n, \mathbb{Z})$. Obviously the Petersson scalar product is positive definite on the space of squareintegrable Jacobi Forms. Furthermore we have the following lemma:

2.3. Lemma. *The Peterson scalar product* $(\Phi, \Psi)_\Gamma$ *is well defined and finite for* $\Phi, \Psi \in J_{\varrho,\mathcal{M}}(\Gamma)$ *and at least one of* Φ *and* Ψ *a cusp form.*

Corollary. *Cusp forms are squareintegrable.*

Proof. The condition cusp form for $\Phi|_{\varrho,\mathcal{M}} M$ $(M \in \Gamma_n)$ enables us to estimate the integral over suitable open neighbourhoods of the corresponding rational boundary components.

In the proof of 2.1. we made use of the decomposition

$$\mathbb{C}^{(j,n)} \cong U \cdot \mathbb{C}^{(l,n)} \oplus V \cdot \mathbb{C}^{(j-l,n)}.$$

Writing $W = UW_1 + VW_2$ had the effect that we could replace $\mathcal{M}$ by $\tilde{\mathcal{M}} := U^t \mathcal{M} U$ which is invertible. Before proceeding further we formalize this kind of argument, so that in future we may assume without loss of generality $\mathcal{M} > 0$.

Let $\Phi \in J_{\varrho,\mathcal{M}}(\Gamma)$ be a Jacobi Form and let

$$\Phi(Z, W) = \sum_{T,R} c(T, R) \cdot e^{2\pi i \sigma(TZ+RW)}$$

be the Fourierexpansion of Φ. We know: $c(T, R) \neq 0$ only if $RV = 0$, so $\sigma(TZ + RW) = \sigma(TZ + \tilde{R}W_1)$ holds whenever $c(T, R) \neq 0$. This shows that $\Phi(Z, UW_1 + VW_2)$ does not depend on W_2, i.e. Φ may be considered as a function of Z and W_1 only. We define $\tilde{\Phi} \in \mathcal{O}(H_n \times \mathbb{C}^{(l,n)}, E)$ by

$$\tilde{\Phi}(Z, W_1) := \Phi(Z, UW_1).$$

One easily checks the formulae

$$(\tilde{\Phi}|_{\varrho,\tilde{\mathcal{M}}} M)(Z, W_1) = (\Phi|_{\varrho,\mathcal{M}} M)(Z, UW_1) \qquad \text{for every } M \in Sp(n, \mathbb{R}),$$

$$(\tilde{\Phi}|_{\tilde{\mathcal{M}}} \zeta)(Z, W_1) = (\Phi|_{\mathcal{M}} U\zeta)(Z, UW_1) \qquad \text{for every } \zeta \in H_{\mathbb{R}}^{(n,l)}$$

with $U\zeta := [(U\lambda, U\mu), U\varkappa U^t]$ for $\zeta = [(\lambda, \mu), \varkappa] \in H_{\mathbb{R}}^{(n,l)}$. Especially we observe. $\tilde{\Phi} \in J_{\varrho,\tilde{\mathcal{M}}}(\Gamma)$ for $\Phi \in J_{\varrho,\mathcal{M}}(\Gamma)$. In fact we have the following theorem:

2.4. Theorem. *The mapping*

$$\mathcal{T}_U : J_{\varrho,\mathcal{M}}(\Gamma) \to J_{\varrho,\tilde{\mathcal{M}}}(\Gamma),$$
$$\Phi \to \mathcal{T}_U(\Phi) := \tilde{\Phi}$$

defines an isomorphism $J_{\varrho,\mathcal{M}}(\Gamma) \cong J_{\varrho,\tilde{\mathcal{M}}}(\Gamma)$ mapping cusp forms one to one onto cusp forms. $\mathcal{T}_U$ is compatible with Siegel operator and Petersson scalar product, i.e. $\mathcal{S}(\tilde{\Phi}) = \widetilde{\mathcal{S}(\Phi)}$ and $(\Phi, \Psi)_\Gamma = (\tilde{\Phi}, \tilde{\Psi})_\Gamma$. Furthermore $\widetilde{E_{k,\mathcal{M}}^{(n)}} = E_{k,\tilde{\mathcal{M}}}^{(n)}$.

The proof of Theorem 2.4. is straightforward.

We shall now define Eisenstein Series of Klingen's type. Again we restrict ourselves to the special case $E = \mathbb{C}$ and $\varrho(N) := (\det N)^k$ with $k \in \mathbb{N}$ even. Let

$\det \mathcal{M} \neq 0$ and $\Phi \in J^{\text{cusp}}_{k,\mathcal{M}}(\Gamma_m)$ be a cusp form. For $n > m$ we define a function $F \in \mathcal{O}(H_n \times \mathbb{C}^{(j,n)})$ by

$$F(Z, W) := \Phi(Z_1, W_1) \quad \text{where } Z = \begin{pmatrix} Z_1 & * \\ * & * \end{pmatrix} \text{ and } W = (W_1, *).$$

Our idea is to consider the series

$$E^{(n)}_{k,\mathcal{M}}(Z, W, \Phi) := \sum_{\gamma \in G^m_\infty \backslash G^{(n,j)}_{\mathbb{Z}}} (F|_{k,\mathcal{M}}\, \gamma)(Z, W)$$

with

$$G^m_\infty := \{(M, [(\lambda, \mu), \varkappa]) \in G^{(n,j)}_{\mathbb{Z}} \mid M \in \Gamma_{n,m}, \lambda = (\lambda_1, 0), \lambda_1 \in \mathbb{Z}^{(j,m)}\}$$

and

$$\Gamma_{n,m} := \left\{ M \in Sp(n, \mathbb{Z}) \mid M = \begin{pmatrix} A_1 & 0 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & D_4 \end{pmatrix}, A_1, B_1, C_1, D_1 \in \mathbb{Z}^{(m,m)} \right\}.$$

We observe that $F|_{k,\mathcal{M}}\,\gamma$ does not depend on the choice of the representative, so $E^{(n)}_{k,\mathcal{M}}(*, *, \Phi)$ is formally well defined. If $\mathcal{M}$ is not invertible we define:

$$E^{(n)}_{k,\mathcal{M}}(*, *, \Phi) := \mathcal{T}_U^{-1}(E^{(n)}_{k,\tilde{\mathcal{M}}}(*, *, \mathcal{T}_U(\Phi))).$$

It is clear that $E^{(n)}_{k,\mathcal{M}}(*, *, \Phi)$ formally transforms like a Jacobi Form in $J_{k,\mathcal{M}}(\Gamma_n)$. For $m = 0, \Phi \equiv 1$ we have $E^{(n)}_{k,\mathcal{M}}(*, *, 1) = E^{(n)}_{k,\mathcal{M}}$, i.e. the ordinary Eisenstein Series occur as special cases of Eisenstein Series of Klingen's type. Again the main problem is to show convergence. The condition on the Fourier expansion of $E^{(n)}_{k,\mathcal{M}}(*, *, \Phi)$ is automatically satisfied for $n \geqq 2$ by the Köcher-principle and for $n = 1$, i.e. $m = 0$ we can apply the results of 2.1. We shall prove:

2.5. Theorem. *Let $\Phi \in J^{\text{cusp}}_{k,\mathcal{M}}(\Gamma_m)$ be a cusp form and let $k \in \mathbb{N}$ even. For $k > n + m + \operatorname{rank}(\mathcal{M}) + 1$ the Eisensteinseries $E^{(n)}_{k,\mathcal{M}}(*, *, \Phi)$ converges normally on $H_n \times \mathbb{C}^{(j,n)}$, consequently $E^{(n)}_{k,\mathcal{M}}(*, *, \Phi) \in J_{k,\mathcal{M}}(\Gamma_n)$.*

In order to prove Theorem 2.5. we need some preparations:

2.6. Lemma. *Let $\Phi \in J_{k,\mathcal{M}}(\Gamma_m)$ be a cusp form. Then there exists some constant $C > 0$ such that*

$$|\Phi(Z, W)| \leqq C \cdot (\det Y)^{-\frac{k}{2}} \cdot e^{2\pi\sigma(\mathcal{M}\cdot\beta Y^{-1}\beta^t)}$$

for every $Z = X + iY \in H_m$ and $W = \alpha + i\beta \in \mathbb{C}^{(j,m)}$.

Proof of *2.6.* Without loss of generality we may assume $\det \mathcal{M} \neq 0$. Defining

$$h(Z, W) := (\det Y)^{\frac{k}{2}} \cdot e^{-2\pi\sigma(\mathcal{M}\cdot\beta Y^{-1}\beta^t)} \cdot |\Phi(Z, W)|$$

we have to show that $h(Z, W)$ is bounded on $H_m \times \mathbb{C}^{(j,m)}$. Since $h(Z, W)$ is $G_{\mathbb{Z}}^{(m,j)}$-invariant (compare 2.2.), it suffices to show that $h(Z, W)$ is bounded on some suitable fundamental domain D of $G_{\mathbb{Z}}^{(m,j)} \setminus H_m \times \mathbb{C}^{(j,m)}$. We choose

$$D := \{(Z, \lambda Z + \mu) \mid Z \in \mathscr{F}_m, \lambda, \mu \in \mathbb{R}^{(j,m)}, 0 \leqq |\lambda_{ij}|, |\mu_{ij}| \leqq 1\},$$

where $\mathscr{F}_m$ denotes Siegel's fundamental domain for H_m/Γ_m.

Now $D \cap \{(Z, W) \in H_m \times \mathbb{C}^{(j,m)} \mid \det Y \leqq K, K > 0\}$ is compact. Therefore our conclusion follows from

$$\lim_{\det Y \to \infty} h(Z, W) = 0 \qquad (Z, W) \in D$$

which is valid since Φ is assumed to be a cusp form. $\square$

2.7. Corollary. *For* $Z = \begin{pmatrix} Z_1 & * \\ * & * \end{pmatrix} \in H_n$, $Z_1 \in H_m$, $W = (W_1, *) \in \mathbb{C}^{(j,n)}$, $W_1 \in \mathbb{C}^{(j,m)}$ *let*

$$H_m(Z, W) := (\det Y_1)^{-\frac{k}{2}} \cdot e^{2\pi\sigma(\mathscr{M} \cdot \beta_1 Y_1^{-1} \beta_1^t)}.$$

Then $|F(Z, W)| \leqq C \cdot |H_m(Z, W)|$ *holds with C like in 2.6. We also observe:*

$$|(F|_{k,\mathscr{M}}\, \gamma)(Z, W)| \leqq C \cdot |(H_m|_{k,\mathscr{M}}\, \gamma)(Z, W)| \text{ for every } \gamma \in G_{\mathbb{R}}^{(n,j)}.$$

For $\gamma \in G_\infty^m$ *and* $k \in \mathbb{N}$ *even we easily check the formula* $|(H_m|_{k,\mathscr{M}}\, \gamma)(Z, W)| = |H_m(Z, W)|$. *Therefore the series*

$$\sum_{\gamma \in G_\infty^m \setminus G_Z^{(n,j)}} |(H_m|_{k,\mathscr{M}}\, \gamma)(Z, W)|$$

is well defined and majorizes the Eisenstein Series $E_{k,\mathscr{M}}^{(n)}(Z, W, \Phi)$.

Proof of Theorem 2.5. Obviously it suffices to show that each point $(Z_0, W_0) \in H_n \times \mathbb{C}^{(j,n)}$ admits an open neighbourhood $U = U(Z_0, W_0)$ such that the series

$$\sum_{\gamma \in G_\infty^m \setminus G_{\mathbb{Z}}^{(n,j)}} \int_U |(F|_{k,\mathscr{M}}\, \gamma)(Z, W)|\, dX\, dY\, d\alpha\, d\beta$$

converges for $k > n + m + j + 1$. (Note $\mathscr{M} > 0$). We shall actually show more Let Υ denote a complete system of representatives of the cosets $G_\infty^m \setminus G_{\mathbb{Z}}^{(n,j)}$, then the series

$$R := \sum_{\gamma \in \Upsilon} \int_U |(H_m|_{k,\mathscr{M}}\, \gamma)(Z, W)|\, dX\, dY\, d\alpha\, d\beta$$

converges for k and $U = U(Z_0, W_0)$ like above.

For $U \subset H_n \times \mathbb{C}^{(j,n)}$ let

$$\operatorname{stab}(U) := \{\gamma \in G_{\mathbb{Z}}^{(n,j)} \mid \gamma \cdot (Z, W) = (Z, W) \text{ for every } (Z, W) \in U\}.$$

It is easy to see that each point $(Z_0, W_0) \in H_n \times \mathbb{C}^{(j,n)}$ admits an open neighbourhood $U = U(Z_0, W_0) \subset H_n \times \mathbb{C}^{(j,n)}$ such that

a) $\bar{U} \subset H_n \times \mathbb{C}^{(j,n)}$ is compact,
b) $\gamma \in G_{\mathbb{Z}}^{(n,j)}, \gamma(U) \cap U \neq \emptyset \Rightarrow \gamma \in \operatorname{stab}((Z_0, W_0))$,
c) $\gamma \in \operatorname{stab}((Z_0, W_0)) \Rightarrow U = \gamma(U)$.

By c) the finite group $S := \operatorname{stab}((Z_0, W_0))/\operatorname{stab}(H_n \times \mathbb{C}^{(j,n)})$ acts on U. So we have a decomposition

$$U = U_1 \cup \ldots \cup U_\nu, \quad \nu = \# S,$$

such that each U_j $(j = 1, \ldots, \nu)$ is a fundamental domain for U/S, i.e. each U_j is a fundamental set and $\gamma(U_j) \cap U_j$ is a set of measure zero unless $\gamma \in \operatorname{stab}(H_n \times \mathbb{C}^{(j,n)})$. We shall show the convergence of R for any $U = U(Z_0, W_0)$ satisfying the conditions a)–c) above. Instead of R we consider the series

$$\tilde{R} := \sum_{\gamma \in \Upsilon} \int_U |(H_m|_{k,\mathcal{M}}\, \gamma)(Z, W)| \cdot |H_n(Z, W)|^{-1}\, dV$$

where $dV := (\det Y)^{-(n+j+1)} \cdot dX \wedge dY \wedge d\alpha \wedge d\beta$ denotes the $G_{\mathbb{R}}^{(n,j)}$-invariant volume form on $H_n \times \mathbb{C}^{(j,n)}$. Since $\bar{U}$ is compact in $H_n \times \mathbb{C}^{(j,n)}$ there exist constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$, such that

$$0 < C_1 < |H_n(Z, W)|^{-1} < C_2 < \infty$$

and

$$0 < C_3 < (\det Y)^{-(n+j+1)} < C_4 < \infty$$

for every $(Z, W) \in U$. So $\tilde{R}$ is convergent if and only if R is convergent. Now the substitution $(Z, W) \to \gamma^{-1} \cdot (Z, W)$ yields

$$\begin{aligned}\tilde{R} &= \sum_{\gamma \in \Upsilon} \int_U |(H_m|_{k,\mathcal{M}}\, \gamma)(Z, W)| \cdot |H_n(Z, W)|^{-1}\, dV \\ &= \sum_{\gamma \in \Upsilon} \int_{\gamma(U)} |H_m(Z, W)| \cdot |H_n(Z, W)|^{-1}\, dV.\end{aligned}$$

Using $U = U_1 \cup \ldots \cup U_\nu$, $\nu = \# S$ with U_j $(j = 1, \ldots, \nu)$ like above, we obtain:

$$\begin{aligned}\tilde{R} &= \sum_{j=1}^{\nu} \sum_{\gamma \in \Upsilon} \int_{\gamma(U_j)} |H_m(Z, W)| \cdot |H_n(Z, W)|^{-1}\, dV \\ &= \sum_{j=1}^{\nu} \int_{\tilde{U}_j} |H_m(Z, W)| \cdot |H_n(Z, W)|^{-1}\, dV\end{aligned}$$

where $\tilde{U}_j := \bigcup_{\gamma \in \Upsilon} \gamma(U_j)$. The second identity makes use of the following observation: If $\operatorname{vol}(\gamma_1(U_j) \cap \gamma_2(U_j)) \neq 0$ with $\gamma_1, \gamma_2 \in \Upsilon$, then also $\operatorname{vol}(\gamma_2^{-1}\gamma_1(U_j)) \neq 0$, which implies $\gamma_2^{-1}\gamma_1 \in \operatorname{stab}(H_n \times \mathbb{C}^{(j,n)})$. Since $\operatorname{stab}(H_n \times \mathbb{C}^{(j,n)}) \subset G_\infty^m$ we obtain $\gamma_1 = \gamma_2$. A similar argument shows $\operatorname{vol}(\gamma(\tilde{U}_j) \cap \tilde{U}_j) = 0$ for $\gamma \in G_\infty^m$,

$\gamma \notin \operatorname{stab}(H_n \times \mathbb{C}^{(j,n)})$. So $\tilde{U}_j$ is contained in some fundamental domain of G_∞^m. Furthermore there exists some constant $C \in \mathbb{R}$ such that $\det(\operatorname{Im} Z) < C$ for every $(Z, W) \in \tilde{U}_j$, $j = 1, \dots, \nu$. Observing that the expression $|H_m(Z, W)| \cdot |H_n(Z, W)|^{-1} \, dV$ is G_∞^m-invariant for even k, we obtain

$$\tilde{R} \leqq (\# S) \cdot \int_{\substack{V \\ \det Y < C}} |H_m(Z, W)| \cdot |H_n(Z, W)|^{-1} \, dV$$

$$= (\# S) \cdot \int_{\substack{V \\ \det Y < C}} (\det Y_1)^{-\frac{k}{2}} \cdot (\det Y)^{\frac{k}{2}} \cdot e^{-2\pi\sigma(\mathcal{M} \cdot (\beta Y^{-1} \beta^t - \beta_1 Y_1^{-1} \beta_1^t))} \, dV$$

for every fundamental set V of $G_\infty^m \setminus H_n \times \mathbb{C}^{(j,n)}$. We choose

$$V := \{(Z, W) \mid Z \in \mathcal{F}_{n,m}[u],\, W_1 = \lambda_1 Z_1 + \mu_1,\, 0 \leqq |\lambda_1^{ij}|, |\mu_1^{ij}|, |\alpha_2^{\mu\nu}| \leqq 1\}$$

where $\mathcal{F}_{n,m}[u]$ denotes the fundamental set for $H_n / \Gamma_{n,m}$ described in FREITAG [3], I, 5., (Anhang). Then the above integral becomes

$$\int_{\substack{\mathcal{F}_{n,m}[u] \\ \det Y < C}} (\det Y_1)^{-\frac{k}{2}} \cdot (\det Y)^{\frac{k}{2} - n - j - 1}$$

$$\cdot \left\{ \int_{\mathbb{R}^{(j,n)}/\mathbb{Z}^{(j,n)}} d\alpha \int_{\mathbb{R}^{(j,m)}/\mathbb{Z}^{(j,m)} \cdot Y_1 \times \mathbb{R}^{(j,n-m)}} e^{-2\pi\sigma(\mathcal{M} \cdot (\beta Y^{-1} \beta^t - \beta_1 Y_1^{-1} \beta_1^t))} \, d\beta_1 \, d\beta_2 \right\} dX \, dY.$$

We compute the integral within the brackets: Let

$$Y = \begin{pmatrix} E & 0 \\ B^t & E \end{pmatrix} \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \begin{pmatrix} E & B \\ 0 & E \end{pmatrix}.$$

Then the substitution $(\tilde{\beta}_1, \tilde{\beta}_2) := (\beta_1, \beta_2) \cdot \begin{pmatrix} E & -E \\ 0 & E \end{pmatrix} = (\beta_1, -\beta_1 B + \beta_2)$ yields

$$\int_{\mathbb{R}^{(j,m)}/\mathbb{Z}^{(j,m)} \cdot Y_1 \times \mathbb{R}^{(j,n-m)}} e^{-2\pi\sigma(\mathcal{M} \cdot (\beta Y^{-1} \beta^t - \beta_1 Y_1^{-1} \beta_1^t))} \, d\beta_1 \, d\beta_2$$

$$= \int_{\mathbb{R}^{(j,m)}/\mathbb{Z}^{(j,m)} \cdot Y_1 \times \mathbb{R}^{(j,n-m)}} e^{-2\pi\sigma(\mathcal{M} \cdot \tilde{\beta}_2 Y_2^{-1} \tilde{\beta}_2^t)} \, d\tilde{\beta}_1 \, d\tilde{\beta}_2$$

$$= \text{const} \cdot (\det Y_1)^j \cdot (\det Y_2)^{\frac{j}{2}} = \text{const} \cdot (\det Y_1)^{\frac{j}{2}} \cdot (\det Y)^{\frac{j}{2}}.$$

So

$$\tilde{R} \leqq \text{const} \cdot \int_{\substack{\mathcal{F}_{n,m}[u] \\ \det Y < C}} (\det Y_1)^{-\frac{(k-j)}{2}} \cdot (\det Y)^{\frac{(k-j)}{2}} \cdot [(\det Y)^{-(n+1)} \cdot dX \, dY]$$

and the latter integral is convergent for $k - j > n + m + 1$ (compare FREITAG [3], Chapter I, 5.10), i.e. $k > n + m + j + 1$. □

2.8. Theorem. *For* $k > n + m + \operatorname{rank}(\mathcal{M}) + 1$ *even we have*

$$J_{k,\mathcal{M}}^{\text{cusp}}(\Gamma_m) \subset \mathcal{S}^{(n-m)}(J_{k,\mathcal{M}}(\Gamma_n)),$$

where $\mathcal{S}^{(n-m)}: J_{k,\mathcal{M}}(\Gamma_n) \to J_{k,\mathcal{M}}(\Gamma_m)$ *denotes the* $(n-m)$*-times iterated Siegel operator.*

Proof. In the following we shall show

$$\mathcal{S}^{(n-m)}(E_{k,\mathcal{M}}^{(n)}(*, *, \Phi)) = \Phi,$$

which obviously implies Theorem 2.8. Since the Siegel operator commutes with the mapping $\mathcal{T}_U$ (compare 2.4.) we may assume without loss of generality $\det \mathcal{M} \neq 0$.

Now

$$\mathcal{S}^{(n-m)}(E_{k,\mathcal{M}}^{(n)}(*, *, \Phi)) = \mathcal{S}^{(n-m)}\left(\sum_{\gamma \in \Upsilon} F|_{k,\mathcal{M}}\,\gamma\right)$$
$$= \mathcal{S}^{(n-m)}(F) + \mathcal{S}^{(n-m)}(\Omega)$$

where

$$\Omega(Z, W) := \sum_{\substack{\gamma \in \Upsilon \\ \gamma \notin G_\infty^m}} (F|_{k,\mathcal{M}}\,\gamma)(Z, W).$$

Since $\mathcal{S}^{(n-m)}(F) = \Phi$ by definition of F, it remains to show $\mathcal{S}^{(n-m)}(\Omega) \equiv 0$. Let

$$\Omega(Z, W) = \sum_{T,R} c_\Omega(T, R) \cdot e^{2\pi i \sigma(TZ + RW)}$$

be the Fourierexpansion of Ω. We shall show

$$c_\Omega\left(\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, R\right) = 0 \text{ for every } T \in \mathbb{R}^{(m,m)} \text{ half integer},$$

which implies $\mathcal{S}^{(n-m)}(\Omega)(Z, W) \equiv 0$, since the Siegel operator of a absolutely convergent Fourier series may be computed termwise. Now

$$c_\Omega(T, R) = \int\limits_{\substack{P \\ \operatorname{Im}(Z,W) = (Y_0, \beta_0)}} \Omega(Z, W) \cdot e^{-2\pi i \sigma(TZ + RW)}\, dZ\, dW$$

where P denotes some fundamental parallelotope of the lattice $t(\Gamma_n) \times \mathbb{Z}^{j,n}$ with $t(\Gamma_n) := \{S \in \mathbb{Z}^{(n,n)} \mid S = S^t\}$. Especially if $T = \begin{pmatrix} T^{(m)} & 0 \\ 0 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} Z_1 & * \\ * & * \end{pmatrix}$ with $Z_1 \in H_m$ we obtain:

$$c_\Omega\left(\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, R\right) = \int\limits_{\substack{P \\ \operatorname{Im}(Z,W) = (Y_0, \beta_0)}} \Omega(Z, W) \cdot e^{-2\pi i \sigma(TZ_1)} \cdot e^{-2\pi i \sigma(RW)}\, dZ\, dW$$
$$= \sum_{\substack{\gamma \in \Upsilon \\ \gamma \notin G_\infty^m}} \int\limits_{\substack{P \\ \operatorname{Im}(Z,W) = (Y_0, \beta_0)}} (F|_{k,\mathcal{M}}\,\gamma)(Z, W) \cdot e^{-2\pi i \sigma(TZ_1)} \cdot e^{-2\pi i \sigma(RW)}\, dZ\, dW$$

since $\Omega = \sum_{\substack{\gamma \in \Upsilon \\ \gamma \notin G_\infty^m}} F|_{k,\mathcal{M}}\, \gamma$ converges uniformly on the area of integration. The above integrals are independent of (Y_0, β_0), so we may deform the area of integration to infinity. Now our assertion follows from the formula

$$\lim_{t \to \infty} (H_m |_{k,\mathcal{M}}\, \gamma)\left(X + i\begin{pmatrix} E & 0 \\ 0 & tE \end{pmatrix}, \alpha\right) = 0 \quad \text{for } \gamma \notin G_\infty^m.$$

The latter may be shown by an argument analogous to the Siegelian case (compare FREITAG [3], page 72). We leave the details to the reader. Hint: The matrix norm of $(CZ + D)^{-1}$ (and consequently any exponential factor occuring) remains bounded as $t \to \infty$. $\square$

3. Theta Series

Let $S \in \mathbb{Z}^{(j,j)}$ be symmetric, positive definite and let $a, b \in \mathbb{Q}^{(j,n)}$. We consider the theta series

$$\theta_{S,a,b}(Z, W) := \sum_{\lambda \in \mathbb{Z}^{(j,n)}} e^{\pi i \sigma(S\cdot((\lambda+a)Z(\lambda+a)^t + 2(\lambda+a)(W+b)^t))}$$

with characteristic (a, b) converging normally on $H_n \times \mathbb{C}^{(j,n)}$.

For $Z_0 \in H_n$ and $\mathcal{M} \in \mathbb{R}^{(j,j)}$ symmetric, positive definite and half integer let $T_{\mathcal{M}}(Z_0)$ denote the vectorspace of all holomorphic functions $g: \mathbb{C}^{(j,n)} \to \mathbb{C}$ satisfying

$$g(W + \lambda Z_0 + \mu) = e^{-2\pi i \sigma(\mathcal{M}\cdot(\lambda Z_0 \lambda^t + 2\lambda W^t))} \cdot g(W)$$

for every $\lambda, \mu \in \mathbb{Z}^{(j,n)}$. Writing this functional equation in terms of Fourier coefficients yields:

3.1. Lemma. *Let $\mathcal{N}$ be a complete system of representatives of the cosets $(2\mathcal{M})^{-1}\, \mathbb{Z}^{(j,n)}/\mathbb{Z}^{(j,n)}$. Then the functions*

$$\{\theta_{2\mathcal{M},a,0}(Z_0, W) \mid a \in \mathcal{N}\}$$

form a basis of $T_{\mathcal{M}}(Z_0)$. Especially $D := \dim_C T_{\mathcal{M}}(Z_0) = \{\det(2\mathcal{M})\}^n$.

As an easy application of the poisson summation formula we obtain:

3.2. Lemma. *For $a \in \mathcal{N}$ we have*

$$\theta_{2\mathcal{M},a,0}(-Z^{-1}, WZ^{-1}) = \{\det(2\mathcal{M})\}^{-\frac{n}{2}} \cdot \left\{\det\left(\frac{Z}{j}\right)\right\}^{\frac{j}{2}} \cdot e^{2\pi i \sigma(\mathcal{M}\cdot WZ^{-1}W^t)}$$

$$\sum_{b \in \mathcal{N}} e^{-2\pi i \sigma(2\mathcal{M}\cdot ba^t)} \cdot \theta_{2\mathcal{M},b,0}(Z, W);$$

furthermore $\theta_{2\mathcal{M},a,0}(Z+S, W) = e^{2\pi i\sigma(\mathcal{M}\cdot aSa^t)} \cdot \theta_{2\mathcal{M},a,0}(Z, W)$ *holds for every* $S \in \mathbb{Z}^{(n,n)}$ *symmetric.*

Using 3.1. and 3.2. we obtain a result of Shimura establishing an isomorphism between $J_{\varrho,\mathcal{M}}(\Gamma_n)$ and a certain space of vector valued Siegel modular forms of half integral weight. By Theorem 2.4. we have $J_{\varrho,\mathcal{M}}(\Gamma_n) \cong J_{\varrho,\tilde{\mathcal{M}}}(\Gamma_n)$, so we may assume without loss of generality $\det \mathcal{M} \neq 0$.

For fixed $Z_0 \in H_n$ and $\Phi \in J_{\varrho,\mathcal{M}}(\Gamma_n)$ each component of the mapping $g(W) := \Phi(Z_0, W)$ is contained in $T_{\mathcal{M}}(Z_0)$. So we may write

$$\Phi(Z, W) = \sum_{a\in\mathcal{N}} f_a(Z) \cdot \theta_{2\mathcal{M},a,0}(Z, W) \tag{*}$$

with uniquely determined holomorphic mappings $f_a : H_n \to E$. The holomorphicity of the f_a's is an immediate consequence of the linear independence of the $\theta_{2\mathcal{M},a,0}(Z, *)$'s $(a \in \mathcal{N})$. Now $\Phi \in J_{\varrho,\mathcal{M}}(\Gamma_n)$ is a Jacobi Form, so $\Phi|_{\varrho,\mathcal{M}} M = \Phi$ holds for every $M \in \Gamma_n$. This together with Lemma 3.2. implies

$$f_a(-Z^{-1}) = \left\{\det\left(\frac{Z}{i}\right)\right\}^{-\frac{j}{2}} \cdot \{\varrho(-Z)\} \cdot \{\det(2\mathcal{M})\}^{-\frac{n}{2}} \cdot \sum_{b\in\mathcal{N}} e^{2\pi i\sigma(2\mathcal{M}\cdot ab^t)} \cdot f_b(Z), \tag{1}$$

$$f_a(Z+S) = e^{-2\pi i\sigma(\mathcal{M}\cdot aSa^t)} \cdot f_a(Z) \text{ for every } S \in \mathbb{Z}^{(n,n)} \text{ symmetric.} \tag{2}$$

Furthermore the Fourier coefficients $c(T, R)$ of $\theta_{2\mathcal{M},a,0}(Z, W)$ are given by

$$c(T, R) = \begin{cases} 1, & \text{if } \exists\, \lambda \in \mathbb{Z}^{(j,n)} \text{ such that } R^t = 2\mathcal{M}(\lambda + a) \text{ and } T = (\lambda+a)^t\,\mathcal{M}(\lambda+a) \\ 0, & \text{otherwise.} \end{cases}$$

Especially $c(T, R) \neq 0$ only if $4T - R\mathcal{M}^{-1}R^t = 0$. Now the cusp condition for the Jacobi Form Φ implies that the functions $\{f_a \mid a \in \mathcal{N}\}$ necessarily must have Fourier expansions of the form

$$f_a(Z) = \sum_{\substack{T=T^t\geq 0\\ \text{half integer}}} c(T) \cdot e^{2\pi i\sigma(TZ)}.$$

Conversely suppose given a family $\{f_a \mid a \in \mathcal{N}\}$ of holomorphic mappings $f_a : H_n \to E$ satisfying the functional equations (1), (2) and the cusp condition (3). Then we obtain a Jacobi Form in $J_{\varrho,\mathcal{M}}(\Gamma_n)$ by defining $\Phi(Z, W)$ via equation (*). We have shown:

3.3. Theorem (Shimura). *Equation* (*) *gives an isomorphism between* $J_{\varrho,\mathcal{M}}(\Gamma_n)$ *and the space of vector valued Siegel modular forms of half integral weight satisfying the transformation laws* (1), (2) *and the cusp condition* (3).

Remark. Of course Theorem 3.3. may also be formulated for Jacobi Forms on subgroups $\Gamma \subset \Gamma_n$ of finite index. If rank $(\mathcal{M})$ is even, then Theorem 3.3. gives an isomorphism of $J_{\varrho,\mathcal{M}}(\Gamma)$ onto a certain space of vector valued Siegel modular forms of integral weight. For more details we refer to SHIMURA [8]—he especially gives a precise definition of vector valued Siegel modular forms of half integral weight.

Corollary 1. *For* $k < \frac{1}{2}\operatorname{rank}(\mathcal{M})$ *we have* $\dim_C J_{k,\mathcal{M}}(\Gamma) = 0$.

Corollary 2. *For* $(2\mathcal{M})$ *unimodular and* $k \cdot n$ *even we have*

$$J_{k,\mathcal{M}}(\Gamma) = \left[\Gamma, k - \frac{j}{2}\right] \cdot \theta_{2\mathcal{M},0,0}(Z, W) \simeq \left[\Gamma, k - \frac{j}{2}\right].$$

We now show that the isomorphism described in Theorem 3.3. is in some sense compatible with the Petersson scalar product defined in 2.2. More precisely:

3.4. Lemma. *Let* $\Phi, \Psi \in J_{\varrho,\mathcal{M}}(\Gamma_n)$ *be two Jacobi Forms such that*

$$\Phi(Z, W) = \sum_{a\in\mathcal{N}} f_a(Z) \cdot \theta_{2\mathcal{M},a,0}(Z, W) \text{ and } \Psi(Z, W) = \sum_{b\in\mathcal{N}} g_b(Z) \cdot \theta_{2\mathcal{M},b,0}(Z, W).$$

Then

$$(\Phi, \Psi)_{\Gamma_n} = \{\det(4\mathcal{M})\}^{-\frac{n}{2}} \cdot \int_{H_n/\Gamma_n} \sum_{a\in\mathcal{N}} \langle\varrho(Y^{\frac{1}{2}})f_a(Z), \varrho(Y^{\frac{1}{2}})\, g_a(Z)\rangle \cdot (\det Y)^{-\frac{j}{2}}\, d\omega,$$

where $d\omega := [(\det Y)^{-(n+1)}\, dX\, dY]$.

Remark. The right hand side is just the expression for the Petersson scalar product $(f, g)_{\Gamma_n}$ of the vector valued Siegel modular forms of half integral weight $f := \{f_a \mid a \in \mathcal{N}\}$ and $g := \{g_b \mid b \in \mathcal{N}\}$.

Proof of 3.4. Our assertion follows immediately from the formula

$$\int_{\mathbb{R}^{(j,n)}/\mathbb{Z}^{(j,n)} \times \mathbb{R}^{(j,n)}/\mathbb{Z}^{(j,n)} \cdot Y} e^{-4\pi\sigma(\mathcal{M}\cdot\beta Y^{-1}\beta^t)} \cdot \theta_{2\mathcal{M},a,0}(Z, W) \cdot \overline{\theta_{2\mathcal{M},b,0}(Z, W)}\, d\alpha\, d\beta$$

$$= \delta_{a,b} \cdot \{\det(4\mathcal{M})\}^{-\frac{n}{2}} \cdot \{\det Y\}^{\frac{j}{2}}$$

where $\delta_{a,b}$ denotes the Kronecker delta of a and b. □

Our next aim is to construct Jacobi Forms in $J_{k,\mathcal{M}}(\Gamma_n)$ by means of theta series. We have already seen the example $\theta_{2\mathcal{M},0,0}(Z, W)$, which is a Jacobi Form in $J_{\frac{j}{2},\mathcal{M}}(\Gamma_n)$ for $2\mathcal{M}$ unimodular. For our purpose the following lemma is useful:

3.5. Lemma. *Let* $\Phi \in J_{\varrho,\mathcal{M}_1}(\Gamma)$ $(2\mathcal{M}_1 \in \mathbb{Z}^{(j_1,j_1)})$ *be a Jacobi Form and let* $c \in \mathbb{Z}^{(j_1,j_2)}$. *Then the mapping*

$$\Phi^c : H_n \times \mathbb{C}^{(j_2,n)} \to E,$$
$$(Z, W) \to \Phi(Z, cW)$$

defines a Jacobi Form in $J_{\varrho,\mathcal{M}_2}(\Gamma)$ *with* $\mathcal{M}_2 := c^t\mathcal{M}_1 c$.

The proof of Lemma 3.5. is straightforward.

Corollary. *For* $U \in Gl(j, \mathbb{Z})$ *we have* $J_{\varrho,\mathcal{M}}(\Gamma) \cong J_{\varrho,U^t\mathcal{M}U}(\Gamma)$.

3.6. Definition. Let $S \in \mathbb{Z}^{(2k,2k)}$ be symmetric, positive definite, unimodular, even and let $c \in \mathbb{Z}^{(2k,j)}$. We define the theta series $\vartheta_{S,c}^{(n)}$ by

$$\vartheta_{S,c}^{(n)}(Z, W) := \sum_{\lambda \in \mathbb{Z}^{(2k,n)}} e^{\pi i\{\sigma(S\lambda Z\lambda^t) + 2\sigma(c^t S\lambda W^t)\}} \quad \text{for } Z \in H_n \text{ and } W \in \mathbb{C}^{(j,n)}.$$

We observe $\vartheta_{S,c}^{(n)} \in J_{k,\mathcal{M}}(\Gamma_n)$ with $\mathcal{M} := \frac{1}{2} c^t S c$ since $\vartheta_{S,c}^{(n)}(Z, W) = \theta_{S,0,0}(Z, cW)$. The Fourier coefficients $c(T, R)$ of $\vartheta_{S,c}^{(n)}$ are given by

$$\begin{aligned} c(T, R) &:= \#\{\lambda \in \mathbb{Z}^{(2k,n)} \mid \lambda^t S \lambda = 2T, \lambda^t S c = R\} \\ &= \#\left\{\lambda \in \mathbb{Z}^{(2k,n)} \,\middle|\, \frac{1}{2}(\lambda, c)^t S(\lambda, c) = \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}R^t & \mathcal{M} \end{pmatrix}\right\}. \end{aligned}$$

Especially we have $\mathcal{S}^{(n-m)}(\vartheta_{S,c}^{(n)}) = \vartheta_{S,c}^{(m)}$, which may also be checked directly.

3.7. Definition. A Jacobi Form $\Phi \in J_{\varrho,\mathcal{M}}(\Gamma)$ is called singular, if it admits a Fourier expansion such that the Fourier coefficients $c(T, R)$ are zero, unless $\det(4T - \tilde{R}\tilde{\mathcal{M}}^{-1}\tilde{R}^t) = 0$.

We say Φ is singular of degree d, if

$$\max\{\operatorname{rank}(4T - \tilde{R}\tilde{\mathcal{M}}^{-1}\tilde{R}^t) \mid c(T, R) \neq 0\} = n - d \quad (0 < d \leqq n).$$

Examples of singular Jacobi Forms are given by theta series: Let $2k < n + \operatorname{rank}(\mathcal{M})$, then $\vartheta_{S,c}^{(n)} \in J_{k,\mathcal{M}}(\Gamma_n)$ is singular, since suppose given T, R with $c(T, R) \neq 0$, there exists some $\lambda \in \mathbb{Z}^{(2k,n)}$ with

$$\frac{1}{2}(\lambda, c)^t S(\lambda, c) = \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}R^t & \mathcal{M} \end{pmatrix}.$$

So $\operatorname{rank}\begin{pmatrix} T & \frac{1}{2}\tilde{R} \\ \frac{1}{2}\tilde{R}^t & \tilde{\mathcal{M}} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}R^t & \mathcal{M} \end{pmatrix} \leqq 2k < n + \operatorname{rank}(\mathcal{M}) =$ $n + \operatorname{rank}(\tilde{\mathcal{M}})$, which implies $\det(4T - \tilde{R}\tilde{\mathcal{M}}^{-1}\tilde{R}^t) = 0$.

In the case $2\mathcal{M}$ unimodular and kn even we have $J_{k,\mathcal{M}}(\Gamma_n) \simeq \left[\Gamma_n, k - \frac{j}{2}\right]$ (Corollary 2 of Theorem 3.3). Using this isomorphism we obtain the following two lemmata, which carry over in a more or less trivial manner from the theory of Siegel modular forms: Lemma 3.8. is a first hint to our result 3.12., while Lemma 3.9. gives us an idea about generalisations of Siegel's Hauptsatz.

3.8. Lemma. *Let $\Phi \in J_{k,\mathcal{M}}(\Gamma_n)$ be a non-vanishing singular Jacobi Form and suppose $2\mathcal{M}$ unimodular, kn even. Then:*

1. $2k < n + \operatorname{rank}(\mathcal{M})$, $k \equiv 0 \pmod 4$,
2. *Φ may be written as linear combination of theta series $\vartheta^{(n)}_{S,c}$.*

Proof. Lemma 3.8. is essentially a restatement of the well known result concerning singular Siegel modular forms in $\left[\Gamma_n, k - \frac{j}{2}\right]$. □

3.9. Lemma. *Let $(2\mathcal{M})$ be unimodular and let* $k \equiv 0 \pmod 4$. *Then the ordinary Jacobi Eisenstein series $E^{(n)}_{k,\mathcal{M}}$ may be written as linear combination of theta series $\vartheta^{(n)}_{S,c}$.*

Proof. $E^{(n)}_{k,\mathcal{M}}(Z, W) = \vartheta^{(n)}_{2\mathcal{M},E}(Z, W) \cdot E^{(n)}_{k-\frac{j}{2}}(Z)$. □

Now the question arises whether we have general theorems like 3.8. and 3.9. valid for arbitrary index $\mathcal{M}$. The answer seems to be affirmative in both cases: Singular Jacobi Forms shall be investigated in the remainder of this chapter, while a more general result of type 3.9. is presented in Chapter 4.

First we note that using the standard argument 2.4. we may restrict our investigation of singular Jacobi forms without loss of generality to the case of an invertible index $\mathcal{M} > 0$. The main problem is to handle the complicated linear relations between the $\vartheta^{(n)}_{S,c}$'s, which seems to be rather difficult in general. The result we shall prove is valid for a more special type of singular Jacobi Forms:

3.10. Definition. A Jacobi Form $\Phi \in J_{k,\mathcal{M}}(\Gamma_n)$ is called strongly singular, if:

1. $\det \mathcal{M} \neq 0$,
2. $2k < n$,
3. Φ is singular of degree n, i.e.

$$c(T, R) \neq 0 \text{ only if } 4T - R\mathcal{M}^{-1} R^t = 0.$$

Examples are given by means of theta series $\vartheta^{(n)}_{S,C}$ with non singular $C \in \mathbb{Z}^{(2k,2k)}$, $(2k < n)$. We have $\mathcal{M} = \frac{1}{2} C^t S C$, so $\operatorname{rank}(\mathcal{M}) = 2k$ and

$$c(T, R) = \#\{\lambda \in \mathbb{Z}^{(2k,n)} \mid \lambda^t S \lambda = 2T, \lambda^t S C = R\}$$
$$= \begin{cases} 1, & \text{if } R \in \mathbb{Z}^{(n,2k)} \cdot C \text{ and } 4T - R\mathcal{M}^{-1} R^t = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Observe that the matrix $\frac{1}{2} R\mathcal{M}^{-1} R^t$ is symmetric, positive and even for $R \in \mathbb{Z}^{(n,2k)} \cdot C$. Of particular interest are the Fourier coefficients $c\left(\frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & S\end{pmatrix}, \begin{pmatrix}0\\ R_S\end{pmatrix}\right)$ with $R_S \in \Delta_R = \Delta_R(S, \mathcal{M}) := \{R_S \in \mathbb{Z}^{(2k,2k)} \mid R_S \mathcal{M}^{-1} R_S^t = 2S\}$.

Note that $\Delta_R \neq \emptyset$ since $SC \in \Delta_R$. We have

$$c\left(\frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & S\end{pmatrix}, \begin{pmatrix}0\\ R_S\end{pmatrix}\right) = \begin{cases}1 & \text{if } \exists\ U \in \operatorname{Aut}(S) \text{ with } USC = R_S\\ 0 & \text{otherwise,}\end{cases}$$

where $\operatorname{Aut}(S) := \{U \in Gl(2k, \mathbb{Z}) \mid USU^t = S\}$ denotes the group of automorphisms of S. It is clear that $\operatorname{Aut}(S)$ acts on Δ_R by multiplication from the left. The set

$$\left\{R_S \in \Delta_R \mid c\left(\frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & S\end{pmatrix}, \begin{pmatrix}0\\ R_S\end{pmatrix}\right) = 1\right\}$$

is precisely the coset $\operatorname{Aut}(S) \cdot SC \in \operatorname{Aut}(S) \setminus \Delta_R$.

This important observation gives us complete information about linear relations between the $\vartheta_{S,C}^{(n)}$'s in the strongly singular case: We set

$$\Delta_C = \Delta_C(S, \mathcal{M}) := \left\{C \in \mathbb{Z}^{(2k,2k)} \,\middle|\, \frac{1}{2} C^t SC = \mathcal{M}\right\}$$

and consider the mapping $\Psi: \Delta_C \to \operatorname{Aut}(S) \setminus \Delta_R$ defined by $\Psi(C) := \operatorname{Aut}(S) \cdot SC$. It is easy to verify that $\Psi(C_1) = \Psi(C_2)$ holds if and only if $C_1 = U^{-t} C_2$ for some $U \in \operatorname{Aut}(S)$. In other words Ψ induces an injective mapping $\overline{\Psi}$: $\operatorname{Aut}(S^{-1}) \setminus \Delta_C \to \operatorname{Aut}(S) \setminus \Delta_R$. So given some complete system of representatives $(C_i)_{i=1,\dots,\gamma}$ of the cosets $\operatorname{Aut}(S^{-1}) \setminus \Delta_C$, the corresponding theta series $\vartheta_{S,C_1}^{(n)}, \dots, \vartheta_{S,C_\gamma}^{(n)}$ are linearly independent. On the other side the substitution $\lambda \to U^t\lambda$ for $U \in Gl(2k, \mathbb{Z})$ yields

$$\vartheta_{S,C}^{(n)}(Z, W) = \vartheta_{USU^t, U^{-t}C}^{(n)}(Z, W) \quad \text{for every } U \in Gl(2k, \mathbb{Z}).$$

Furthermore tde theta series

$$\vartheta_{S_1}^{(n)}(Z), \dots, \vartheta_{S_l}^{(n)}(Z)$$

are linearly independent for $2k \leq n$, if only $S_1, \dots, S_l$ are pairwise inequivalent. Combining these results we obtain:

3.11. Lemma. *For $2k < n$ and $k \equiv 0 \pmod 4$ let $S_1, \dots, S_h$ be a complete system of representatives of the unimodular classes of symmetric, positive definite, unimodular, even matrices $S \in \mathbb{Z}^{(2k,2k)}$; furthermore let $C_1^{(\alpha)}, \dots, C_{\gamma_\alpha}^{(\alpha)}$ $(\alpha = 1, \dots, h)$ be a complete system of representatives of the cosets $\operatorname{Aut}(S_\alpha^{-1}) \setminus \Delta_C(S_\alpha, \mathcal{M})$. Then the theta series*

$$\{\vartheta_{S_\alpha, C_\beta^{(\alpha)}}^{(n)} \mid \alpha \in \{1, \dots, h\}, \beta \in \{1, \dots, \gamma_\alpha\}\}$$

are linearly independent. Furthermore each strongly singular theta series $\vartheta^{(n)}_{S,C} \in J_{k,\mathcal{M}}(\Gamma_n)$ *coincides with precisely one of the above listed ones.*

Remark. The above description of the linear relations between the $\vartheta^{(n)}_{S,C}$'s depends essentially on the strongly singular case—in general there exist much more linear relations.

Lemma 3.11. provides a good deal of information necessary to prove:

3.12. Theorem. *Let* $\Phi \in J_{k,\mathcal{M}}(\Gamma_n)$ *be a non-vanishung strongly singular Jacobi Form. Then*:

1. $2k = \operatorname{rank}(\mathcal{M}),\ k \equiv 0 \pmod 4$,
2. Φ *may be written as linear combination of theta series* $\vartheta^{(n)}_{S,C}$.

Before proving Theorem 3.12. we state two easy auxiliary lemmata without proof:

3.13. Lemma. *Let* $\Phi \in J_{k,\mathcal{M}}(\Gamma_n)$ *be a Jacobi Form and let* $c(T, R)$ *denote its Fourier coefficients. Then*

$$c(UTU^t, UR) = c(T, R), \textit{ for every } U \in Gl(n, \mathbb{Z}), \tag{1}$$

$$c\left(T + \frac{1}{2}R\lambda + \frac{1}{2}\lambda^t R^t + \lambda^t \mathcal{M}\lambda,\ R + 2\lambda^t \mathcal{M}\right) = c(T, R) \textit{ for every } \lambda \in \mathbb{Z}^{(j,n)}. \tag{2}$$

3.14. Lemma. (Fourier Jacobi expansion of Jacobi Forms). *Let* $\Phi \in J_{k,\mathcal{M}}(\Gamma_n)$ *be a Jacobi Form and let* $\varrho < n$. *We suppose* $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$ *with* $Z_2 \in H_\varrho$ *and* $W = (W_1, W_2)$ *with* $W_2 \in \mathbb{C}^{(j,\varrho)}$. *The coefficients* Ψ_{T_2,R_2} *of the partial Fourier expansion*

$$\Phi\left(\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}, (W_1, W_2)\right) = \sum_{T_2,R_2} \Psi_{T_2,R_2}(Z_1, W_1) \cdot e^{2\pi i\sigma(T_2 Z_2)} \cdot e^{2\pi i\sigma(R_2 W_2)}$$

define Jacobi Forms in $J_{k,\mathcal{M}}(\Gamma_{n-\varrho})$ *for every* T_2, R_2.

Proof of 3.12. Let $c(T, R)$ be a non-vanishing Fourier coefficient of Φ. Then $4T - R\mathcal{M}^{-1}R^t = 0$ implies $\operatorname{rank}(T) < n$ since $\operatorname{rank}(\mathcal{M}) \leqq 2k < n$ by Corollary 1 of 3.3. Let

$$\varrho := \max\{\operatorname{rank}(T) \mid \exists\, R \text{ such that } c(T, R) \neq 0\}.$$

By Lemma 3.13. we have $c(UTU^t, UR) = c(T, R)$ for every $U \in Gl(n, \mathbb{Z})$, so there exists some $S \in \mathbb{Z}^{(\varrho,\varrho)}$ with $\det S \neq 0$ and some $R_S \in \mathbb{Z}^{(\varrho,j)}$ such that

$$c\left(\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}, \begin{pmatrix} 0 \\ R_S \end{pmatrix}\right) \neq 0.$$

The formula $2S - R_S \mathcal{M}^{-1} R_S^t = 0$ implies $\varrho \leqq j = \operatorname{rank}(\mathcal{M})$. We now choose and fix some pair S, R_S like above with det S minimal. Analogously to the theory of singular Siegel modular forms our main idea is to consider the Fourier Jacobi expansion of Φ (compare 3.14.) for $Z_2 \in H_\varrho$, $W_2 \in \mathbb{C}^{(j,\varrho)}$ with ϱ like above:

$$\Phi\left(\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}, (W_1, W_2)\right) = \sum_{T_2, R_2} \Psi_{T_2,R_2}(Z_1, W_1) \cdot e^{2\pi i \sigma(T_2 Z_2)} \cdot e^{2\pi i \sigma(R_2 W_2)}.$$

Especially we shall compute the coefficient $\Psi_{\frac{1}{2}S,R_S} \in J_{k,\mathcal{M}}(\Gamma_{n-\varrho})$. We have

$$\Psi_{\frac{1}{2}S,R_S}(Z_1, W_1) = \sum_{T_1,T_{12},R_1} c\left(\begin{pmatrix} T_1 & T_{12}^t \\ T_{12} & \frac{1}{2}S \end{pmatrix}, \begin{pmatrix} R_1 \\ R_S \end{pmatrix}\right) \cdot e^{2\pi i \sigma(T_1 Z_1)} \cdot e^{2\pi i \sigma(R_1 W_1)}.$$

Like in the Siegelian case we obtain

$$\Psi_{\frac{1}{2}S,R_S}(Z_1, W_1) = c\left(\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}S \end{pmatrix}, \begin{pmatrix} 0 \\ R_S \end{pmatrix}\right) \cdot \sum_{T_1,T_{12},R_1} e^{2\pi i \sigma(T_1 Z_1)} \cdot e^{2\pi i \sigma(R_1 W_1)},$$

where the summation is taken over all T_1, T_{12}, R_1 for which there exists some $U \in Gl(n, \mathbb{Z})$ such that

$$\begin{pmatrix} T_1 & T_{12}^t \\ T_{12} & \frac{1}{2}S \end{pmatrix} = U \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}S \end{pmatrix} U^t$$

and

$$\begin{pmatrix} R_1 \\ R_S \end{pmatrix} = U \cdot \begin{pmatrix} 0 \\ R_S \end{pmatrix}.$$

The set of all such T_1, T_{12}, R_1 can be described explicitely: We necessarily must have

$$\left(\begin{pmatrix} T_1 & T_{12}^t \\ T_{12} & \frac{1}{2}S \end{pmatrix}, \begin{pmatrix} R_1 \\ R_S \end{pmatrix}\right) = \left(\frac{1}{2}\begin{pmatrix} \lambda^t S \lambda & \lambda^t S \\ S\lambda & S \end{pmatrix}, \begin{pmatrix} \lambda^t R_S \\ R_S \end{pmatrix}\right)$$

for some $\lambda \in \mathbb{Z}^{(\varrho, n-\varrho)}$. Therefore

$$\Psi_{\frac{1}{2}S,R_S}(Z_1, W_1) = c\left(\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}S \end{pmatrix}, \begin{pmatrix} 0 \\ R_S \end{pmatrix}\right) \cdot \sum_{\lambda \in \mathbb{Z}^{(\varrho, n-\varrho)}} e^{\pi i \{\sigma(S\lambda T Z_1 \lambda^t) + 2\sigma(R_S^t \lambda W_1^t)\}}$$

Now $\Psi_{\frac{1}{2}S,R_S} \in J_{k,\mathcal{M}}(\Gamma_{n-\varrho})$ implies $\Psi_{\frac{1}{2}S,R_S}(Z_1, 0) \in [\Gamma_{n-\varrho}, k]$, so $\det S = 1$, rank $(S) = 2k$ and $k \equiv 0 \pmod 4$ follow immediately. Furthermore defining

$C := S^{-1} \cdot R_S \in \mathbb{Z}^{(\varrho,j)}$ we observe

$$\Psi_{\frac{1}{2}S,R_S}(Z_1, W_1) = \text{const} \cdot \vartheta_{S,C}^{(n-\varrho)}(Z_1, W_1).$$

Consequently $\frac{1}{2}\, C^t SC = \mathcal{M}$ and $\varrho = \operatorname{rank}(S) = \operatorname{rank}(\mathcal{M}) = j$. This proves the first part of our theorem.

For the second part let $S_1, \dots, S_h, C_1^{(\alpha)}, \dots, C_{\gamma_\alpha}^{(\alpha)}$ $(\alpha = 1, \dots, h)$ be like in Lemma 3.11. Furthermore let $c_\beta^\alpha(T, R)$ denote the Fouriercoefficients of $\vartheta^{(n)}_{S_\alpha, C_\beta^{(\alpha)}}$. We consider the function

$$\Phi_0(Z, W) := \Phi(Z, W) - \sum_{\alpha=1}^{h} \sum_{\beta=1}^{\gamma_h} \mu_{\alpha,\beta} \cdot \vartheta^{(n)}_{S_\alpha, C_\beta^{(\alpha)}}(Z, W)$$

with

$$\mu_{\alpha,\beta} := \frac{c\left(\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & S_\alpha \end{pmatrix}, S_\alpha C_\beta^{(\alpha)}\right)}{c_\beta^\alpha\left(\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & S_\alpha \end{pmatrix}, S_\alpha C_\beta^{(\alpha)}\right)}$$

which is also strongly singular. By definition each Fourier coefficient $c_0(T, R)$ of Φ_0 vanishes for $T = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}$, $S \in \mathbb{Z}^{(2k,2k)}$, $\det S = 1$ and $R = \begin{pmatrix} 0 \\ R_S \end{pmatrix}$ with $2S = R_S \mathcal{M}^{-1} R_S^t$. So recalling the beginning of our proof $\Phi_0 \equiv 0$ follows. □

4. Applications

1. A non-surjectiveness theorem

Let $F \in [\Gamma_{n+1}, k]$ be a Siegel modular form of degree $n + 1$ and weight k. We consider the mapping

$$\alpha : [\Gamma_{n+1}, k] \to J_{k,1}(\Gamma_n)$$

defined by sending F to its first Fourier Jacobi coefficient Φ_1 obtained from the expansion

$$F\begin{pmatrix} Z_1 & W^t \\ W & z_2 \end{pmatrix} = \sum_{m=0}^{\infty} \Phi_m(Z_1, W) \cdot e^{2\pi i m z_2}.$$

If $n = 1$, there exists a certain subspace $[\Gamma_2, k]^* \subset [\Gamma_2, k]$, namely the Maaß Spezialschar, which maps isomorphically onto $J_{k,1}(\Gamma_1)$ (compare Eichler/Zagier [2], 6). In the following we shall show that in general the mapping $\alpha : [\Gamma_{n+1}, k] \to J_{k,1}(\Gamma_n)$ is not surjective, so there exists no generalisation of the Maaß Spezialschar to arbitrary degrees in the above sense.

We start with the investigation of the Fourier Jacobi expansion of theta series defining Siegel modular forms. For $S \in \mathbb{Z}^{(2k,2k)}$ symmetric, positive definite, unimodular, even and $\tilde{Z} \in \boldsymbol{H}_{n+j}$ the theta series

$$\vartheta_S^{(n+j)}(\tilde{Z}) := \sum_{\tilde{\lambda} \in \mathbb{Z}^{(2k,n+j)}} e^{\pi i \sigma(S\tilde{\lambda}\tilde{Z}\tilde{\lambda}^t)}$$

defines a Siegel modular form in $[\Gamma_{n+j}, k]$. It is easy to verify that the partial Fourier expansion with respect to the variable $Z_2 \in \boldsymbol{H}_j$ obtained from the decomposition $\tilde{Z} = \begin{pmatrix} Z & W^t \\ W & Z_2 \end{pmatrix}$ is of the form $\vartheta_S^{(n+j)}(\tilde{Z}) = \sum_{\mathcal{M} \geq 0} \Phi_{\mathcal{M}}(Z, W) \cdot e^{2\pi i \sigma(\mathcal{M} Z_2)}$ with $\mathcal{M}$-th Fourier Jacobi coefficient

$$\Phi_{\mathcal{M}}(Z, W) = \sum_{\{c \in \mathbb{Z}^{(2k,j)} | \frac{1}{2} c^t S c = \mathcal{M}\}} \vartheta_{S,c}^{(n)}(Z, W).$$

Next we shall show that for fixed S there may exist $c_1, c_2 \in \mathbb{Z}^{(2k,1)}$, such that $\vartheta_{S,c_1}^{(n)}, \vartheta_{S,c_2}^{(n)} \in J_{k,1}(\Gamma_n)$ are linearly independent. In order to verify that two such theta series are linearly independent, it suffices to show that one Fourier coefficient $c(T, R)$ takes different values for $\vartheta_{S,c_1}^{(n)}$ and $\vartheta_{S,c_2}^{(n)}$, since $c(0, 0) = 1$ for each $\vartheta_{S,c}^{(n)}$. We recall that

$$c(T, R) = \#\{\lambda \in \mathbb{Z}^{(2k,n)} \mid \lambda^t S \lambda = 2T, \lambda^t S c_i = R\} \text{ for } i = 1, 2.$$

We shall construct an example for $k = 16$. Let $R = 0$ and $T = T_0 := \begin{pmatrix} 1 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ \vdots & \vdots \ddots \vdots \\ 0 & 0 \dots 0 \end{pmatrix}$ then

$$c(T_0, 0) = \#\{\lambda \in \mathbb{Z}^{(32,1)} \mid \lambda^t S \lambda = 2, \lambda^t S c_i = 0\} \text{ for } i = 1, 2.$$

So in order to construct our desired example, we have to find a suitable $S \in \mathbb{Z}^{(32,32)}$ and two vectors $c_1, c_2 \in \mathbb{Z}^{(32,1)}$ with $c_1^t S c_1 = c_2^t S c_2 = 2$, such that $c(T_0, 0)$ takes different values for $i = 1, 2$. For $S \in \mathbb{Z}^{(32,32)}$ we choose the matrix obtained from the even unimodular lattice $E := E_8 \oplus E_{24}$, where

$$E_{8m} := \left\{ \frac{1}{2}(x_1, \dots, x_{8m}) \in \mathbb{R}^{(8m)} \mid x_i \in \mathbb{Z}, x_j - x_k \equiv 0 \ (\mathrm{mod}\ 2), \sum_{i=1}^{8m} x_i \equiv 0 \ (\mathrm{mod}\ 4) \right\}$$

for $m \in \mathbb{N}$ (compare SERRE [9]). For $L \in \{E, E_8, E_{24}\}$, $t \in \mathbb{N}$ and $y \in L$ we define:

$$N(L, 2t) := \#\{x \in L \mid x^t x = 2t\},$$

$$N(L, 2t, y) := \#\{x \in L \mid x^t x = 2t, x^t y = 0\}.$$

Our problem is then equivalent to finding two vectors $y_1, y_2 \in E$ with $y_1^t y_1 =$

$y_2^t y_2 = 2$, such that $N(E, 2, y_1) \neq N(E, 2, y_2)$. We claim that the vectors

$$y_1^t := \frac{1}{2}(\underbrace{1, 1, 1, 1, 1, 1, 1, 1}_{E_8}, \underbrace{0, 0, \ldots, 0, 0, 0}_{E_{24}}) \in E$$

$$y_2^t := \frac{1}{2}(\underbrace{0, 0, 0, 0, 0, 0, 0, 0}_{E_8}, \underbrace{0, 0, \ldots, 0, 2, 2}_{E_{24}}) \in E$$

have the desired properties. Using the notation $\tilde{y}_1$ (resp. $\tilde{y}_2$) for the canonical projections of y_1 onto E_8 and y_2 onto E_{24} some easy combinatorial calculations yield:

$$N(E, 2, y_1) = N(E_8, 2, \tilde{y}_1) + N(E_{24}, 2) = 126 + 1104 = 1230,$$

$$N(E, 2, y_2) = N(E_8, 2) + N(E_{24}, 2, \tilde{y}_2) = 240 + 926 = 1166.$$

We have shown:

4.1. Lemma. *There exists some $S \in \mathbb{Z}^{(32,32)}$ symmetric, positive definite, unimodular, even and $c_1, c_2 \in \mathbb{Z}^{(32,1)}$ with $c_1^t S c_1 = c_2^t S c_2 = 2$, such that the theta series $\vartheta_{S,c_1}^{(n)}$ and $\vartheta_{S\,c_2}^{(n)}$ define linearly independent Jacobi Forms in $J_{16,1}(\Gamma_n)$.*

Combining Lemma 4.1. with the description of the Fourier Jacobi expansion of the theta series $\vartheta_S^{(n+j)}(\tilde{Z}) \in [\Gamma_{n+j}, k]$ for $k = 16, j = 1$ we obtain:

4.2. Theorem. *For $n \geqq 32$ the mapping $\alpha : [\Gamma_{n+1}, 16] \to J_{16,1}(\Gamma_n)$ defined by Fourier Jacobi expansion is not surjective.*

Proof. We have $n + 1 > 2k$, so each $F \in [\Gamma_{n+1}, 16]$ is singular (compare FREITAG [3], III, 5. and Anhang IV). Therefore the theta series

$$\vartheta_{S_1}^{(n+1)}(\tilde{Z}), \ldots, \vartheta_{S_{h(32)}}^{(n+1)}(\tilde{Z})$$

form a basis of $[\Gamma_{n+1}, 16]$, whenever $\mathscr{R} := \{S_1, \ldots, S_{h(32)}\}$ is a complete system of representatives of the unimodular classes of symmetric, positive definite, unimodular, even matrices $S \in \mathbb{Z}^{(32,32)}$. Especially $\dim_{\mathbb{C}}[\Gamma_{n+1}, 16] = h(32)$ for $n \geqq 32$. Furthermore

$$\dim_{\mathbb{C}} \alpha([\Gamma_{n+1}, 16]) = \#\{S \in \mathscr{R} \mid \exists\, c \in \mathbb{Z}^{(32,1)} \text{ such that } c^t S c = 2\}.$$

On the other side Lemma 4.1. implies

$$\dim_{\mathbb{C}} J_{16,1}(\Gamma_n) \geqq \#\{S \in \mathscr{R} \mid \exists\, c \in \mathbb{Z}^{(32,1)} \text{ such that } c^t S c = 2\} + 1,$$

since $\vartheta_{S,c}^{(n)}(Z, 0) = \vartheta_S^{(n)}(Z)$ and $n \geqq 2k$. Comparison of the dimension estimates shows that α is not surjective.

2. Siegel's Hauptsatz for Jacobi Forms of index $m \in \mathbb{Z}$.

The result we shall proof is the following:

4.3. Theorem. *Let* $k \equiv 0 \pmod 4$, $k > n + 2$. *Then the Eisenstein series* $E_{k,m}^{(n)}(Z, W)$ *may be written as linear combination of theta series* $\vartheta_{S,c}^{(n)}(Z, W)$.

The proof of 4.3. will contain explicite description of the above linear combinations and furthermore yields a stability theorem for Poincaré Square Series, which we shall state now: Let $k, m, n \in \mathbb{N} \setminus \{0\}$ and $Z \in H_n$, then the Poincaré Square Series $P_{k,m}^{(n)}(Z)$ is defined as

$$P_{k,m}^{(n)}(Z) := \sum_{M \in \Gamma_{n,0} \setminus \Gamma_n} \det (CZ + D)^{-k} \sum_{\lambda \in \mathbb{Z}^{(1,n)}} e^{2\pi i m \lambda M<Z> \lambda^t}.$$

For $k > n + 2$ this series is well known to converge normally on H_n and to define a non vanishing Siegel modular form of weight k. Our result may be formulated as follows:

4.4. Theorem. *Let* $k \equiv 0 \pmod 4$, $k > n + 2$ *and let* $S_1, \ldots, S_h$ *denote a complete system of representatives of the unimodular classes of symmetric, positive definite, unimodular, even matrices* $S \in \mathbb{Z}^{(2k,2k)}$. *Then* $P_{k,m}^{(n)}$ *may be written as linear combination of the corresponding theta series* $\vartheta_{S_\nu}^{(n)}$, *i.e.*

$$P_{k,m}^{(n)}(Z) = \sum_{\nu=1}^{h} \tilde{m}_\nu \cdot \vartheta_{S_\nu}^{(n)}(Z).$$

The coefficients $\tilde{m}_\nu$ *are given by*

$$\tilde{m}_\nu := \delta(m)^{-1} \cdot \frac{A(S_\nu, S_\nu)^{-1}}{A(S_1, S_1)^{-1} + \ldots + A(S_h, S_h)^{-1}} \cdot \sum_{\substack{t^2 | m \\ t > 0}} \mu(t) \cdot A\left(S_\nu, \frac{2m}{t^2}\right),$$

where

$$A(S^{(2k,2k)}, T^{(q)}) := \#\{G \in \mathbb{Z}^{(2k,q)} \mid G^t S G = T\}$$

and

$$\delta(m) := \sum_{\substack{d^2 | m \\ d > 0}} \mu(d)\, a_k\left(\frac{m}{d^2}\right).$$

Here $\mu(d)$ *denotes the Möbius function and* $a_k(n)$ *denotes the n-th Fouriercoefficient of the Eisenstein series* $E_k^{(1)} \in [\Gamma_1, k]$.

Proof of 4.3. and 4.4. We consider the Fourier Jacobi expansion of the ordinary Siegelian Eisenstein series of degree $n + 1$ and weight k:

$$E_k^{(n+1)}(\tilde{Z}) = \sum_{m \geq 0} e_{k,m}^{(n)}(Z, W) \cdot e^{2\pi i m z_2}$$

where $\tilde{Z} \in H_{n+1}$ is written as $\tilde{Z} = \begin{pmatrix} Z & W^t \\ W & z_2 \end{pmatrix}$ with $Z \in H_n$ and $z_2 \in H_1$. A result of Böcherer (compare [1], Satz 7) states:

$$\sum_{m>0} e_{k,m}^{(n)}(Z, W) \cdot e^{2\pi i n m z_2}$$
$$= \sum_{\substack{t \in \mathbb{N} \\ n>0}} \sum_{\substack{\alpha \in \mathbb{N} \\ \alpha>0}} \sum_{\substack{\lambda \in \mathbb{Z}^{(1,n)} \\ (\alpha,\lambda) \text{ primitive}}} \sum_{M \in \Gamma_{n,0} \backslash \Gamma_n} a_k(t) \cdot \det(CZ + D)^{-k}$$
$$\cdot e^{2\pi i t \alpha^2 W(CZ+D)^{-1}CW^t} \cdot e^{2\pi i t\{\lambda M<Z>\lambda^t + 2\alpha\lambda(CZ+D)^{-t}W^t\}} \cdot e^{2\pi i t \alpha^2 z_2}$$
$$= \sum_{\substack{t \in \mathbb{N} \\ n>0}} \sum_{\substack{\alpha \in \mathbb{N} \\ \alpha>0}} \sum_{\substack{d \in \mathbb{N} \\ d>0}} \mu(d)\, a_k(t) \cdot E_{k,td^2}^{(n)}(Z, \alpha W) \cdot e^{2\pi i t d^2 \alpha^2 z_2},$$

where $E_{k,td^2}^{(n)}$ denotes the Jacobi Eisenstein series of weight k and index $t\,d^2$ (compare 2.1.). Using Böcherer's result we immediately obtain:

$$e_{k,m}^{(n)}(Z, W) = \sum_{\substack{s^2|m \\ s>0}} \delta\left(\frac{m}{s^2}\right) \cdot E_{k,\frac{m}{s^2}}^{(n)}(Z, sW) \tag{1}$$

where we have introduced $\delta(\nu) := \sum_{\substack{d^2|\nu \\ d>0}} \mu(d)\, a_k\left(\frac{\nu}{d^2}\right)$ in order to simplify our notation. Since $a_k(n) = \text{const.} \cdot \sum_{\substack{d|n \\ d>0}} d^{k-1}$ for some positive constant, it is easy to verify that $\delta(\nu) > 0$ for every $\nu > 0$. Therefore we may invert the system of equations (1) and obtain:

$$E_{k,m}^{(n)}(Z, W) = \delta(m)^{-1} \left\{ e_{k,m}^{(n)}(Z, W) - \sum_{\substack{s^2|m \\ s>1}} \delta\left(\frac{m}{s^2}\right) \cdot E_{k,\frac{m}{s^2}}^{(n)}(Z, sW) \right\}$$
$$= \delta(m)^{-1} \left\{ e_{k,m}^{(n)}(Z, W) - \sum_{\substack{s_1^2|m \\ s_1>1}} e_{k,\frac{m}{s_1^2}}^{(n)}(Z, s_1 W) + \sum_{\substack{s_1^2|m \\ s_1>1}} \sum_{\substack{s_2^2|\frac{m}{s_1^2} \\ s_2>1}} e_{k,\frac{m}{s_1^2 s_2^2}}^{(n)}(Z, s_1 s_2 W) - \dots \right\}$$
$$= \delta(m)^{-1} \cdot \sum_{\substack{t^2|m \\ t>0}} p(t) \cdot e_{k,\frac{m}{t^2}}^{(n)}(Z, tW)$$

with $p(t) := \sum_{\gamma \geqq 0} (-1)^\gamma \# \left\{ (s_0, s_1, \dots, s_\gamma) \in \mathbb{Z}^{\gamma+1} \mid s_0 = 1,\, s_1 > 1, \dots, s_\gamma > 1, \prod_{i=0}^{\gamma} s_i = t \right\}$. We observe that $p(t)$ is nothing but the Möbius function $\mu(t)$, since

$$\sum_{t \geqq 1} \frac{p(t)}{t^s} = \sum_{\gamma \geqq 0} (-1)^\gamma \cdot (\zeta(s) - 1)^\gamma = \sum_{\gamma \geqq 0} (1 - \zeta(s))^\gamma$$
$$= \frac{1}{1 - (1 - \zeta(s))} = \frac{1}{\zeta(s)} = \sum_{t \geqq 1} \frac{\mu(t)}{t^s},$$

where $\zeta(s)$ denotes the Riemann zeta function. Therefore we obtain:

$$E_{k,m}^{(n)}(Z, W) = \delta(m)^{-1} \cdot \sum_{\substack{t^2|m \\ t>0}} \mu(t) \cdot e_{k,\frac{m}{t^2}}^{(n)}(Z, tW). \tag{2}$$

By Siegel's Hauptsatz we have

$$E_k^{(n+1)}(\tilde{Z}) = \sum_{\nu=1}^{h} m_\nu \cdot \vartheta_{S_\nu}^{(n+1)}(\tilde{Z}) \tag{3}$$

with

$$m_\nu := \frac{A(S_\nu, S_\nu)^{-1}}{A(S_1, S_1)^{-1} + \ldots + A(S_h, S_h)^{-1}}.$$

Fourier Jacobi expansion of both sides of (3) yields

$$e_{k,m}^{(n)}(Z, W) = \sum_{\nu=1}^{h} m_\nu \sum_{\substack{c \in \mathbb{Z}^{(2k,1)} \\ c^t S_\nu c = 2m}} \vartheta_{S_\nu,c}^{(n)}(Z, W). \tag{4}$$

Inserting (4) into (2) we obtain

$$E_{k,m}^{(n)}(Z, W) = \delta(m)^{-1} \cdot \sum_{\substack{t^2|m \\ t>0}} \mu(t) \sum_{\nu=1}^{h} m_\nu \sum_{\substack{c \in \mathbb{Z}^{(2k,1)} \\ c^t S_\nu c = \frac{2m}{t^2}}} \vartheta_{S_\nu,c}^{(n)}(Z, tW) \tag{5}$$

and Theorem 4.4. follows immediately from the formula

$$P_{k,m}^{(n)}(Z) = E_{k,m}^{(n)}(Z, 0)$$

and

$$\vartheta_{S_\nu}^{(n)}(Z) = \vartheta_{S_\nu,c}^{(n)}(Z, 0) \text{ for every } c \in \mathbb{Z}^{(2k,1)}.$$

Actually we have shown more: The formula $\vartheta_{S_\nu,c}^{(n)}(Z, tW) = \vartheta_{S_\nu,tc}^{(n)}(Z, W)$ implies

$$E_{k,m}^{(n)}(Z, W) = \delta(m)^{-1} \cdot \sum_{\nu=1}^{h} \sum_{\substack{t^2|m \\ t>0}} \sum_{\substack{c \in t\mathbb{Z}^{(2k,1)} \\ c^t S_\nu c = 2m}} m_\nu \cdot \mu(t) \cdot \vartheta_{S_\nu,c}^{(n)}(Z, W),$$

which also proves 4.3. □

Remark. Theorem 4.4. may also be obtained directly without using Jacobi forms but deeper results of Böcherer and Klingen—the above indicated proof has the advantage of being more elementary.

Appendix. An open problem

In order to obtain Jacobi Forms on the modular group Γ_n we made use of several construction principles:

1. Fourier Jacobi expansion of Siegel modular forms (Introduction).
2. Eisenstein Series.
   a) ordinary (2.1.).
   b) of Klingen's type (2.5.).
3. Theta Series (3.6.).
4. Linear combinations and linear transformations of the torus variable (3.5.).

As we have seen principle 1. is not enough in order to obtain all Jacobi Forms. For example there are theta series $\vartheta^{(n)}_{S,c}$ which are no Fourier Jacobi coefficients of Siegel modular forms. Nevertheless each theta series may be obtained by combining principles 1. and 4., since $\vartheta^{(n)}_{S,c}(Z, W) = \vartheta^{(n)}_{S,E}(Z, cW)\,(E = E^{(2k)})$ and $\{\#\mathrm{Aut}(S)\}\cdot\vartheta^{(n)}_{S,E}$ occurs as Fourier-Jacobi coefficient of $\vartheta^{(n+2k)}_S$. Furthermore we have seen that the ordinary Eisenstein Series $E^{(n)}_{k,m}(Z, W)$ may be obtained by combining 1. and 4. too. So it seems that we do not know any examples of Jacobi Forms, which may not be obtained in this way. Therefore we formulate the following

Problem. Can one construct all Jacobi Forms by combining principles 1. and 4.?

Of course our theory is far away from giving an answer to that question.

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Eingegangen am 12. 1. 1989

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