Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64

On the Stability of the Quadratic Mapping in Normed Spaces

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Abstract. The problem of the stability of the quadratic functional equation (including Ulam-Hyers and Rassias types of stability) in normed spaces is investigated.

1. The problem of the stability of functional equations has been posed by S.M. Ulam in [7]. D.H. Hyers in [3] has given a positive answer for the stability of the linear functional equation

$$f(x+y) = f(x) + f(y). \tag{1}$$

From that time many other related questions have been studied (see e.g. [2], [4], [6]).

For the quadratic functional equation some results are contained in [1]. In this paper we prove theorems concerning some other types of stability of the quadratic functional equation in normed spaces.

Let us note that $f: X \to Y$, where X and Y are groups is called a quadratic function iff f satisfies the following functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \text{for all} \quad x, y \in X. \tag{2}$$

2. Let $(E_1, \|\cdot\|)$ and $(E_2, \|\cdot\|)$ be two normed spaces and let $\mathbb{R}$ stand for the set of all real numbers.

We shall prove the following

Lemma. *Assume that there exist $\xi \geq 0$, $\eta \geq 0$ and $v \in \mathbb{R}$ such that a function $f: E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \xi + \eta(\|x\|^v + \|y\|^v) \tag{3}$$

for all $x, y \in E_1 \backslash \{0\}$. Then for $x \in E_1 \backslash \{0\}$ and $n \in \mathbb{N}$ (the set of all natural numbers)

$$\begin{aligned} &\|f(2^n x) - 4^n f(x)\| \\ &\quad \leq 3^{-1}(4^n - 1)(\xi + c) + 2 \cdot 4^{n-1} \eta \|x\|^v (1 + a + \cdots + a^{n-1}), \end{aligned} \tag{4}$$

$$\begin{aligned} &\|f(x) - 4^n f(2^{-n} x)\| \\ &\quad \leq 3^{-1}(4^n - 1)(\xi + c) + 2^{1-v} \eta \|x\|^v (1 + b + \cdots + b^{n-1}), \end{aligned} \tag{5}$$

where $a = 2^{v-2}$, $b = 2^{2-v}$, $c = \|f(0)\|$.

AMS (1980) subject classification: Primary 39C05. Key words and phrases: the stability of functional equations.

Proof. Put $x = y \neq 0$ in (3). We get

$$\|f(2x) - 4f(x)\| \leq \|f(0)\| + \xi + 2\eta\|x\|^v$$

which proves (4) for $n = 1$. Now, let us assume that (4) is true for $k \leq n$ and $x \in E_1 \backslash \{0\}$. Then for $n+1$ we have

$$\begin{aligned}
&\|f(2^{n+1}x) - 4^{n+1}f(x)\| \\
&\leq \|f(2 \cdot 2^n x) - 4f(2^n x)\| + 4\|f(2^n x) - 4^n f(x)\| \\
&\leq (\xi + c) + 2\eta\|2^n x\|^v + \frac{4}{3}(\xi + c)(4^n - 1) + 2 \cdot 4^n \eta \|x\|^v (1 + a + \cdots + a^{n-1}) \\
&= 3^{-1}(4^{n+1} - 1)(\xi + c) + 2 \cdot 4^n \eta \|x\|^v (1 + a + \cdots + a^n),
\end{aligned}$$

which proves the inequality (4) for all natural n.

Analogously, taking $x = y = \frac{t}{2}$, we can verify the inequality (5) for $n = 1$. Applying the induction principle we get the result for all $n \in \mathbb{N}$, which completes the proof. □

Having done this, we are able to prove the following

Theorem 1. *Let E_1 be a normed space and E_2 a Banach space and let $f: E_1 \to E_2$ be a function satisfying the inequality* (3) *for all $x, y \in E_1 \backslash \{0\}$ and let $v < 2$. Then there exists exactly one quadratic mapping $g: E_1 \to E_2$ such that*

$$(6) \qquad \|g(x) - f(x)\| \leq 3^{-1}(\xi + c) + 2(4 - 2^v)^{-1}\eta\|x\|^v, \quad x \in E_1 \backslash \{0\}.$$

If, moreover, f is measurable (i.e. $f^{-1}(G)$ is a Borel set in E_1 for every open set G in E_2) or $\mathbb{R} \ni t \to f(tx)$ is continuous for each fixed $x \in E_1$, then g satisfies the condition

$$(7) \qquad g(tx) = t^2 g(x) \quad \textit{for all} \quad x \in E_1 \quad \textit{and} \quad t \in \mathbb{R}.$$

Proof. Define the sequence of functions $\{g_n\}$ by the formula

$$(8) \qquad g_n(x) = 4^{-n} f(2^n x), \quad x \in E_1, n \in \mathbb{N}.$$

Then $\{g_n\}$ is a Cauchy sequence for every $x \in E_1$. Really, for $x = 0$ it is trivial. Let $0 \neq x \in E_1$. We have for $n > m$,

$$\begin{aligned}
&\|g_n(x) - g_m(x)\| \\
&= 4^{-n}\|f(2^{n-m} \cdot 2^m x) - 4^{n-m} f(2^m x)\| \\
&\leq 4^{-n} \cdot 3^{-1}(4^{n-m} - 1)(\xi + c) + 2 \cdot 4^{-m-1}\eta\|2^m x\|^v (1 + a + \cdots + a^{n-m-1}) \\
&\leq 3^{-1} \cdot 4^{-m}(\xi + c) + 2^{m(v-2)-1}(1 - a)^{-1}\eta\|x\|^v.
\end{aligned}$$

Since $v < 2$, we guess that $\{g_n(x)\}$ is a Cauchy sequence. Define

$$g(x) := \lim_{n \to \infty} g_n(x) \quad x \in E_1.$$

We shall check that g is a quadratic function. If $x = y = 0$, since $g(0) = 0$, it is clear. For $y = 0$, $x \neq 0$ we have

$$g(x+0) + g(x-0) - 2g(x) - 2g(0) = 0.$$

Let us now consider the case $x, y \in E_1 \backslash \{0\}$. We get the following estimations

$$\begin{aligned} &\|g_n(x+y) + g_n(x-y) - 2g_n(x) - 2g_n(y)\| \\ &= 4^{-n}\|f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y)\| \\ &\leq 4^{-n}(\xi + \eta(\|2^n x\|^v + \|2^n y\|^v)) \\ &\leq 4^{-n}\xi + 2^{n(v-2)}\eta(\|x\|^v + \|y\|^v). \end{aligned}$$

For $n \to \infty$ we get the equality

$$g(x+y) + g(x-y) - 2g(x) - 2g(y) = 0.$$

Consider the case $x = 0$, $y \neq 0$, we put into that equation $x = y$ getting $g(2x) = 4g(x)$ for $x \in E_1$. Moreover, setting $y = -x$, we obtain $g(-x) = g(x)$ for $x \in E_1$. Therefore,

$$g(y) + g(-y) - 2g(0) - 2g(y) = 0,$$

i.e. g is a quadratic function.

The estimation (6) we may obtain directly from the inequality (4).

To prove the uniqueness assume that there exist two quadratic functions $g_i: E_1 \to E_2$, $i = 1, 2$ such that

$$\|g_i(x) - f(x)\| \leq c_i + b_i\|x\|^v, \quad x \in E_1 \backslash \{0\}, i = 1, 2,$$

where c_i, b_i, $i = 1, 2$ are given nonnegative constants. Then we have

$$g_i(2^n x) = 4^n g_i(x), \quad x \in E_1, n \in \mathbb{N}, i = 1, 2.$$

Now we obtain for every $x \in E_1$ $(g_1(0) = g_2(0) = 0)$

$$\begin{aligned} \|g_1(x) - g_2(x)\| &\leq 4^{-n}\left(\|g_1(2^n x) - f(2^n x)\| + \|g_2(2^n x) - f(2^n x)\|\right) \\ &\leq 4^{-n}(c_1 + c_2) + 2^{n(v-2)}(b_1 + b_2)\|x\|^v, \end{aligned}$$

whence, if $n \to \infty$, we get $g_1(x) = g_2(x)$ for all $x \in E_1$.

Let L be any continuous linear functional defined on the space E_2. Consider the mapping $\varphi: \mathbb{R} \to \mathbb{R}$ defined as follows

$$\varphi(t) = L[g(tx)] \quad \text{for all } t \in \mathbb{R} \text{ and } x \in E_1,\ x \text{ fixed}.$$

It is easy to see that φ is a quadratic function. If we assume that f is measurable, then φ as the pointwise limit of the sequence of measurable functions

$$\varphi_n(t) = 4^{-n}L[f(2^n tx)], \quad n \in \mathbb{N}, t \in \mathbb{R}$$

is measurable. Therefore φ as a measurable quadratic function is continuous (see [5]), so has the form

$$\varphi(t) = t^2\varphi(1) \quad \text{for} \quad t \in \mathbb{R}.$$

Consequently

$$L[g(tx)] = \varphi(t) = t^2\varphi(1) = t^2 L[g(x)] = L[t^2 g(x)],$$

whence taking into account that L is any continuous linear functional, we get the property (7). This ends the proof of the theorem. □

Theorem 2. *Let E_1 be a normed space and E_2 a Banach space and let $\eta \geq 0$ and $v > 2$ be given real numbers. Let $f: E_1 \to E_2$ satisfy the condition*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \eta(\|x\|^v + \|y\|^v) \tag{9}$$

for all $x, y \in E_1$. Then there exists exactly one quadratic mapping $h: E_1 \to E_2$ such that

$$\|h(x) - f(x)\| \leq 2(2^v - 4)^{-1}\eta\|x\|^v \quad \text{for all} \quad x \in E_1. \tag{10}$$

If, moreover, f is measurable or $\mathbb{R} \ni t \to f(tx)$ is contunuous for each fixed $x \in E_1$, then h satisfies the condition (7).

Proof. Define the sequence

$$h_n(x) = 4^n f(2^{-n}x), \quad x \in E_1, n \in \mathbb{N}. \tag{11}$$

Since $f(0) = 0$, applying (5), we get for $x \in E_1$ and $n > m$,

$$\|h_n(x) - h_m(x)\| \leq 2^{1-v} \cdot 2^{m(2-v)}(1-b)^{-1}\eta\|x\|^v,$$

which implies for $v > 2$ that $\{h_n(x)\}$ is a Cauchy sequence for every $x \in E_1$. Define

$$h(x) = \lim_{n\to\infty} h_n(x), \quad x \in E_1.$$

Then in view of (8) we may verify that h is a quadratic function. Using again the estimation (5) we obtain (10).

It is not difficult to prove that

$$h(\lambda x) = \lambda^2 h(x) \quad \text{for all } x \in E_1 \text{ and all rational numbers } \lambda. \tag{12}$$

Now assume that there exist two quadratic functions $h_i: E_1 \to E_2$, $i = 1, 2$ such that

$$\|h_i(x) - f(x)\| \leq d_i\|x\|^v, \quad i = 1, 2, x \in E_1,$$

where d_i, $i = 1, 2$ are nonnegative constants. We get by (12) for $x \in E_1$

$$\begin{aligned} \|h_1(x) - h_2(x)\| &= 4^n\|h_1(2^{-n}x) - h_2(2^{-n}x)\| \\ &\leq 4^n(d_1 + d_2)\|2^{-n}x\|^v \\ &= 2^{n(2-v)}(d_1 + d_2)\|x\|^v, \end{aligned}$$

from which we guess that $h_1 = h_2$. The proof of the last statement of the theorem is quite similar to that one given for theorem 1. □

3. Now we shall present an example concerning the special case $v = 2$. This is a modification of the example contained in [2].

Define

$$\varphi(x) = \begin{cases} \alpha, & x \in (-\infty, -1] \cup [1, +\infty), \\ \alpha x^2, & x \in (-1, 1), \end{cases}$$

where $\alpha > 0$. Put

$$f(x) = \sum_{n=0}^{\infty} 4^{-n} \varphi(2^n x), \quad x \in \mathbb{R}.$$

Then f is bounded by $\frac{4}{3}\alpha$ and satisfies the condition

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \le 32\alpha(x^2 + y^2) \tag{13}$$

for all $x, y \in \mathbb{R}$.

Really, for $x = y = 0$ or $x, y \in \mathbb{R}$ such that $x^2 + y^2 \ge \frac{1}{4}$ it is obvious. Consider the case $0 < x^2 + y^2 < \frac{1}{4}$. Then there exists $k \in \mathbb{N}$ such that

$$4^{-k-1} \le x^2 + y^2 < 4^{-k}, \tag{14}$$

whence $4^{k-1}x^2 < 4^{-1}$ and $4^{k-1}y^2 < 4^{-1}$ and consequently

$$2^{k-1x}, 2^{k-1}y, 2^{k-1}(x+y), 2^{k-1}(x-y) \in (-1, 1).$$

Therefore for every $n = 0, 1, \dots, k-1$ we have

$$2^n x, 2^n y, 2^n(x+y), 2^n(x-y) \in (-1, 1)$$

and

$$\varphi(2^n(x+y)) + \varphi(2^n(x-y)) - 2\varphi(2^n x) - 2\varphi(2^n y) = 0$$

for $n = 0, 1, \dots, k-1$. Now we obtain applying (14)

$$\begin{aligned} &|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \\ \le\ & \sum_{n=0}^{\infty} 4^{-n} |\varphi(2^n(x+y)) + \varphi(2^n(x-y)) - 2\varphi(2^n x) - 2\varphi(2^n y)| \\ \le\ & \sum_{n=k}^{\infty} 6\alpha 4^{-n} = 2 \cdot 4^{1-k}\alpha \\ \le\ & 32\alpha(x^2 + y^2), \end{aligned}$$

i.e. the condition (13) holds true.

Assume that there exist a quadratic function $g: \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - g(x)| \le \beta x^2 \quad \text{for all} \quad x \in \mathbb{R}.$$

Since g is locally bounded, then (see [5]) it is of the form $g(x) = \gamma x^2$, $x \in \mathbb{R}$, where γ is a constant. Therefore we have

$$|f(x)| \le (\beta + |\gamma|)x^2 \quad \text{for} \quad x \in \mathbb{R}. \tag{15}$$

Let $k \in \mathbb{N}$ be such that $k\alpha > \beta + |\gamma|$. Then if $x \in (0, 2^{1-k})$, $2^n x \in (0,1)$ for $n \leq k-1$ and we have

$$f(x) = \sum_{n=0}^{\infty} 4^{-n} \varphi(2^n x) \geq \sum_{n=0}^{k-1} \alpha 4^{-n} (2^n x)^2 = k\alpha x^2 > (\beta + |\gamma|) x^2,$$

which in comparison with (15) is a contradiction.

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Eingegangen am: 05.03.1991

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