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The stability of Homomorphisms and Amenability, with applications to functional equations.

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1. *Introduction.*

The investigation in respect of the stability of homomorphisms, i. e. of the Cauchy functional equation, was proposed in 1940 by S. M. ULAM during a talk before the Mathematics Club of the University of Wisconsin. D. H. HYERS solved the problem in 1941 (see [8] and Theorem 1 of the present paper).

In the last ten years this result has been used by many authors, especially by people working in the field of functional equations.

This led to a great number of papers dealing with generalizations of HYERS' result in different directions (see, for instance, the vast bibliography in [9]).

Others important investigations have used the stability of homomorphisms in order to solve some non-homogeneous functional equations and in particular alternative functional equations of Cauchy type. As far as I know the first result of this kind is in [3].

The original HYERS' theorem holds when the mapping involved is defined on an abelian group. This yields to the following question: is this fact indeed essential? The answer is no. L. SZÉKELYHIDI in [16] proved that the amenability of the group is enough to ensure stability. Now another question naturally arises: what are the connections between stability of homomorphisms and amenability of the group where they are defined?

In the present paper we try to give answers to the previous questions.

In Section 2 the notion of stability is defined and some consequences of it are proved. The results are not all new, but they are reported in order to make the paper self-containing.

The third section deals with the connections between stability and amenability of groups (or semigroups) and a necessary and sufficient condition for amenability in term of stability is proved.

In Section 4 some results of the former two sections are used to solve an alternative functional Cauchy equation, that is an equation of the form

$$f(xy) - f(x) - f(y) \in V$$

where V is a particular set. The result obtained extends known results to a more general frame.

2. *Stability of Homomorphisms.*

We start this section with the following

Definition 1. *Let G be a group (or a semigroup) and B a Banach space. We say that the couple (G, B) has the property of the stability of homomorphisms (shortly (G, B) is HS) if for every function $f: G \to B$ such that*

$$\|f(xy) - f(x) - f(y)\| \leqq K$$

for every $x, y \in G$ and for some K, there exist $\Phi \in Hom(G, B)$ and K' depending only on K such that

$$\|f(x) - \Phi(x)\| \leqq K' \tag{1}$$

for all $x \in G$.

Different definitions of stability for the homomorphisms (or, that's the same, stability for the Cauchy functional equation) are given and studied in several paper, see for example [14] and [11].

The definition above is suggested by the following theorem ([8]):

Theorem 1. (D. H. Hyers)–*Let E and E′ be Banach spaces and let $f: E \to E'$ be such that $\|f(x+y) - f(x) - f(y)\| \leqq \delta$, $x, y \in E$. Then the limit $l(x) = \lim_{n \to +\infty} 2^{-n} f(2^n x)$ exists for each $x \in E$, $l(x)$ is an additive function, and the inequality $\|f(x) - l(x)\| \leqq \delta$ is true for all x in E. Moreover $l(x)$ is the only additive function satisfying this inequality.*

A glance to the proof of Theorem 1 shows that it remains true if the Banach space *E* is substituted by an arbitrary abelian group or semigroup; so we can say that for all Banach spaces *B* and all abelian groups (or semigroups) *G* the couple (*G*, *B*) is *HS*.

From Theorem 1 we get easily the following

Proposition 1. *Let G be an arbitrary group (or semigroup) and let B be a Banach space. Assume that $f: G \to B$ satisfies the inequality*

$$\|f(xy) - f(x) - f(y)\| \leqq K$$

for all $x, y \in G$.

Then the limit $g(x) = \lim_{n \to +\infty} 2^{-n} f(x^{2^n})$ exists for all $x \in G$ and

$$\|f(x) - g(x)\| \leqq K \text{ and } g(x^2) = 2g(x) \text{ for all } x \in G. \tag{2}$$

The function g is the unique satisfying conditions (2).

Proof. The existence of the limit and the first of the (2) are contained in the first part of the proof of Hyers' theorem. The second of the (2) is an immediate consequence of the definition of *g*. Let now $h: G \to B$ satisfying conditions (2), then

$$\|f(x^{2^n}) - h(x^{2^n})\| = \|f(x^{2^n}) - 2^n h(x)\| \leqq K,$$

dividing by 2^n and letting $n \to +\infty$, we have $g(x) = h(x)$.
Proposition 1 shows that the existence of the limit

$$\lim_{n \to +\infty} 2^{-n} f(x^{2^n})$$

depends only on the completness of the space B. Whether the function g is additive or not depends on the group G.
These simple remarks enable us to prove the following

Theorem 2. *Let the couple* (G, B) *be HS, then the smallest constant* K' *fulfilling inequality* (1) *is equal to* K, *moreover the homomorphism* Φ *satisfying* (1) *is unique.*

Proof. Let g be the function defined as in Proposition 1, it fulfills the conditions (2). Since (G, B) is *HS*, there exists $\Phi \in Hom(G, B)$ such that $\|f(x) - \Phi(x)\| \leqq K'$ for all $x \in G$. $\|f(x^{2^n}) - \Phi(x^{2^n})\| = \|f(x^{2^n}) - 2^n \Phi(x)\| \leqq K'$, dividing by 2^n and taking the limit as $n \to +\infty$, we have $g(x) = \Phi(x)$, thus if K' is the smallest constant fulfilling (1), we get $K' \leqq K$. On the other hand, if we take $f(x) = \Phi(x) + c$, where $\Phi \in Hom(G, B)$ and $\|c\| = K$, we get $\|f(xy) - f(x) - f(y)\| = \|c\| = K$ and $\|f(x) - \Phi(x)\| = K$, whence the smallest constant fulfilling (1) must be K.
The uniqueness of Φ follows from the Proposition 1.

Corollary 1. *If* G *is a finite group* (*or semigroup*), *then* (G, B) *is HS for every Banach space* B, *and* $\Phi = 0$.

Theorem 3. *Assume that the couple* $(G, \mathbb{C})$ (*or* $(G, \mathbb{R})$) *is HS. Then for every complex* (*real*) *Banach space* B, *the couple* (G, B) *is HS.*

Proof. Let $f: G \to B$ be such that $\|f(xy) - f(x) - f(y)\| \leqq K$ for all $x, y \in G$. If B' is the (topological) dual of B, for every $L \in B'$ we have $|L\{f(xy) - f(x) - f(y)\}| = |L(f(xy)) - L(f(x)) - Lf(y))| \leqq K\|L\|$, thus there exists $\Phi_L \in Hom(G, \mathbb{C})$ such that $|L(f(x)) - \Phi_L(x)| \leqq K\|L\|$. Define $\Phi(x) = \lim_{n \to +\infty} 2^{-n} f(x^{2^n})$ (the limit exists by Proposition 1); we have $\Phi_L(x) = \lim_{n \to +\infty} 2^{-n} L(f(x^{2^n})) = L(\Phi(x))$, by the continuity of L. So $L(\Phi(xy)) = \Phi_L(xy) = \Phi_L(x) + \Phi_L(y) = L(\Phi(x)) + L(\Phi(y))$ for every $L \in B'$, hence $\Phi(xy) = \Phi(x) + \Phi(y)$. By Proposition 1 we have that (G, B) is *HS*.

From now in this section we assume that the couple (G, B) (G group or semigroup) be *HS*, and if f is a function from G into B, by $\mathscr{C}f(x,y)$ we denote the Cauchy difference $\mathscr{C}f(x,y) = f(xy) - f(x) - f(y)$. If $\mathscr{C}f(x,y)$ is bounded, by Φ_f we intend the homomorphism approximating f as indicated in Definition 1.

We now study the relations between the range of $\mathscr{C}f$ (when bounded) and the range of $f - \Phi_f$. The results we achieve will be used to solve some alternative functional equations.

Definition 2. *Let* M *be an arbitrary subset of a real or complex vector space, by* $C(M)$

we denote the convex hull of M, *that is the set of all elements of the form* $\sum_1^n \alpha_i x_i, \alpha_i \geqq 0$, $\sum_1^n \alpha_i = 1, x_i \in M$.

The following useful theorem is a straightforword generalization (with its proof) of Theorem 2 in [3].

Theorem 4. *Assume that* $\mathscr{C}f(x,y) \in M$, *where* M *is a bounded subset of a Banach space* B. *Then* $h(x) \in \overline{C(-M)}$, *where* $h = f - \Phi_f$. *(By* $\bar{A}$ *we intend the closure of* A*).*

Proof. Let $x \in G$ and $u = h(x)$. Then for every positive integer s we have

$$h(x^s) = su + \sum_{i=1}^{s-1} m_i, m_i \in M. \tag{3}$$

This can be proved by induction over s:

$$h(x^2) - 2h(x) = f(x^2) - 2f(x) \in M, \text{ so } h(x^2) = 2h(x) + m_1, m_1 = 2u + m_1 \in M;$$
$$h(x^{s+1}) - h(x^s) - h(x) = f(x^{s+1}) - f(x^s) - f(x) \in M,$$

so

$$h(x^{s+1}) = h(x^s) + h(x) + m_s = (s+1)u + \sum_{i=1}^{s} m_i.$$

Dividing (3) by s and taking the limit as
$s \to +\infty$, we have

$$u = -\lim_{n \to +\infty} s^{-1} \sum_{i=1}^{s-1} m_i$$

(the limit exists since h is bounded), hence $u \in \overline{C(-M)}$.

As a consequence of Theorem 4, the range of $f - \Phi_f$ is contained in the closed subspace of B spanned by M.
A result of this kind, under different assumptions, has been obtained by K. Baron in [1]. He proves the following

Theorem 5. (K. Baron) *Let* G *be an abelian group,* Y *a vector space over* $\mathbb{Q}$ *and* Z *an arbitrary subspace of* Y. *A function* $f: G \to Y$ *satisfies the condition*

$$f(x_1 + x_2) - f(x_1) - f(x_2) \in Z$$

for all $x_1, x_2 \in G$, *if and only if there exists an additive function* $g: G \to Y$ *such that* $f(x) - g(x) \in Z$ *for every* $x \in G$.

It has to be noticed that the additive function g in Baron's theorem is not uniquely determined.
The following theorem gives a stronger result than Theorem 4 and it will be used in Section 4 in order to solve some functional equations.

Theorem 6. *Assume that* $\mathscr{C}f(x,y) \in M$ $(x,y \in G)$ *where* M *is a bounded subset of* B *and let* $h = f - \Phi_f$. *If* ε *is the identity of* G, *let* $-h(\varepsilon) = m_o \in M$.
Then $h(x) \in \{-(m_o + M) + \overline{C(M)}\} \cap \overline{C(-M)}$.

Proof. Since $h(\varepsilon) - h(x) - h(x^{-1}) \in M$ and $-h(\varepsilon) = m_0 \in M$, we have, for some $m \in M$, $h(x) = -h(x^{-1}) - m_0 - m$, hence $h(x) \in -h(x^{-1}) - (m_0 + M)$. By Theorem 4, $-h(x^{-1}) \in \overline{C(M)}$ and $h(x) \in \overline{C(-M)}$; thus we get the desired result.

3. *Stability and Amenability.*

In the previous section we obtained some consequences from the fact that the couple (G, B) has the property of the stability of homormorphisms.

In this section we intend to analyse for which groups G the couple (G, B) is HS. We have already remarked that, as a consequence of Hyers' theorem and of Corollary 1, the couple (G, B) is HS for all abelian and all finite groups (or semigroups), whatever be the Banach space B.

Definition 3. — *Let* G *be a group or a semigroup and* $B(G)$ *be the space of all bounded complex—valued functions on* G, *equipped with the supremum norm* $\|f\|_\infty$.
A linear functional m *on* $B(G)$ *is a left invariant mean (LIM) if:*

(α) $m(\bar{f}) = \overline{m(f)}$, $f \in B(G)$;
(β) $\inf\{f(x)\} \leqq m(f) \leqq \sup\{f(x)\}$, *for all real-valued* $f \in B(G)$;
(γ) $m({}_xf) = m(f)$, *for all* $x \in G$ *and* $f \in B(G)$, *where* ${}_xf(t) = f(xt)$.

Likewise we say m *is a right invariant mean if* $m(f_x) = m(f)$ *for all* $x \in G$, *where* $f_x(t) = f(tx)$, *and we define two-sided invariance in the usual way. Condition* (β) *is equivalent to* $m(f) \geqq 0$ *if* $f \geqq 0$, *and* $m(1) = 1$, *hence* $\|m\| = 1$ *for every mean.*

The following two propositions hold (see [7]):

Proposition 2. *If* G *is a semigroup with a left invariant mean and a right invariant mean on* $B(G)$, *then there exists a two-sided invariant mean on* $B(G)$.

Proposition 3. *If* G *is a group, there is a left invariant mean on* $(B(G)$ *if and only if there is a right invariant mean on* $B(G)$. *Hence, by Proposition* 2, *there is a two-sided invariant mean on* $B(G)$.

Definition 4. *A semigroup* G *is left (right) amenable if there is a left (right) invariant mean on* $B(G)$; *if* G *is a group these conditions are the same and we say that* G *is amenable.*

The following theorem due to L. Székelyhidi ([16]) shows that amenability implies the stability of homomorphisms. We give here a direct proof.

Theorem 7. (L. Székelyhidi) *Let* G *be a left (right) amenable semigroup, then* $(G, \mathbb{C})$ *is* HS.

Proof. Let $f: G \to \mathbb{C}$ be such that $|f(xy) - f(x) - f(y)| \leqq K$ for all $x, y \in G$; then, for each fixed $x \in G$, the function $f(xy) - f(y)$, as a function of y, is in $B(G)$. Let m_y be a left invariant mean on $B(G)$ (the suffix y denotes that m_y acts on functions of the variable y) and define

$$\Phi(x) = m_y\{{}_xf - f\}, x \in G.$$

$$\Phi(xz) = m_y\{{}_{xz}f - f\} = m_y\{{}_{xz}f - {}_xf + {}_xf - f\} = m_y\{{}_x({}_zf - f)\} + m_y\{{}_xf - f\}$$
$$= m_y\{{}_zf - f\} + \Phi(x) = \Phi(z) + \Phi(x), \text{ so } \Phi \in Hom(G, \mathbb{C}).$$

$$|\Phi(x) - f(x)| = |m_y\{{}_xf - f\} - f(x)| = |m_y\{{}_xf - f - f(x)\}| \leqq \sup_{y \in G} |f|(xy) - f(x)$$
$$- f(y)| \leqq K, x \in G, \text{ thus } (G, \mathbb{C}) \text{ is } HS.$$

Corollary 2. *Let G be a left (right) amenable semigroup, then for all Banach spaces B, (G, B) is HS.*

Proof. From Theorem 7 and Theorem 3.

Now a question arises naturally: are there groups or semigroups G such that (G, B) is not HS for some B?
In view of Theorem 7 we must look among non amenable groups or semigroups. The following theorem (presented by the author to the 22^{nd} International Symposium on Functional Equations) gives an answer.

Theorem 8. *Let $F(a, b)$ ($S(a,b)$) be the free group (semigroup) generated by the elements a and b. The couple $(F(a,b), \mathbb{R})$ ($(S(a, b), \mathbb{R})$) is not HS.*

Proof. We shall construct a function $f: F(a,b) \to \mathbb{R}$ such that $|\mathscr{C}f(x,y)| \leqq 1$ and for every $\Phi \in Hom(F(a,b), \mathbb{R})$ the difference $f - \Phi$ is unbounded. To do this, if $x \in F(a,b)$, we assume that the "word" x be reduced, that is it does not contain pairs of the forms $aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b$ and it is written without exponents different from 1 and -1. We now define a function $f: F(a,b) \to \mathbb{R}$ in the following way: if $r(x)$ is the number of pairs of the form ab contained in x and $s(x)$ is the number of pairs of the form $b^{-1}a^{-1}$ contained in x, then $f(x) = r(x) - s(x)$.

The function f is unbounded and for each $x, y \in F(a,b)$ we have

$$f(xy) - f(x) - f(y) \in \{-1, 0, 1\}$$

so $|\mathscr{C}f(x,y)| \leqq 1$.

Assume now that $\Phi \in Hom(F(a,b), \mathbb{R})$ exists so that $f - \Phi$ is bounded. Φ is completely determined by the values $\Phi(a)$ and $\Phi(b)$ and f is identically zero on the subgroups generated by a and b respectively. Hence the boundedness of $f - \Phi$ on these two subgroups implies $\Phi = 0$; thus $f - \Phi = f$, a contradiction since f is unbounded.

If instead of the free group $F(a,b)$ we consider the free semigroup $S(a,b)$, we get the analogous result by defining $f(x) = r(x)$.

It is well known that a group containing $F(a,b)$ as a subgroup, is not amenable. Hence a question arises: can Theorem 8 be extended to groups containing $F(a,b)$?
This can be formulated in the following way:

let $G \supset F(a,b)$ and let $f: F(a,b) \to \mathbb{R}$ be defined as in Theorem 8; is it possible to extend f to $\tilde{f}: G \to \mathbb{R}$ such that $|\mathscr{C}\tilde{f}(x,y)| \leqq K$ for all $x, y \in G$?

This problem is till now open.

Theorem 7 and 8 suggest to study the connections between the stability of homomorphisms and the amenability of groups or semigroups.
In fact we obtain a theorem giving a necessary and sufficient condition for the amenability in terms of a kind of multi-stability.

We denote with $B^r(G)$ the space of all bounded real-valued functions on G and with $B^r\mathscr{C}(G)$ the space of all real-valued functions f on G for which $\mathscr{C}f$ is bounded on $G \times G$.

Theorem 9. *Let G be a group. G is amenable if and only if for every n-tuple $f_1, f_2, \ldots, f_n \in B^r\mathscr{C}(G)$, there exist $\Phi_1, \Phi_2, \ldots, \Phi_n \in Hom(G, \mathbb{R})$, such that $f_i - \Phi_i \in B^r(G)$ and for all n-tuples $x_1, x_2, \ldots, x_n \in G$, the inequality*

$$H(x_1, x_2, \ldots, x_n) \leqq \sum_{i=1}^{n} \{\Phi_i(x_1) - f_i(x_i)\} \leqq K(x_1, x_2, \ldots, x_n) \tag{4}$$

holds, where

$$H(x_1, x_2, \ldots, x_n) = \inf_{y \in G} \sum_{i=1}^{n} \mathscr{C}f_i(x_1, y) \text{ and } K(x_1, x_2, \ldots, x_n) = \sup_{y \in G} \sum_{i=1}^{n} \mathscr{C}f_i(x_i, y).$$

Proof. Assume that G is amenable and let m_y be a left invariant mean on $B(G)$. If $f_i \in B^r\mathscr{C}(G)$ we put $\Phi_i(x) = m_y\{{}_x f_i - f_i\}$, $x \in G$; by Theorem 7 we have $\Phi_i \in Hom(G, \mathbb{R})$ and $f_i - \Phi_i \in B^r(G)$. Fix now $x_1, x_2, \ldots, x_n \in G$, by ($\beta$) of Definition 3 we have

$$\inf_{y \in G} \{ \sum_{i=1}^{n} (f_i(x_i y) - f_i(y))\} \leqq m_y\{ \sum_{i=1}^{n} ({}_{x_i}f_i - f_i)\} \leqq \sup_{y \in G} \{ \sum_{i=1}^{n} f_i(x_i y) - f_i(y))\},$$

hence from the definition of Φ_i and of f_i we get

$$\inf_{y \in G} \{\sum_{i=1}^{n} f_i(x_i, y)\} + \sum_{i=1}^{n} f_i(x_i) \leqq \sum_{i=1}^{n} \Phi_i(x_i) \leqq \sup_{y \in G} \{\sum_{i=1}^{n} f_i(x_i, y)\} + \sum_{i=1}^{n} f_i(x_i)$$

thus we get inequality (4).
Conversely let $f_1, f_2, \ldots, f_n \in B^r(G)$, then $f_1, f_2, \ldots, f_n \in B^r\mathscr{C}(G)$ and the corresponding homomorphisms (existing by hypothesis) Φ_i are equal to zero. We now show that the inequality (4) implies the condition of Dixmier (see [7]). We must show that

$$\sup_{y \in G} \{ \sum_{i=1}^{n} (f_i(x_i y) - f_i(y))\} \geqq 0. \tag{5}$$

Assume that the supremum in (5) be equal to $-\varepsilon < 0$, then

$$\sum_{i=1}^{n} \mathscr{C}f_i(x_i, y) < -\varepsilon - \sum_{i=1}^{n} f_i(x_i),$$

hence by (4) (the Φ_i's are zero) we get

$$-\sum_{i=1}^{n} f_i(x_i) < -\varepsilon - \sum_{i=1}^{n} f_i(x_i),$$

a contradiction.

Remark 1. Theorem 9 is stated for groups, but obviously it is true also for semigroups and for left or right amenability.

4. *Applications to Functional Equations*

In this section we shall use some results of the previous sections in order to solve some alternative Cauchy functional equations.

The problem, in its general form, can be formulated as follows: *to find all functions f such that*

$$(*)_V \qquad f(xy) - f(x) - f(y) \in V$$

where $f: G \to H$, G and H groups and $V \subset H$.

The problem of solving an equation of this kind appeared the first time in a paper of R. Ger ([6]) in the special case $G = H = \mathbb{R}$ and $V = \{0,1\}$.
Results have been obtained in [4] and in [12] when G and H are abelian groups and V has only two elements, in [3] when G is abelian, H is a Banach space and $V = \{0, b, 2b, \ldots, Mb\}$ for $b \in H$ and M positive integer, and in [13] when G is abelian and H and V satisfies some suitable hypotheses.

Herein we assume that H is a Banach space B and that the couple (G, B) is HS and we study $(*)_V$ for special sets V. We shall use techniques similar to those contained in [3] and generalize some results of [13].
It is easy to see that one can always suppose, without loss of generality, that $0 \in V$; indeed if $a \in V$ and we put $g(x) = f(x) - a$, then f is a solution of $(*)_V$ if and only if g is a solution of $(*)_{V-a}$.

Let $v_1, v_2, \ldots, v_n$ be n independent vectors in B and let $V = \{0, v_1, v_2, \ldots, v_n\}$. If we consider equation $(*)_V$, thanks to Theorem 4 of section 2, we can assume that the unknown function f is bounded and, identifying the vectors v_i with the standard basis in $\mathbb{R}^n$ via an isomorphism, that f assumes values in $\mathbb{R}^n$ or, more precisely, in the convex hull (contained in the unit cube) of the set $-V$ (we use the same names for vectors v_i and the set V either as elements or subset of B or as elements or subset of $\mathbb{R}^n$). Since $f: G \to \mathbb{R}^n$, by f_i we denote the i-th component of f.

Lemma 1. *Let $f: G \to C(-V)$ be a solution of $(*)_V$. Let $F_{j,\alpha} = \{v \in C(-V): p_j(v) = \alpha\}$, $j = 1, 2, \ldots, n$; $\alpha = 0,1$, where p_j is the projection on the j-th coordinate axis. Then*

i) the sets $S_{j,\alpha} = \{x \in G: f(x) \in F_{j,\alpha}\}$, $\alpha = 0,1$, are either empty or subsemigroups of G;
ii) $H_j = S_{j,0} \cup S_{j,1}$ is a (non empty) normal subgroup of G;
iii) if $y \in H_j$ and $x \in G \backslash H_j$, then $f_j(xy) = f_j(x) = f_j(xy)$.

Proof. The proof is similar to that of Theorem 6 in [3].

Lemma 2. *Let $f: G \to C(-V)$ be a solution of $(*)_V$. Define $H = \{x \in G: f(x) \in -V\}$. Then H is a normal subgroup of G.*

Proof. Since $H = \bigcap_{j=1}^{n} H_j$, the theorem follows from (ii) of Lemma 1.

Theorem 10. *Let $f: G \to C(-V)$ be a solution of $(*)_V$. The range of f is contained in the segments joining the $n+1$ points $-v_0, -v_1, \ldots, -v_n$ (where $v_0 = 0$).*

Proof. The theorem follows from Theorem 6 of section 2. We give here a direct proof. If ε is the identity of G, we have $-f(\varepsilon) \in V$, so for any $x \in G$ we get $f(\varepsilon) - f(x) - f(x^{-1}) \in V$, so $f(x^{-1}) = -f(x) + f(\varepsilon) - v_i$ for some $i = 0, \ldots, n$. Let $f(\varepsilon) = -v_k$, then $f(x^{-1}) = -f(x) - v_k - v_i$ and $-f(x) = \sum_{j=0}^{n} \lambda_j v_j, \lambda_j \geqq 0$ and $\sum_{j=0}^{n} \lambda_j = 1$. This yields

$$f(x^{-1}) = \sum_{\substack{j=0 \\ j \neq i,k}}^{n} \lambda_j v_j + (\lambda_k - 1) v_k + (\lambda_i - 1) v_i$$

and since $f(x^{-1}) \in C(-V)$, this implies $\lambda_j = 0$ for $j \neq i, k$ and $2 - \lambda_k - \lambda_i \leqq 1$, that is $\lambda_k + \lambda_i = 1$ and $f(x) = -\lambda_k v_k - \lambda_i v_i$.

Theorem 11. *Let $f: G \to C(-V)$ be a solution of $(*)_V$.*

i) The range of f is contained in only one of the segments joining the points of $-V$;
ii) f is constant on the non-zero cosets of H (H is the subgroup of G defined in Lemma 2).

Proof. (i) Let $x \in G$ be such that $f(x) = -\lambda_k v_k - \lambda_i v_i, \lambda_k + \lambda_i = 1, \lambda_k > 0, \lambda_i > 0$, and let $y \in H$ be such that $-f(y) = v_j, j \neq k, i$. Then, by $(*)_V$, we have

$$f(y) - f(x) - f(yx^{-1}) = -v_j + \lambda_k v_k + \lambda_i v_i - f(yx^{-1}) \in V,$$

that is

$$-f(yx^{-1}) \in v_j - \lambda_k v_k + \lambda_i v_i + V;$$

if $v_k \neq v_0, v_i \neq v_0$ this implies that either the coefficient of v_k or that of v_i is negative; a contradiction.

If $v_k = v_0 = 0$, then $-f(yx^{-1}) = v_j + (1 - \lambda_i) v_i$, a contradiction by Theorem 10. Let now $y \in G$ be such that $f(y) = -\mu_j v_j - \mu_l v_l, \mu_j + \mu_l = 1, \mu_j > 0, \mu_l > 0$, where the pairs (v_k, v_i) and (v_j, v_l) are different. By $(*)_V$ we have

$f(y)-f(x)-f(yx^{-1}) = -\mu_j v_j - \mu_l v_l + \lambda_k v_k + \lambda_i v_i - f(yx^{-1}) \in V,$

that is

$$-f(yx^{-1}) = \sum_{r=0}^{n} \sigma_r v_r \in \mu_j v_j + \mu_l v_l - \lambda_k v_k - \lambda_i v_i + V.$$

If the four vectors v_j, v_l, v_k, v_i are different and $v_k \neq v_0, v_i \neq v_0$, then either σ_k or σ_i is negative, a contradiction. If $v_k = v_0 = 0$, we must have $-f(yx^{-1}) = \mu_j v_j + \mu_l v_l + (1-\lambda_i) v_i$, again a contradiction, since, by Theorem 10, all σ_r but two must be zero. Assume now $v_j = v_k$; we have $-f(yx^{-1}) \in (\mu_j - \lambda_k) v_j + \mu_l v_l - \lambda_i v_i + V$. If v_j, v_l, v_i are different from $v_0 = 0$, we have two possibilities: $\mu_j \neq \lambda_k$, then $-f(yx^{-1}) = (\mu_j - \lambda_k) v_j + \mu_l v_l + (1-\lambda) v_i$, a contradiction as above; $\mu_j = \lambda_k$, then $\mu_l = \lambda_l$ and $-f(yx^{-1}) = \mu_l v_l + (1-\mu_l) v_i$, this means that the three points $f(x), f(y), f(yx^{-1})$ lie on three different segments joining the points of $-V$, thus, by the first part of the proof, H must be empty; a contradiction.

If $v_j = v_0 = 0$, then $-f(yx^{-1}) = \mu_l v_l + (1-\lambda_i) v_i$; a contradiction as above.
If $v_l = v_0 = 0$, then $-f(yx^{-1}) = (\mu_j - \lambda_k) v_j + (1-\lambda_i) v_i$; again a contradiction.
If $v_i = v_0 = 0$, then $-f(yx^{-1}) \in (\mu_j - \lambda_k) v_j + \mu_l v_l + V$, so either $-f(yx^{-1}) = (\mu_j - \lambda_k) v_j + \mu_l v_l$ or $-f(yx^{-1}) = (1 + \mu_j - \lambda_k) v_j + \mu_l v_l$, but this two points are not on the required segments.

(ii) If $f = (0, \ldots, f_i, 0, \ldots, 0)$, the theorem follows from Lemma 1.
If $f(x) = f_i(x) v_i + f_j(x) v_j$, where $f_i, f_j : G \to [-1,0]$ are solutions of $(*)_{\{0,1\}}$ such that $f_i(x) + f_j(x) = -1$, then by (i), $H = \{x : f(x) = -v_i\} \cup \{x : f(x) = -v_j\} = \{x : f_j(x) = 0\} \cup \{x : f_i(x) = 0\} = \{x : f_j(x) = 0\} \cup \{x : f_j(x) = -1\}$ and, by Lemma 1, f_j is constant on the non-zero cosets of H; since $f_i + f_j = -1$, also f_i is constant on the non-zero cosets of H and so this happens for f.

Theorem 12. *A function $f : G \to C(-V)$ is a solution of equation $(*)_V$ if and only if it has one of the following forms:*

i) $f = (0, \ldots, 0, f_i, 0, \ldots, 0)$ for some $i = 1, \ldots, n$ and $f_i : G \to [-1,0]$ is a solution of $(*)_{\{0,1\}}$;
ii) $f = (0, \ldots, 0, f_i, 0, \ldots, 0, f_j, 0, \ldots, 0)$ for some i and j, where $f_i, f_j : G \to [-1,0]$ are solutions of $(*)_{\{0,1\}}$, such that $f_i(x) + f_j(x) = -1$ for all $x \in G$.

Proof. The proof follows immediately from Theorem 11.

Remark 2. It has to be observed that if there are solutions of $(*)_V$ of the form (i) of Theorem 12, then there are also of the form (ii); indeed let $f : G \to [-1,0]$ be a solution of $(*)_{\{0,1\}}$, it is immediately seen that the function $g(x) = -f(x) - 1$ is a solution of $(*)_{\{0,1\}}$.

Till now the set V was a special finite set; we intend to extend, when possible, the previous results to the case of V infinite.

Consider as before the Banach space B and assume it is a real space of infinite dimension. As any vector space, B has a Hamel basis, i.e. a set $V = \{v_j\}_{j \in J}$ of independent vectors spanning B, in the sense that any element of B has a unique representation as a finite linear combination of elements of V. We can always assume that all v_j's have norm 1, so V is a bounded subset of B.

Now we consider the equation $(*)_V$ for the set V described above. As in the previous case we can suppose that the unknown function f is bounded and takes values in $C(-V) = \overline{C(-V)}$.

For any subset I of J, by p_I we denote the coordinate projection given by

$$p_I(x) = p_I(\sum_{j\in J}\alpha_j v_j) = \sum_{j\in I}\alpha_j v_j, x\in B,$$

(where only a finite number of α_j's are different from zero).
For any $I \subset J, f_I = p_I \circ f$. Obviously if $f: G \to C(-V)$ is a solution of $(*)_V$, then for any $I \subset J, f_I$ is a solution of $(*)_{p_I(v)}$.

Then from Theorem 12 we get immediately the following

Theorem 13. *Let $f: G \to C(-V)$ be a solution of $(*)_V$; then f has only these possible forms:*

i) *there exists $k\in J$ such that for every $i\in J$, $i \neq k, f_i = 0$ and $f_k = \lambda_k v_k$, where $\lambda_k: G \to [-1,0]$ is a solution of $(*)_{\{0,1\}}$;*

ii) there exist $k, h\in J$ *such that for every $i\in J$, $i \neq k$, $i \neq h, f_i = 0$,* and $f_k = \lambda_k v_k$, $f_h = \lambda_h v_h$ *where $\lambda_k, \lambda_h: G \to [-1,0]$ are solutions of $(*)_{\{0,1\}}$ such that* $\lambda_k(x) + \lambda_h(x) = -1$.

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