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Cohomology of Graded Lie Algebras of Cartan Type of Characteristic p

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Abstract

Let F be an algebraically closed field of characteristic $p > 3$. We study the cohomology of graded Lie algebras of Cartan type over F. Let $L = \bigoplus_{i \geqslant -1} L_{[i]}$ be a graded Lie algebra of Cartan type. For every irreducible $L_{[0]}$-module V_0, a graded L-module $\tilde{V}_0$ is constructed from which all irreducible graded L-modules are derived [9, 10, 11]. We determine the structures of $H^1(L, \tilde{V}_0)$, where $L = W(2, \mathbf{m})$, $W(3, \mathbf{m})$ or $H(2, \mathbf{m})$ and the structures of $H^1_*(L, V)$, where $L = W(2, (1,1))$, $W(3, (1,1,1))$ or $H(2, (1,1))$ and V is an irreducible restricted L-module.

§ 0. *Introduction*

In [3], DZHUMADIL'DAEV gave the structure of the cohomology groups $H^1(W(1, \mathbf{n}), U_t)$ of the Zassenhaus algebra $W(1, \mathbf{n})$. In [9], [10] and [11], SHEN GUANGYU constructed the graded modules $\tilde{V}_0$ of graded Lie algebras of Cartan type and determined the structure of the irreducible graded modules.

In this paper we study the cohomology of graded Lie algebras of Cartan type. In particular, we determine the structure of the first cohomology of the graded Lie algebra $L = \bigoplus_{i \geqslant -1} L_{[i]} (\cong W(2, \mathbf{m}))$, $W(3, \mathbf{m}))$ or $H(2, \mathbf{m}))$ of Cartan type with coefficients in $\tilde{V}_0$ where V_0 is an irreducible $L_{[0]}$-module. If moreover L is restricted, that is, $L = W(2, (1,1))$, $W(3, (1,1,1))$ or $H(2, (1,1))$, then we determine the structure of the cohomology groups $H^1(L, V)$, where V is an irreducible L-module, and the restricted cohomology groups $H^1_*(L, V)$, where V is an irreducible restricted L-module.

In § 1, we shall review the notions and the results of SHEN GUANGYU [9, 10, 11] and give a general discussion of the cohomology of graded Lie algebras of Cartan type. In § 2, we discuss some cohomology properties of $W(n, \mathbf{m})$. In § 3, we reduce the computation of $H^1(W(n, \mathbf{m}), \tilde{V}_0)$ to the computation of $H^1(sl(n), V_0)$. Thus we determine the structures of $H^1(W(n, \mathbf{m}), \tilde{V}_0)$ for $n = 2,3$ (see Theorem 3.1) and $H^1_*(W(n, (1,1 \ldots, 1)), V)$ for $n = 2,3$ (see Theorem 3.2). In § 4, we determine the structures of $H^1(H(2, \mathbf{m}), \tilde{V}_0)$ (see Theorem 4.1) and $H^1_*(H(2, (1,1), V)$ (see Theorem 4.2).

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§ 1. *The Graded Modules* $\tilde{V}_0$

Let F be an algebraically closed field, char $F = p > 0$. All Lie algebras and modules treated in the present article are assumed to be finite-dimensional.

The following is a direct implication of DZHUMADIL'DAEV [3 Theorem 1].

Proposition 1.1. *Let L be a Lie algebra over F and ϱ an irreducible representation of L in the module M. If there exists a p-polynomial $z(t) = \sum_{i=0}^{k} c_i t^{p^i}$ and an element $x \in L$ such that $z(ad\, x) = 0$ and $z(\varrho(x)) \neq 0$, then $H^*(L, M) = 0$.*

A Lie algebra L is a graded Lie algebra if $L = \bigoplus_{i \in \mathbf{Z}} L_{[i]}$ where the $L_{[i]}$ are subspaces of L and $[L_{[i]}, L_{[j]}] \subset L_{[i+j]}$. An L-module V is graded if $V = \bigoplus_{i \geq 0} V_i$ and $L_{[j]} V_i \subset V_{i+j}$ (We assume $V_0 \neq 0$ if $V \neq 0$ and V_0 is called the base space of V).

Proposition 1.2 [10] *Let $L = \bigoplus_{i \in \mathbf{Z}} L_{[i]}$ be a graded Lie algebra.*

(1) *An irreducible L-module V is isomorphic to a graded module if and only if the elements of $L^+ := \bigoplus_{i>0} L_{[i]}$ and $L^- := \bigoplus_{i<0} L_{[i]}$ act nilpotently on V.*

(2) *If V is an irreducible graded module then $V \mapsto V_0$ is, up to isomorphism, a bijective map of the class of irreducible graded L-modules onto the class of irreducible $L_{[0]}$-modules.*

(3) *If L is centerless and restricted, then every irreducible restricted L-module V is graded and $V \mapsto V_0$ is, up to isomorphism, a bijective map of the class of irreducible restricted L-modules onto the class of irreducible restricted $L_{[0]}$-modules.*

Theorem 1.1. Let $L = \bigoplus_{i=-k}^{l} L_{[i]}$ *be a graded Lie algebra over F, $L \neq L_{[0]}$, and ϱ an irreducible representation of L in the module V. If V is not isomorphic to a graded module, then $H^*(L, V) = 0$.*

Proof. By Proposition 1.2, there is $x \in L^+ \cup L^-$ such that $\varrho(x)$ is not nilpotent. Choose a positive integer i such that $p^i > k + l + i$. Then $(ad x)^{p^i} = 0$ and the conclusion follows from Proposition 1.1.

We now give a brief description of Lie algebras of Cartan type $W(n, \mathbf{m})$, $S(n, \mathbf{m})$ and $H(n, \mathbf{m})$. Let $A(n)$ be the set of n-tuples of non-negative integers. For α

$=(\alpha_1,\ldots,\alpha_n)\in A(n)$, let $|\alpha| = \sum_{i=1}^{n} \alpha_i$. Set $\varepsilon_i = (\delta_{1i},\ldots,\delta_{ni})\in A(n)$. Let $\mathfrak{U}(n)$ be the divided power algebra with basis $\{x^{(\alpha)}|\alpha\in A(n)\}$ and multiplication

$$x^{(\alpha)}x^{(\beta)} = C_{\alpha}^{\alpha+\beta}x^{(\alpha+\beta)}, \alpha,\beta\in A(n),$$

where

$$C_{\beta}^{\alpha} = \prod_{i=1}^{n} C_{\beta_i}^{\alpha_i} \text{ for } \alpha = (\alpha_1,\ldots,\alpha_n), \beta = (\beta_1,\ldots,\beta_n)\in A(n)$$

and

$$C_{\beta_i}^{\alpha_i} = \frac{\alpha_i!}{\beta_i!\,(\alpha_i-\beta_i)!}.$$

If $\mathbf{m} = (m_1,\ldots,m_n)$ is an n-tuple of positive integers and $A(n,\mathbf{m}) = \{\alpha\in A(n)|\alpha_i < p^{m_i}\}$, then $\mathfrak{U} = \mathfrak{U}(n,\mathbf{m}) = \langle x^{(\alpha)}|\alpha\in A(n,\mathbf{m})\rangle$ is a subalgebra of $\mathfrak{U}(n)$. Write $\pi = (p^{m_1}-1,\ldots,p^{m_n}-1)\in A(n,\mathbf{m})$. Define derivations $D_i, i = 1,\ldots,n$, of $\mathfrak{U}(n,\mathbf{m})$ by

$$D_i x^{(\alpha)} = x^{(\alpha-\varepsilon_i)}$$

(We set $x^{(\alpha)} = 0$ if $\alpha\notin A(n)$). Then $W = W(n,\mathbf{m}) := \{\sum_{j=1}^{n} a_j D_j \,|\, a_j\in\mathfrak{U}(n,\mathbf{m})\}$ is a derivation algebra of $\mathfrak{U}(n,\mathbf{m})$. The bracket operation of $W(n,\mathbf{m})$. is

$$\left[\sum_i a_i D_i, \sum_i b_i D_i\right] = \sum_i \sum_j (a_j D_j(b_i) - b_j D_j(a_i))\,D_i. \tag{1.1}$$

Set $\mathfrak{U}_{[i]} = \langle x^{(\alpha)}|\,|\alpha| = i\rangle$ and $W_{[i]} = \langle x^{(\alpha)}D_j\,|\,x^{\alpha)}\in\mathfrak{U}_{[i+1]}, j = 1,\ldots,n\rangle$. Then $W = \bigoplus_{i\geqslant -1} W_{[i]}$ is a graded Lie algebra of depth 1.

The subspace $S(n,\mathbf{m})$ spanned by

$$D_{i,j}(f) := D_j(f)D_i - D_i(f)D_j, f\in\mathfrak{U}(n,\mathbf{m}), i,j = 1,\ldots,n,$$

is a Lie subalgebra of $W(n,\mathbf{m})$ (see [7]). If $n = 2r$, let

$$\sigma(i) = \begin{cases} 1, 0\leqslant i\leqslant r, \\ -1, r < i\leqslant n, \end{cases}$$

$$\tilde{\imath} = i + \sigma(i)r, i = 1,\ldots,n.$$

The subspace $H(n,\mathbf{m})$ spanned by

$$\mathscr{D}(x^{(\alpha)}) := \sum \sigma(\tilde{\imath})\,D_{\tilde{\imath}}(x^{(\alpha)})\,D_i, \alpha\in A(n,\mathbf{m}), \alpha \neq \pi,$$

is a Lie subalgebra of $S(n,\mathbf{m})$ (see [7]).

If L is any one of $W(n,\mathbf{m}), S(n,\mathbf{m})$ and $H(n,\mathbf{m})$, then $L = \bigoplus_{i\geqslant -1} L_{[i]}$ is a graded Lie algebra of depth 1 and under the linear map $x^{(\varepsilon_i)}D_j\mapsto E_{ij}$, $L_{[0]}$ is isomorphic to $gl(n)$, $sl(n)$ and $sp(n)$ respectively, where E_{ij} is the matrix whose (k,l) – component is $\delta_{ik}\delta_{jl}$.

We proceed to construct a class of graded modules of $L = W(n,\mathbf{m})$, $S(n,\mathbf{m})$ or $H(n,\mathbf{m})$. Let ϱ_0 be a representation of $L_{[0]}$ in the module V_0 and $\tilde{V}_0 = \mathfrak{U}\otimes V_0$. If D

$= \Sigma a_i D_i \in L$, then $\tilde{D} := \Sigma D_i(a_j) \otimes E_{ij} \in \mathfrak{U} \otimes L_{[0]}$ (see [9]). Let $\tilde{D} = \Sigma g_i \otimes l_i$, where $g_i \in \mathfrak{U}, l_i \in L_{[0]}$. Define a linear transformation $\tilde{\varrho}_0(D)$ of $\tilde{V}_0$ by

$$\tilde{\varrho}_0(D)(f \otimes v) = D(f) \otimes v + \Sigma g_i f \otimes \varrho_0(l_i) v, f \in \mathfrak{U}, v \in V_0.$$

We have (cf. [9], [10], [11])

Proposition 1.3.

(1) *$\tilde{\varrho}_0$ is a representation of L in $\tilde{V}_0$.*

(2) *$\tilde{V}_0 = \bigoplus_{i \geqslant 0} V_i$ is a graded L-module where $V_i = \langle x^{(\alpha)} \otimes V_0 \mid |\alpha| = i \rangle$. The base space of $\tilde{V}_0$ is $1 \otimes V_0 \cong V_0$.*

(3) *$\tilde{V}_0$ is transitive (i.e., Ann $L_{[-1]} = 1 \otimes V_0$).*

(4) *If V_0 is an irreducible $L_{[0]}$-module, then the irreducible graded L-module with base space V_0 is isomorphic to the (unique) minimum submodule $(\tilde{V}_0)_{min}$ of $\tilde{V}_0$.*

(5) *If V_0 is $L_{[0]}$-irreducible, then $\tilde{V}_0$ is L-irreducible unless V_0 is trivial or a highest weight module with a fundamental weight as its highest weight.*

(6) *If $\mathbf{m} = (1, \ldots, 1)$, i.e., L is restricted, and V_0 is an irreducible restricted $L_{[0]}$-module, then $(\tilde{V}_0)_{min}$ is the unique irreducible restricted L-module whose base space is isomorphic to V_0.*

Note 1.1. All irreducible graded modules of L are determined in [11]. (1) If $L = W(n, \mathbf{m})$, let $\mathfrak{H} = \langle E_{11}, \ldots, E_{nn} \rangle$ be the standard Cartan subalgebra of $gl(n)$, $\Lambda_i, i = 1, \ldots, n$, be linear functions on $\mathfrak{H}$ such that $\Lambda_i(E_{jj}) = \delta_{ij}, \lambda_0 = 0$ and $\lambda_i = \sum_{j=1}^{i} \Lambda_j, \mathrm{i} = 1, \ldots, n$ (the fundamental weights of $gl(n)$). *If* $\lambda \in \mathfrak{H}^*$, denote by $V_0(\lambda)$ the irreducible module of $gl(n)$ with highest weight λ. If $V_0 = V_0(\lambda_i)$, $\tilde{V}_0(n, \mathbf{m})$ and $\tilde{V}_0(n, \mathbf{m})_{min}$ will be denoted by $\tilde{V}_0(\lambda_i, n, \mathbf{m})$ and $M(\lambda_i, n, \mathbf{m})$ respectively. By [11], we have the following facts. The $W(n, \mathbf{m})$-module $\tilde{V}_0(\lambda_i, n, \mathbf{m})$ is isomorphic to the module of differential i-forms with coefficients in $\mathfrak{U}(n, \mathbf{m})$. Let $d_i: \tilde{V}_0(\lambda_i, n, \mathbf{m}) \to \tilde{V}_0(\lambda_{i+1}, n, \mathbf{m})$ be the exterior differential operator which is a $W(n, \mathbf{m})$-module homomorphism. We have $M(\lambda_i, n, \mathbf{m}) = d_{i-1} \tilde{V}_0(\lambda_i n, \mathbf{m})$ and $\ker d_i \mid M(\lambda_i, n, \mathbf{m}) \cong F^{(C_i^n)}$ (cf. [11, Proposition 2.1 and Lemma 2.1]). Hence we have an exact sequence

$$0 \to \ker d_i \to \tilde{V}_0(\lambda_i, n, \mathbf{m}) \to M(\lambda_{i+1}, n, \mathbf{m}) \to 0 \tag{1.2}$$

which induces the exact sequences

$$\begin{cases} 0 \to M(\lambda_0, n, \mathbf{m}) (= F) \to \tilde{V}_0(\lambda_0, n, \mathbf{m}) \to \mathrm{M}(\lambda_1, n, \mathbf{m}) \to 0 \\ 0 \to F^{(C_i^n)} \to \tilde{V}_0(\lambda_i, n, \mathbf{m}) / M(\lambda_i, n, \mathbf{m}) \to M(\lambda_{i+1}, n, \mathbf{m}) \to 0 \end{cases} \tag{1.3}$$

where $M(\lambda_{n++1}, n, \mathbf{m}) = 0$.

(2) If $L = H(2, \mathbf{m})$, let Z_0 and N_0 be the one-dimensional trivial module and the natural module of $H_{[0]} = sl(2)$, respectively, then $\tilde{V}_0$, whose base space is $H_{[0]}$-irreducible, is reducible if and only if $V_0 = Z_0$ or N_0. We have $\tilde{Z}_0 = \mathfrak{U}, (\tilde{Z}_0)_{min} = F$ and $(\tilde{N}_0)_{min} \cong \mathfrak{U}'/F \cdot l$, where $\mathfrak{U}' := \bigoplus_{\alpha \neq \pi} \langle x^{(\alpha)} \rangle$.

Lemma 1.1. *If $i \neq j$, then $ad_W(x^{(\varepsilon_i)} D_j)$ is nilpotent.*

Proof. From (1.1) we have $ad(x^{(\varepsilon_i)} D_j) \cdot x^{(\alpha)} D_k = x^{(\varepsilon_i)} x^{(\alpha - \varepsilon_j)} D_k - x^{(\alpha)} x^{(\varepsilon_i \varepsilon_k)} D_j$. The assertion follows from an induction on α_j.

Corollary 1.1. *If X is a root vector of $L_{[0]}$ (with respect to the standard Cartan subalgebra consisting of diagonal matrices), then $ad_L X$ is nilpotent.*

Proof. It is obvious for $L_{[0]} = gl(n)$ or $sl(n)$. For $L = sp(n)$, a glance at the expressions of the root vectors (cf. [12, chap. 1, § 3]) shows the result.

Theorem 1.2. *Let $L = W(n, \mathbf{m}), S(n, \mathbf{m})$ or $H(n, \mathbf{m})$, and V_0 an irreducible $L_{[0]}$-module. If V_0 is not a highest weight module, then $H^*(L, \tilde{V}_0) = 0$.*

Proof. It is shown in [11, Lemma 1.1] that there exists a root vector X of $L_{[0]}$ such that $\tilde{\varrho}_0(X)^p = a.\ 1 \neq 0$. Our conclusion follows directly from Proposition 1.1 and Corollary 1.1.

Let $L_{[0]} = gl(n), sl(n)$ or $sp(n)$. An $L_{[0]}$-module V_0 with highest weight λ is called integral if $\lambda(h_i) \in \{0,1, \ldots, p-1\} \subset F$, for all h_i, where $\{h_i\}$ is the standard Chevalley basis of the standard Cartan subalgebra of $L_{[0]}$.

Proposition 1.4. *Let V_0 be an irreducible highest weight module of $L_{[0]}$ which is not integral, then*

$$H^*(L, \tilde{V}_0) = 0,\ L = W(n, \mathbf{m}),\ S(n, \mathbf{m}) \text{ or } H(n, \mathbf{m}).$$

Proof. If $L = W(n, \mathbf{m})$, we have $ad\, x^{(\varepsilon_i)} D_i(x^{(\alpha)} D_j) = x^{(\varepsilon_i)} x^{(\alpha - \varepsilon_i)} D_j - \delta_{ij} x^{(\alpha)} D_j = (\alpha_i - \delta_{ij}) x^{(\alpha)} D_j$. Hence $(ad\, x^{(\varepsilon_i)} D_i)^p - ad\, x^{(\varepsilon_i)} D_i = 0$. On the other hand, suppose $\lambda(E_{ii}) = a \notin \{0,1, \ldots, p-1\}$ and v_λ is a highest weight vector. We have $\tilde{\varrho}_0(x^{(\varepsilon_i)} D_i)(x^{(\alpha)} \otimes v_\lambda) = (\alpha_i + a) x^{(\alpha)} \otimes v_\lambda$. Hence $\tilde{\varrho}_0(x^{(\varepsilon_i)} D_i)^p - \tilde{\varrho}_0(x^{(\varepsilon_i)} D_i) \neq 0$, and our conclusion follows from Proposition 1.1 ($\tilde{V}_0$ is irreducible by Proposition 1.3 (5)). For $L = S(n, \mathbf{m})$ or $H(n, \mathbf{m})$ the argument is similar.

§ 2. *The properties of cohomology of $W(n, \mathbf{m})$*

In section 2 we give some general discussion of the cohomology of $W(n, \mathbf{m})$ with coefficients in $\tilde{V}_0$. For convenience, let $W = W(n, \mathbf{m}), \mathfrak{U} = \mathfrak{U}(n, \mathbf{m})$ and $W_i = \bigoplus_{j \geq i} W_{[j]}$. By Proposition 1.3 (2), we identify V_0 with the subspace $1 \otimes V_0$ of $\tilde{V}_0$. Suppose that W_1 acts trivially on the $W_{[0]}$-module V_0, then we may regard V_0 as a W_0-module. Now we generalize Lemma 3 in [3] and obtain

Lemma 2.1. *The relative cohomology $H^*(W, W_{[-1]}, \tilde{V}_0)$ is a direct summand of $H^*(W, \tilde{V}_0)$ and $H^*(W, W_{[-1]}, \tilde{V}_0) \cong H^*(W_0, V_0)$.*

Proof. We denote the projection from $\tilde{V}_0$ onto V_0 by Pr_{V_0}. Let $\mathscr{A}: C^*(W, \tilde{V}_0) \to C^*(W_0, V_0)$ be a linear map such that $\mathscr{A}\tilde{v} = Pr_{V_0}(\tilde{v})$ for $\tilde{v} \in C^0(W, \tilde{V}_0) = \tilde{V}_0$, and

$$\mathscr{A}\,\psi(l_1,\ldots,l_k) = Pr_{V_0}(\psi(l_1,\ldots,l_k)).$$

for $l_1,\ldots,l_k \in W_0$, where $k > 0$ and $\psi \in C^k(W,\tilde{V}_0)$. We shall show that the following diagram is commutative.

$$\begin{array}{ccc} C^k(W,\tilde{V}_0) & \xrightarrow{d} & C^{k+1}(W,\tilde{V}_0) \\ \mathscr{A}\downarrow & & \mathscr{A}\downarrow \\ C^k(W_0,V_0) & \xrightarrow{d} & C^{k+1}(W_0,V_0) \end{array} \qquad k \geq 0$$

It is clear that $Pr_{V_0}(l(\tilde{v})) = Pr_{V_0}(l\,Pr_{V_0}(\tilde{v}))$, for $l \in W_0, \tilde{v} \in \tilde{V}_0$, and $l(\mathscr{A}\,\psi(\ldots,l,\ldots)) = \mathscr{A}\,(l(\psi(\ldots,\hat{l},\ldots))$, for $\psi \in C^k(W,\tilde{V}_0)$ and $l \in W_0$. Thus we have

$$d\,\mathscr{A}\,\psi = \mathscr{A}\,d\,\psi.$$

We now prove that $\mathscr{A}|_{C^*(W,W_{[-1]},\tilde{V}_0)}$ is injective. By the definition of relative cohomology and Proposition 1.3 (3), we have $C^0(W,W_{[-1]},\tilde{V}_0) = \tilde{V}_0^{\,W[-1]} = V_0$. Thus $\mathscr{A}|_{C^0(W,W_{[-1]},\tilde{V}_0)}$ is the identity map. For $k > 0$, let $\psi \in C^k_{(W,W_{[-1]},\tilde{V}_0)}$ and $\mathscr{A}\,\psi = 0$. Assume $Pr_{V_{j_1}}(\psi(\ldots)) \neq 0$, where j_1 is the smallest positive integer such that the inequality is valid. Since $\psi \in C^k(W,W_{[-1]},\tilde{V}_0)$, we have

$$D_j(\psi(l_1,\ldots,l_k)) = \sum_{i=1}^{k}(-1)^i\,\psi([D_j,l_i],l_1,\ldots,\hat{l}_i,\ldots,l_k),$$

for $l_1,\ldots,l_k \in W_0$ and $j = 1,2,\ldots,n$. Applying $Pr_{V_{j-1}}$ to the both sides of the equality, the right side becomes zero, but the left side doesn't, and we get a contradiction. Thus $\psi = 0$.

Next we define a linear map $\mathscr{A}': C^*(W_0,V_0) \to C^*(W,\tilde{V}_0)$. Thus, we set

$$(\mathscr{A}'\,\varphi)(x^{(\alpha_{(1)})}\otimes_{(1)},\ldots,x^{(\alpha_{(k)})}D_{(k)})$$

$$= \sum_{|\beta_{(1)}|+\ldots+|\beta_{(k)}|-k=r} D^{\beta_{(1)}}(x^{(\alpha_{(1)})})\ldots D^{\beta_{(k)}}(x^{(\alpha_{(k)})})\otimes\varphi\,(x^{(\beta_{(1)})}D_{(1)},\ldots,x^{(\beta_{(k)})}D_{(k)})$$

for $k > 0$ and $\varphi \in C^k(W_0,V_0)$, where $\alpha_{(i)},\beta_{(i)} \in A(n,\mathbf{m}), D_{(i)} \in \{D_1,\ldots,D_n\}$, and let $\beta_{(i)} = (\beta_{i1},\ldots,\beta_{in})$, then $D^{\beta_{(i)}} = D_1^{\beta_{i1}}\ldots D_n^{\beta_{in}}$, and

$$\mathscr{A}'\,v = v, \textit{for } v \in C^0(W_0,V_0) = V_0.$$

It is clear that $\mathscr{A}'\,C^*(W_0,V_0) \subseteq C^*(W,W_{[-1]},\tilde{V}_0)$. We need only to show that $\mathscr{A}\,\mathscr{A}': C^*(W_0,V_0) \to C^*(W_0,V_0)$ is the identity map. Then $C^*(W,W_{[-1]},\tilde{V}_0) \cong C^*(W_0,V_0)$ and the lemma is proved.

For $l \in W_0$, if $l \in W_{[i]}$, we write $|l| = i$. For $r \geqq 0$ and $k > 0$, let $C^k_r(W_0,V_0) = \{\varphi \in C^k(W_0,V_0)|$ if $|l_1| + \ldots + |l_k| \neq r$, then $\varphi(l_1,\ldots,l_k) = 0\}$. Then $C^k(W_0,V_0) = \bigoplus_{r\geqslant 0} C^k_r(W_0,V_0)$. If $\varphi \in C^k_r(W_0,V_0)$, then

$$(\mathscr{A}'\,\varphi)(x^{(\alpha_{(1)})}D_{(1)},\ldots,x^{(\alpha_{(k)})}D_{(k)})$$

$$= \sum_{|\beta_{(1)}|+\ldots+|\beta_{(k)}|-k=r} D^{\beta_{(1)}}(x^{(\alpha_{(1)})})\ldots D^{\beta_{(k)}}(x^{(\alpha_{(k)})}) \otimes \varphi(x^{(\beta_{(1)})}D_{(1)},\ldots,x^{(\beta_{(k)})}D_{(k)}).$$

It is obvious that $\mathscr{A}'\varphi(l_1,\ldots,l_k) = 0$, if $|l_1|+\ldots+|l_k| < r$, $\mathscr{A}\mathscr{A}'\varphi(l_1,\ldots,l_k) = 0$, if $|l_1|+\ldots+|l_k| > r$, and $\mathscr{A}\mathscr{A}'\varphi(l_1,\ldots,l_k) = \varphi(l_1,\ldots,l_k)$, if $|l_1|+\ldots+|l_k| = r$.

We now wish to compute the cohomology $H^*(W_0, V_0)$ in special cases.

Let $T = \langle x^{(\varepsilon_1)}D_1,\ldots,x^{(\varepsilon_n)}D_n\rangle$, V_0 a W_0-module and $V_0^T = \{v\in V_0 | T\cdot v = 0\}$. Then we have

Lemma 2.2. *If* $V_0^T = 0$, *then*

$$H^1(W_0, V_0) \cong H^1(W_0, T, V_0)$$

Proof. Let

$$\mathbf{C}: 0 \to C^0(W_0, V_0) \overset{d^0}{\to} C^1(W_0, V_0) \overset{d^1}{\to} C^2(W_0, V_0) \to \ldots \tag{2.1}$$

be the standard cohomology complex. Let us abbreviate $C^j(W_0, V_0)$ as C^j. Let

$$F_tC^j = 0, \text{ if } t \geqq l, t > j,\ F_tC^j = \{\psi\in C^j | \psi(l_1,\ldots,l_j) = 0,$$
$$\text{if } j-t+1 \text{ of variables } l_i\in T\}, \text{ if } j \geqq t \geqq l$$
$$F_tC^j = C^j, \text{ if } t \leqq 0.$$

Then we obtain a filtration of the complex $\mathbf{C}$ (2.1)

$$\ldots \subset F_{t+1}\mathbf{C} \subset F_t\mathbf{C} \subset F_{t-1}\mathbf{C} \subset \ldots.$$

As in [8, Corollary 11. 12], it determines a spectral sequence $\{E_r | r=1,2,\ldots\}$, where $E_1^{i,j} = H^{i+j}(F_i\mathbf{C}/F_{i+1}\mathbf{C})$, for $i,j\in\mathbb{Z}$. By [4, § 2, Corollary to Theorem 2]. We have

$$E_1^{i,j} \cong H^j(T, C^i(W_0/T, V_0)).$$

Since the filtration $\{F_t\mathbf{C}\}$ is bounded, the spectral sequence $\{E_r\}$ converges to $H^*(\mathbf{C})$ with the obvious filtration

$$\psi_t H^n(\mathbf{C}) = im(H^n(F_t\mathbf{C}) \to H^n(\mathbf{C})),$$

where the map is induced by the inclusion $F_t\mathbf{C}\to\mathbf{C}$, we have

$$E_\infty^{i,j} \cong \psi_i H^{i+j}(\mathbf{C}) / \psi_{i+1} H^{i+j}(\mathbf{C}).$$

In particular, we have $E_\infty^{0,1} \cong H^1(\mathbf{C})/E_\infty^{1,0}$ since $\psi_2 H^1(\mathbf{C}) = 0$. Hence

$$E_\infty^{0,1} \cong H^1(W_0, V_0) / E_\infty^{1,0}. \tag{2.2}$$

By [4, p. 594], we have

$$E_2^{0,1} \cong H^1(T, V_0),$$
$$E_2^{1,0} \cong H^1(W_0, T, V_0).$$

Since T is a commutative Lie algebra and $V_0{}^T = 0$, we have

$$H^1(T, V_0) = 0.$$

Thus we have $E_\infty^{1,0} = 0$. Hence $H^1(W_0, V_0) \cong E_\infty^{1,0}$ is a subquotient module of $E_2^{1,0} \cong H^1(W_0, T, V_0)$. It forces $H^1(W_0, V_0) \cong H^1(W_0, T, V_0)$.

§ 3. *The computation of* $H^1(W, \tilde{V}_0)$

Let W be $W(n, \mathbf{m})$ over an algebraically closed field F, char $F = p > 2$. In particular, if $\mathbf{m} = (1,1,\ldots,1)$, then $W(n,(1,1\ldots,1))$ is a restricted Lie algebra. In this section we shall reduce the computation of the cohomology groups $H^1(W(n,\mathbf{m}), \tilde{V}_0)$ to the computation of $H^1(sl(n), V_0)$. Hence we can determine the structure of $H^1(W(n,\mathbf{m}), \tilde{V}_0)$ for $n = 2,3$. Thanks to Theorem 1.1 and 1.2, we need to consider only the cases where V_0 is an irreducible highest weight module of $W_{[0]}$. We shall also determine the structure of $H^1(W(n,\mathbf{m}), M(\lambda_i, n, \mathbf{m}))$ for $i = 0,1,\ldots,n$. In particular, we shall determine the structure of the cohomology group $H^1(W(n,\mathbf{m})\, V)$, where $V$is an irreducible $W(n,\mathbf{m})$-module, and the restricted cohomology group $H^1_*(W(n,(1,1,\ldots,1)), V)$, where V is an irreducible restricted $W(n,(1,1,\ldots,1))$-module for $n = 2$ and 3. We divide the computation into several steps.

Step 1*:* In this step we shall prove:

a) If $V_0 = F$ with trivial action (i.e. $\tilde{V}_0 = \mathfrak{U}$) then there is an exact sequence $0 \to H^1(W, W_{[-1]}, \mathfrak{U}) \to H^1(W, \mathfrak{U}) \to H^1(W_{[-1]}, \mathfrak{U}) \to 0;$

b) If $V_0 \neq F$, then $H^1(W, W_{[-1]}, \tilde{V}_0) \tilde{\to} H^1(W, \tilde{V}_0)$.

In case a), we need to show only that the restriction map $H^1(W, \mathfrak{U}) \to H^1(W_{[-1]}, \mathfrak{U})$ is surjective. Indeed, any 1-cocycle $\psi \in Z^1(W_{[-1]}, \mathfrak{U})$ can be extended to a 1-cocycle on Wvia $\psi(x^{(\alpha)} D_i) = x^{(\alpha)} \psi(D_i)$. This yields the surjectivity of the restriction map. It implies that natural action of $W_{[0]} = gl(n)$ on $H^1(W_{[-1]}, \mathfrak{U})$, is trivial.

In case b), if $V_0 \neq F$, then the restriction map $H^1(W, \tilde{V}_0) \to H^1(W_{[-1]}, \tilde{V}_0)$ takes values in

$$H^1(W_{[-1]}, \tilde{V}_0)^{gl(n)} = (H^1(W_{[-1]}, \mathfrak{U}) \otimes V_0)^{gl(n)} = H^1(W_{[-1]}, \mathfrak{U}) \otimes (V_0{}^{gl(n)}) = 0.$$

Hence the restriction map is 0. It implies that

$$H^1(W, W_{[-1]}, \tilde{V}_0) \tilde{\to} H^1(W, \tilde{V}_0).$$

Using the cohomology five-term sequence, we have.

$$0 \to H^1(\langle D_1\rangle, \mathfrak{U}) \to H^1(W_{[-1]}, \mathfrak{U}) \to H^1(\langle D_2, \ldots, D_n\rangle, \mathfrak{U})^{<D_1>} \to 0.$$

It is obvious that $H^1(\langle D_1\rangle, \mathfrak{U}) = C^1(\langle D_1\rangle, \mathfrak{U})/d^o(\mathfrak{U}) = \langle[\beta_1]\rangle$, where $\beta_1(D_1) = x^{(p^{m_1}-1,0,0,\ldots,0)}$ and $[\beta_1]$ is the cohomology class of β_1. On the other hand, let $\mathfrak{U}_i = \langle x^{(i_1,\ldots,i_n)} \in \mathfrak{U} | i_1 = i\rangle$, then we have

$$\begin{aligned} H^1(\langle D_2, \ldots, D_n\rangle, \mathfrak{U})^{<D_1>} &= \bigoplus_{i=0}^{p^{m_1}-1} H^1(\langle D_2, \ldots, D_n\rangle, \mathfrak{U}_i)^{<D_1>} \\ &= H^1(\langle D_2, \ldots, D_n\rangle, \mathfrak{U}_0). \end{aligned}$$

To repeat this procedure, we obtain

$$H^1(W_{[-1]}, \mathfrak{U}) \cong \bigoplus_{i=1}^{n} \langle[\beta_i]\rangle,$$

where

$$\beta_i(D_j) = \begin{cases} x^{(0,\ldots,p^{m_i}-1,\ldots,0)}, \text{if } j = i, \\ 0, \text{ if } j \neq i, \end{cases}$$

for $i,j = 1,2,\ldots,n$. Thus we have proved

Lemma 3.1. *If V_0 is a nontrivial irreducible $gl(n)$-module with a highest weight, then*

$$H^1(W, \tilde{V}_0) \cong H^1(W, W_{[-1]}, \tilde{V}_0).$$

If $V_0 = F$ with trivial action, then

$$H^1(W, \tilde{V}_0) \cong \bigoplus_{i=1}^{n} \langle[\beta_i]\rangle \oplus H^1(W, W_{[-1]}, \mathfrak{U}).$$

Our next task is to compute $H^1(W, W_{[-1]}, \tilde{V}_0)$ in Lemma 3.1.

Step 2: By Lemma 2.1, we have

$$H^*(W, W_{[-1]}, \tilde{V}_0) \xrightarrow{\sim} H^*(W_0, V_0).$$

Step 3: Let $I = E_{11} + E_{22} + \ldots + E_{nn} \in gl(n) = W_{[0]}$. We shall prove

Lemma 3.2. *If I does not act trivially on V_0, then*

$$H^*(gl(n), V_0) = 0$$

and

$$H^1(W_0, V_0) \cong H^1(W_1, V_0)^{gl(n)}.$$

Proof. The first claim is a special case of Proposition 1.1 (Take $z(t) = t, x = I$). Using the cohomology five-term sequence, we have

$$\begin{aligned} 0 \to H^1(gl(n), V_0^{W_1}) &\to H^1(W_0, V_0) \to H^1(W_1, V_0)^{gl(n)} \\ &\to H^2(gl(n), V_0^{W_1}) \to H^2(W_0, V_0), \end{aligned} \tag{3.1}$$

since $W_0/W_1 \cong W_{[0]} \cong gl(n)$. Using (3.1) the second claim follows from the first one.

Step 4. In this step we shall compute $H^1(W_1, V_0)^{gl(n)}$ for any V_0.

As W_1 acts trivially on V_0, this is equal to

$$Hom_{gl(n)}(W_1/[W_1, W_1], V_0) = \bigoplus_{i \geqslant 1} Hom_{gl(n)}(Y_i, V_0), \tag{3.2}$$

where Y_i is the contribution to $W_1/[W_1, W_1]$ coming from $W_{[i]}$. By (1.1) and Lucas Theorem, that is,

$$\binom{\sum_{\mu \geq 0} b_\mu \ p^\mu}{\sum_{\nu=0}^{t} a_\nu \ p^\nu} \equiv \prod_{\mu=0} \binom{b_\mu}{a_\mu} \equiv \prod_{\nu=0}^{t} \binom{b_\nu}{a_\nu} \ (mod\ p),$$

where $\sum_{v\geq 0} b_\mu p^v$ and $\sum_{\mu=0}^{t} a_v p^v$ are the p-adic expressions, we can verify directly

Lemma 3.3. $[W_1, W_1] = \langle x^{(\alpha)} D_k \in W_1 \| \alpha| > 2, k = 1,2,\ldots,n, (\alpha) \neq (p^{\mu_1}, 0\ldots,0), (0, p^{\mu_2}, 0,,,,0),\ldots,(0,\ldots,0,p^{\mu_n})$ *for* $0 < \mu_i < m_i, i = 1,2,\ldots,n\rangle$.

By Lemma 3.3, we have

$$W_1/[W_1, W_1] \cong \langle \bar{W}_{[1]}, \overline{x^{(\alpha)} D_k} | \alpha = (0,\ldots,0,p^{\mu_i},0,\ldots,0, \text{ for } 0 < \mu_i < m_i \text{ and } i = 1,2,\ldots,n\rangle, \tag{3.3}$$

where the congruent classes modulo $[W_1, W_1]$ are denoted by the bars. Then $Y_i = 0$ for all $i > 1, i \neq p^\mu - 1$ for all $\mu > 0$ and $Y_{p^\mu - 1}$ $(\mu > 0)$ is isomorphic to a direct sum of $\# \{j | m_j > \mu\}$-copies of $(F^n)^*$ regarded as $gl(n)$-modules, where $(F^n)^* \cong W_{[-1]}$ is the dual of the natural module of $gl(n)$ (see [9, Lemma 2.2]). Then there are $\# \{j | m_j > \mu\}$-isomorphisms, denoted by $\psi_j^{(\mu)}$ for $j = 1,2,\ldots,n$ and $m_j > \mu$, which form a basis of $Hom_{gl(n)}(Y_{p^\mu - 1}, (F^n)^*)$. Hence

$$\bigoplus_{i\geq 2} Hom_{gl(n)}(Y_i, V_0) = \begin{cases} \langle \psi_j^{(\mu)} | \begin{matrix} j = 1,2,\ldots,n, \\ m_j > \mu > 0 \end{matrix} \rangle, & \text{if } V_0 \cong (F^n)^*, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

To compute (3.2), we need to obtain only $Hom_{gl(n)}(Y_1, V_0)$. We can verify easily that the linear map $x^{(\alpha)} D_i \to x^{(\alpha)} \otimes D_i$ is a $gl(n)$-module isomorphism of $W_{[1]}$ onto $\mathfrak{U}_{[2]} \otimes W_{[-1]}$. As $sl(n)$-modules we have $\mathfrak{U}_{[2]} \cong V(2\lambda_1)$ and $W_{[-n]} \cong V(\lambda_{1-1})$, where $\lambda_1, \ldots, \lambda_n$ are the fundamental dominant weights and $V(\lambda)$ is the irreducible $sl(n)$-module with the highest weight λ. In characteristic 0, we have $W_{[1]} \cong \mathfrak{U}_{[2]} \otimes W_{[-1]} \cong V(2\lambda_1 + \lambda_{n-1}) \oplus V(\lambda_1)$, where the action of I on $V(2\lambda_1 + \lambda_{n-1})$ and $V(\lambda_1)$ is the scalar multiplication by 1 (by [6, Ex. 24.12]). In prime characteristic, the formula holds if and only if the Weyl modules $V(2\lambda_1 + \lambda_{n-1})$ and $V(\lambda_1)$ are simple. If $p \geqq \langle 2\lambda_1 + \lambda_{n-1} + \delta, \alpha_0^\vee \rangle = n + 2$, where δ = half the sum of positive roots and $\alpha_0 = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}$ is the highest root, then $V(2\lambda_1 + \lambda_{n-1})$ and $V(\lambda_1)$ are simple and the formula holds. Thus

$$Hom_{gl(n)}(Y_1, V_0) \cong \begin{cases} F, & \text{if } V_0 \cong V(2\lambda_1 + \lambda_{n-1}) \text{ or } V(\lambda_1), \\ 0, & \text{otherwise,} \end{cases} \tag{3.5}$$

where the action of I on V_0 is the scalar multiplication by 1.

By (3.2) – (3.5), we have

Proposition 3.1. *Suppose char $F = p \geqq n + 2$. Let V_0 be any irreducible $gl(n)$-module with a highest weight. Then we have*

$$H^1(W_1, V_0)^{gl(n)} \cong \begin{cases} \langle \psi_j^{(\mu)} | \begin{matrix} j = 1,2,\ldots,n \\ m_j > \mu > 0 \end{matrix} \rangle, & \textit{if } V_0 \cong V(\lambda_{n-1}) \textit{ and the action of } I \textit{ on } V_0 \textit{ is the scalar multiplication by } p-1, \\ F, & \textit{if } V_0 \cong V(2\lambda_1 + \lambda_{n-1}) \textit{ or } V(\lambda_1) \textit{ and the action of } I \textit{ on } V_0 \textit{ is the scalar multiplication by } 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Step 5: If I acts trivially on V_0, then by Proposition 3.1 we have

$$H^1(W_1, V_0)^{gl(n)} = 0. \tag{3.6}$$

Using the cohomology five-term sequence, we have

$$0 \to H^1(gl(n), V_0^{W_1}) \to H^1(W_0, V_0) \to H^1(W_1, V_0)^{gl(n)} \to H^2(gl(n), V_0^{W_1}) \to H^2(W_0, V_0). \tag{3.7}$$

Since $W_0/W_1 \cong W_{[0]} \cong gl(n)$. By (3.6) and (3.7), we have

$$H^1(W_0, V_0) \cong H^1(gl(n), V_0). \tag{3.8}$$

So we have a reduction from W to $gl(n)$ or even to $sl(n)$ as for $V_0 \neq F$

$$H^1(gl(n), V_0) \cong H^1(sl(n), V_0)^{gl(n)/sl(n)}.$$

If $p \nmid n$, then we have

$$\mathrm{H}^1(gl(n), V_0) \cong H^1(sl(n), V_0), \tag{3.9}$$

where $V_0 \neq F$. Let Φ^+ and W be the set of the positive roots and the Weyl group of $sl(n)$, respectively. If $V(\lambda)$ is the irreducible $sl(n)$-module with the highest weight λ and $\lambda \notin W \cdot 0 = \{w(\delta) - \delta \,|\, w \in W\}$, then by [6, Ex. 23.4 and 3, Theorem 3], we have

$$H^*(sl(n), V(\lambda)) = 0, \text{ for } \lambda \notin W.0. \tag{3.10}$$

Now we shall discuss $H^1(sl(n), V(\lambda))$ for $\lambda \in W.0$. Let G be the algebraic group $SL(n)$ over the field F and G_1 the first Frobenius kernel of G. Fix a Borel subgroup B in G and a maximal torus T in B. Let U be the unipotent radical of B. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots, $X(T)$ the lattice of all weights of T, $X(T)^+$ the set of dominant weights in $X(T)$, and $X_1 = \{\lambda \in X(T)^+ \,|\, 0 \leqq \langle \lambda, \alpha \rangle < p$, for all $\alpha \in \Delta\}$. More precisely we ought to replace X_1 by $X(T)/p\,X(T)$. Then X_1 is the set of all weights for $sl(n)$. Suppose $p > h$ (the Coxeter number), i.e., $p > n$. If $w \in W$ has length $l(w)$ and $\lambda = X(T)^+$ with $w \cdot 0 + p\lambda \in X(T)^+$, then by [1, Corollary 5.5] we have

$$H^1(G_1, H^0(w \cdot 0 + p\lambda)) \cong \begin{cases} H^0(S^{(1-l(w))/2}(u^*) \otimes \lambda)^{(1)}, \text{if } l(w) \equiv 1 \mod 2; \\ O, \quad \text{otherwise;} \end{cases} \tag{3.11}$$

where $S(u^*)$ is the symmetriec algebra on $u^* = (Lie\; U)^*$.

Since $V(\lambda)$ is restricted and irreducible, by [5, p. 575], we have

$$H^1(sl(n), V(\lambda)) \cong H^1{}_*(sl(n), V(\lambda)), \text{ for } \lambda \in X_1, \lambda \neq 0. \tag{3.12}$$

Since $H^1(G_1, V) \cong H^1{}_*(sl(n), V)$ for any restricted $sl(n)$-module, by (3.11) and (3.12), we can compute $H^1(sl(n), V(\lambda))$ at least for $n = 3$.

For $n = 2$, by (3.8) – (3.12), we have

$$H^1(W_0, V(\lambda)) \cong \begin{cases} F \oplus F, \text{if } \lambda = p-2, \\ \langle x^{(1,0)} D_1 \rangle, \text{ if } \lambda = 0, \\ 0, \text{ if } \lambda \neq 0, p-2, \end{cases} \tag{3.13}$$

where I acts trivially on $V(\lambda)$ and $\langle x^{(1,0)}D_1\rangle \cong W_0/[W_0, W_0]$.

For $n = 3$, the fundamental weights of $sl(3)$ are λ_1 and λ_2 (cf. Note 1.1 (1)). Then $X_1 = \{a_1\lambda_1 + a_2\lambda_2 \in X(T) | 0 \leqslant a_1, a_2 < p\}$.

Let $X_t = \{a_1\lambda_1 + a_2\lambda_2 | a_1 + a_2 > p-2 \text{ and } 0 < a_1, a_2 < p-1\}$ and $X_b = \{a_1\lambda_1 + a_2\lambda_2 | a_1 + a_2 < p-2 \text{ and } 0 \leqslant a_1, a_2 < p-1\}$. Recall that if $\lambda \in X_b$ then $H^0(\lambda) = V(\lambda)$ and if $\lambda = a_1\lambda_1 + a_2\lambda_2 \in X_t$ then we have a short exact sequence.

$$0 \to V(\lambda) \to H^0(\lambda) \to V(\lambda') \to 0, \tag{3.14}$$

where $\lambda' = (p-2-a_2)\lambda_1 + (p-2-a_1)\lambda_2 \in X_b$. By (3.8) – (3.12) and (3.14), we have

$$H^1(W_0, V(\lambda)) \cong \begin{cases} F, & \text{if } \lambda = (p-2)(\lambda_1+\lambda_2), \\ H^0(\lambda_1)^{(1)}, & \text{if } \lambda = (p-2)\lambda_1+\lambda_2, \\ H^0(\lambda_2)^{(1)}, & \text{if } \lambda = \lambda_1 + (p-2)\lambda_2, \\ \langle x^{(\varepsilon_1)}D_1\rangle & \text{if } \lambda = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.15}$$

where I acts trivially on $V(\lambda)$.

By Lemma 3.1, Lemma 2.1, Lemma 3.2, Proposition 3.1, (3.13) and (3.15), we have

Theorem 3.1 *Suppose char* $F = p \geqslant n + 2$ for $n = 2,3$. *Let* V_0 *be an irreducible* $gl(n)$-*module with a highest weight.*

(1) *When the action of I is trivial,*

$$H^1(W(n,\mathbf{m}), \tilde{V}_0) \cong \begin{cases} \langle[\beta_1]\rangle \oplus \ldots \oplus \langle[\beta_n]\rangle \oplus \langle x^{(\varepsilon_1)}D_1\rangle, \textit{if } V_0 \cong F, \\ F \oplus F, \textit{ if } n = 2 \textit{ and } V_0 \cong V(p-2), \\ F, \textit{ if } n = 3 \textit{ and } V_0 = V((p-2)(\lambda_1+\lambda_2)), \\ H^0(\lambda_1)^{(1)}, \textit{ if } n = 3 \textit{ and } V_0 = V((p-2)\lambda_1 + \lambda_2) \\ H^0(\lambda_2)^{(1)}, \textit{ if } n = 3 \textit{ and } V_0 = V(\lambda_1 + (p-2)\lambda_2), \\ 0, \textit{ otherwise,} \end{cases}$$

where F is the trivial $gl(n)$-*module,* $V(\lambda)$ *is the irreducible* $sl(n)$-*module with highest weight* λ *which is regarded as a* $gl(n)$-*module such that I acts trivially on it.*

(2) *When the action of I is nontrivial,*

$$H^1(W(n,\mathbf{m}), \tilde{V}_0) \cong \begin{cases} \langle \psi_j^{(\mu)} \mid {j=1,2\ldots,n, \atop m_j > \mu > 0}\rangle, & \textit{if } V_0 \cong V(\lambda_{n-1}) \textit{ and the} \\ & \textit{action of I on } V_0 \textit{ is the scalar multiplication by } p-1. \\ F, & \textit{if } V_0 \cong V(2\lambda_1+\lambda_{n-1}) \textit{ or } V(\lambda_1) \textit{ and the action of I on } V_0 \\ & \textit{is the scalar multiplication by } 1, \\ 0, & \textit{otherwise.} \end{cases}$$

Step 6: In this step we shall compute $H^1(W(n,\mathbf{m}), M(\lambda_i, n, \mathbf{m}))$ for $i = 1, \ldots, n$. For convenience, denote $\tilde{V}_0(\lambda_i, n, \mathbf{m}))$ and $M(\lambda_i, n, \mathbf{m})$ by $\tilde{V}_i$ and M_i, respectively. Since I operates nontrivially on $V(\lambda_i)$ for $i > 0$ and $p > n$, we have

$$\begin{aligned} \mathrm{H}^1(W, \tilde{V}_i) &\cong H^1(W, W_{[-1]}, \tilde{V}_i) && \text{(by Lemma 3.1)} \\ &\cong H^1(W_0, V(\lambda_i)) && \text{(by Lemma 2.1)} \\ &\cong H^1(W_1, V(\lambda_i))^{gl(n)} && \text{(by Lemma 3.2)} \\ &\cong Hom_{gl(n)}(W_1/[W_1, W_1], V(\lambda_i)) \\ &\cong \begin{cases} F, \text{ if } i = 1, \\ 0, \text{ if } i > 1. \end{cases} && \text{(by (3.4) and (3.5))} \end{aligned}$$

By the proof of Theorem 3.1 (i), we can obtain

$$dim\, H^1(W, \mathfrak{U}) = n + 1.$$

By (1.3), we have the long exact sequence

$$H^0(W, \tilde{V}_i) \to H^0(W, \tilde{V}_i/M_i) \to H^1(W, M_i) \to H^1(W, \tilde{V}_i).$$

Since $H^0(W, \tilde{V}_i) = 0$ and $H^0(W, \tilde{V}_i/M_i) \cong F^{(C_i^n)}$ (by (1.3)), we have

$$H^1(W, M_i) \cong F^{(C_i^n)} \qquad \text{for } i \geqslant 2$$

and

$$dim\, H^1(W, M_1) \leqslant n + 1.$$

On the other hand, $0 \to F \to \mathfrak{U} \to M_1 \to 0$ yields

$$H^1(W, F) \to H^1(W, \mathfrak{U}) \to H^1(W, M_1).$$

Since $H^1(W, F) = 0$ and $H^1(W, \mathfrak{U}) \cong F^{n+1}$, it implies that $dim\ H^1(W, M_1) \geqslant n + 1$, so $H^1(W, M_1) \cong F^{n+1}$. Now we have obtained $H^1(W, M_i)$for $i = 1, 2, \ldots, n$. By Proposition 1.3 (5), if V_0 is $gl(n)$-irreducible, then $\tilde{V}_0$ is reducible if and only if $\tilde{V}_0 = \tilde{V}_0(\lambda_1, n, \mathbf{m})$ for $i = 0, 1, \ldots, n$. Then by Theorem 3.1 we have

Corollary 3.1. *Suppose char $F = p \geqslant n + 2$ for $n = 2,3$. Let V be an irreducible $W(n, \mathbf{m})$-module. Then*

$$H^1(W(n, \mathbf{m}), V) \cong \begin{cases} F^{n+1}, \text{ if } V \cong M(\lambda_1, n\,\mathbf{m}), \\ F^{(C_i^n)}, \text{ if } \tilde{V} \cong M(\lambda_i, n, \mathbf{m}) \text{ for } i \geqslant 2, \\ F, \text{ if } V \cong V(2\lambda_1 + \lambda_{n-1}) \text{ and the action of } I \text{ on} \\ \quad V(2\lambda_1 + \lambda_{n-1}) \text{ is the scalar multiplication by } 1, \\ F \oplus F, \text{ if } n = 2 \text{ and } V \cong \tilde{V}(p-2) \text{ and the action of } I \text{ is} \\ \quad \text{trivial.} \\ F, \text{ if } n = 3 \text{ and } V \cong \tilde{V}((p-2)(\lambda_1 + \lambda_2)), \text{ and the action of} \\ \quad I \text{ is trivial,} \\ H^0(\lambda_1)^{(1)}, \text{ if } n = 3 \text{ and } V \cong \tilde{V}((p-2)\lambda_1 + \lambda_2) \text{ and the} \\ \quad \text{action of } I \text{ is trivial,} \\ H^0(\lambda_2)^{(1)}, \text{ if } n = 3 \text{ and } V \cong \tilde{V}(\lambda_1 + (p-2)\lambda_2) \text{ and the} \\ \quad \text{action of } I \text{ is trivial,} \\ 0, \text{ otherwise.} \end{cases}$$

Finally, we review some results of G. Hochschild [5].

Lemma 3.4. ([5, *Theorem* 2.1]) *Let L be a restricted Lie algebra and Vbe a restricted L-module. Then the canonical homomorphism of $H^1_*(L, V)$ into $H^1(L, V)$ maps $H^1_*(L, V)$ isomorphically onto a subspace of $H^1(L, V)$.*

Lemma 3.5. ([5, *p.* 575]) *Let L and Vbe as in Lemma 3.4. If $V^L = 0$, then*

$$H^1_*(L, V) \cong H^1(L, V).$$

From Corollary 3.1, Lemma 3.4 and Lemma 3.5, we obtain

Theorem 3.2. *Let char $F = p \geqslant n + 2$ for $n = 2,3$ and V an irreducible restricted $W(n,(1,1,\ldots,1))$-module. Then*

$$H^1_*(W(n,\mathbf{m}),V) \cong \begin{cases} F^{n+1}, \text{ if } V \cong M(\lambda_1,n,(1,1,\ldots,1)), \\ F^{(C_i^n)}, \text{ if } V \cong M(\lambda_i,n,\mathbf{m}) \text{ for } i \geqslant 2, \\ F, \text{ if } V \cong \tilde{V}(2\lambda_1+\lambda_{n-1}) \text{ and the action of } I \text{ on } \\ \quad V(2\lambda_1+\lambda_{n-1}) \text{ is the scalar multiplication by } 1, \\ F \oplus F, \text{ if } n = 2 \text{ and } V \cong \tilde{V}(p-2) \text{ and the action of } I \\ \quad \text{is trivial}, \\ F, \text{ if } n = 3 \text{ and } V \cong \tilde{V}((p-2)(\lambda_1+\lambda_2)) \text{ and the action of} \\ \quad I \text{ is trivial}, \\ H^0(\lambda_1)^{(1)}, \text{ if } n = 3 \text{ and } V \cong \tilde{V}((p-2)\lambda_1+\lambda_2) \text{ and the} \\ \quad \text{action of } I \text{ is trivial}, \\ H^0(\lambda_2)^{(1)}, \text{ if } n = 3 \text{ and } V \cong \tilde{V}(\lambda_1+(p-2)\lambda_2) \text{ and the} \\ \quad \text{action of } I \text{ is trivial}, \\ 0, \text{ otherwise.} \end{cases}$$

§ 4. $H^1(H(2,\mathbf{m}),\tilde{V}_0)$ *and* $H^1_*(H(2(1,1)),V)$

Let H be $H(2,\mathbf{m})$ over an algebraically closed field F, char $F = p > 3$. In particular, if $\mathbf{m} = (1,1)$, then $H(2,(1,1))$ is restricted. Thanks to Theorem 1.1 and Theorem 1.2, we have $H^1(H(2,\mathbf{m}),V) = 0$, where the $H(2,\mathbf{m})$-module V is not isomorphic to a graded module or $V \cong \tilde{V}_0$, where the base space V_0 is an irreducible $H_{[0]}$-module but is not a highest weight module. In this section we use the same methods for $W(2,\mathbf{m})$ to determine the structure of $H^1(H(2,\mathbf{m}),\tilde{V}_0)$, where V_0 is an irreducible highest weight module of $H_{[0]}(\cong sp(2) = sl(2))$, the structure of the cohomology groups $H^1(H(2,(1,1)).V)$ where V is an irreducible $H(2,(1,1))$-module and the structure of the restricted cohomology groups $H^1_*(2,(1,1))V)$, where V is an irreducible restricted $H(2(1,1))$-module.

Let char $F = p > 3$, $H = H(2,\mathbf{m})$, $H_{[i]} = H \cap W(2,\mathbf{m})_{[i]}$, and $H_i = \bigoplus_{j \geqslant i} H_{[i]}$. We have $\mathscr{D}(x^{(\alpha)}) = -D_2(x^{(\alpha)})D_1 + D_1(x^{(\alpha)})D_2$ for $\alpha \in A(2,m), \alpha \neq \pi$. If we modify the proof of Lemma 2.1, then it is not difficult to prove

Lemma 4.1. *The relative cohomology $H^*(H,H_{[-1]},\tilde{V}_0)$ is a direct summand of $H^*(H,\tilde{V}_0)$ and $H^*(H,H_{[-1]},\tilde{V}_0) \cong H^*(H_0,V_0)$, where the $H_{[0]}$-module V_0 is the base space of $\tilde{V}_0$ and $H_{[i]}(i > 0)$ acts trivially on V_0.*

Let $V(m)$ denote the $m + 1$ dimensional irreducible restricted $sl(2)$-module. In the case of $H(2,\mathbf{m})$, we have the following lemma which is similar to Lemma 3.1.

Lemma 4.2. (*i*) *If V is a nontrivial irreducible $sl(2)$-module with a highest weight, then*

$$H^1(H,\tilde{V}_0) \cong H^1(H,H_{[-1]},\tilde{V}_0).$$

(*ii*) *If $V_0 = F$, then*

$$H^1(H,\mathfrak{U}) \cong \langle[\beta_1]\rangle \oplus \langle[\beta_2]\rangle \oplus H^1(H,H_{[-1]},\mathfrak{U}),$$

where $\mathfrak{U} = \tilde{V}(0)$, *and* β_1 *and* β_2 *can be defined similary as in* § 3, *Step* 1.

Let V_0 be an irreducible $sl(2)$-module with a highest weight.

Now we shall compute $H^1(H,H_{[-1]},\tilde{V}_0)$. If $V_0 = V(0) = F$, then

$$H^1(H,H_{[-1]},\tilde{V}_0) \cong H^1(H_0,V_0) \cong H_0/[H_0,H_0]. \tag{4.1}$$

By direct computation, we have

$$H_0/[H_0,H_0] \cong \langle \mathscr{D}(x^{(p^{\mu_1},0)}), \mathscr{D}(x^{(0,p^{\mu_2})}) \,|\, \begin{matrix} 0<\mu_1<m_1 \\ 0<\mu_2<m_2 \end{matrix} \rangle. \tag{4.2}$$

Using the cohomology five-term sequence, we have

$$0 \to H^1(sl(2),V_0{}^{H_1}) \to H^1(H_0,V_0) \to H^1(H_1,V_0)^{sl(2)}$$
$$\to H^2(sl(2),V_0{}^{H_1}) \to H^2(H_0,V_0). \tag{4.3}$$

Now we shall compute $H^1(H_1,V_0)^{sl(2)}$ for $V_0 \neq F$. This is equal to

$$Hom_{sl(2)}(H_1/[H_1,H_1],V_0) = \bigoplus_{i\geqslant 1} Hom_{sl(2)}(Y_i,V_0), \tag{4.4}$$

where Y_i is the contribution to $H_1/[H_1,H_1]$ coming from $H_{[i]}$. $H = H(2,\mathbf{m})$ is spanned by $\mathscr{D}(f) := -D_2(f)D_1 + D(f)D_2, f\in\mathfrak{U}$. We have

$$[\mathscr{D}(f),\mathscr{D}(g)] = \mathscr{D}(D_1(f)D_2(g) - D_2(f)D_1(g)). \tag{4.5}$$

By (4.5) and direct computation, we have

$$[H_1,H_1] = \langle \mathscr{D}(x^{(i,j)}) \in H_1 \,|\, i+j>3, (i,j) \neq (p^{\mu_1},0),$$
$$(0,p^{\mu_2}),(p^{\mu_1}+1,0),(0,p^{\mu_2}+1),(p^{\mu_1},1)$$
$$\text{or } (1,p^{\mu_2}), \text{for } 0<\mu_1<m_1, 0<\mu_2<m_2\rangle. \tag{4.6}$$

By (4.4) and (4.6), we can show easily that

$$H^1(H_1,V_0)^{sl(2)} \cong \begin{cases} F^{m_1+m_2-2}, & \text{if } V_0 = V(1), \\ F, & \text{if } V_0 = V(3), \\ 0, & \text{if } V_0 \neq F, V(1) \text{ or } V(3). \end{cases} \tag{4.7}$$

If $V_0 \ncong F$ or $V(p-2)$, then by (4.3), we have

$$H^1(H_0,V_0) \cong H^1(H_1,V_0)^{sl(2)}. \tag{4.8}$$

If $V_0 \cong V(p-2)$ and $p>5$, then by (4.3) and (4.7), we have

$$H^1(H_0,V_0) \cong H^1(sl(2),V_0) \cong F \oplus F. \tag{4.9}$$

If $V_0 \cong V(p-2)$ and $p=5$, then by (4.3) and (4.7), we have

$$2 = dim\, H^1(sl(2),V_0) \leqslant dim\, H^1(H_0,V_0) \leqslant 3.$$

Since $H_{[1]} = \langle \mathscr{D}(x^{(3,0)}), \mathscr{D}(x^{(2,1)}), \mathscr{D}(x^{(1,2)}), \mathscr{D}(x^{(0,3)}) \rangle \cong V(3)$, regarded as $sl(2)$-module, let

$$\psi(l) = \begin{cases} 0, & \text{if } l \in H_{[i]} \text{ for } i > 1, \\ 1, & \text{if } l \in H_{[1]}, \end{cases}$$

then the cohomology class of ψ is in $H^1(H_1, V(3))^{sl(2)}$, by (4.4) and (4.7). Since $\psi|_{H_{[1]}} \in Hom_{sl(2)}(H_{[1]}, V(3))$, we have

$$l_0\,\psi(l_1) = \psi([l_0, l_1]), \text{ for } l_0 \in H_{[0]} \text{ and } l_1 \in H.$$

$\psi \in Z^1(H_1, V(3))$ can be extended to an 1-cocycle on H_0 via $\psi(l_0) = 0$ for $l_0 \in H_{[0]}$. We denote its cohomology class by $[\psi]$. It implies that

$$H^1(H_0, V_0) \cong F \oplus F \oplus \langle[\psi]\rangle. \tag{4.10}$$

Now we have proved

Theorem 4.1. *Suppose char $F = p > 3$. Let V_0 be an irreducible $sl(2)$-module with a highest weight. Then*

$$H^1(H(2,\mathbf{m}), \tilde{V}_0) \cong \begin{cases} \langle[\beta_1]\rangle \oplus \langle[\beta_2]\rangle \oplus \langle \mathcal{D}(x^{(p^{\mu_1},0)}), \mathcal{D}(x^{(0,p^{\mu_2},0)})\ {}_{0<\mu_2<m_2}^{0<\mu_1<m_1} \rangle, & \\ & \text{if } V_0 \cong F \\ F^{m_1+m_2-2}, & \text{if } V_0 = V(1), \\ F, & \text{if } V_0 = V(3) \text{ and } p \neq 5, \\ F \oplus F, & \text{if } V_0 = V(p-2) \text{ and } p > 5, \\ F \oplus F \oplus \langle[\psi]\rangle, & \text{if } V_0 = V(p-2) \text{ and } p = 5, \\ 0, & \text{otherwise.} \end{cases}$$

If V is an irreducible $H(2,(1,1))$-module, then we can also see that $H^1(H(2,(1,1)), V) = 0$, unless $\tilde{V} \cong \tilde{V}(3)$, $V(p-2)$ or $(\tilde{N}_0)_{min}$. We need to compute only the case of $V = (\tilde{N}_0)_{min}$. From Note 1.1 (2), we have $(\tilde{N}_0)_{min} \cong \mathfrak{U}'/F \cdot 1 = \langle \bar{x}^{(i_1, i_2)} | 0 \leqq i_1, i_2 \leqq p-1, (i_1, i_2) \neq (0,0) \text{ or } (p-1, p-1)\rangle$, where $\bar{x}^{(i_1, i_2)} = x^{(i_1, i_2)} + F \cdot 1$. Now we compute $H^1(H, \mathfrak{U}'/F \cdot 1)$.

It is similar to § 3, Step 1 that there is an exact sequence

$$0 \to H^1(H, H_{[-1]}, \mathfrak{U}'/F \cdot 1) \to H^1(H, \mathfrak{U}'/F \cdot 1) \to H^1(H_{[-1]}, \mathfrak{U}'/F \cdot 1) \to 0 \tag{4.11}$$

and

$$H^1(H_{[-1]}, \mathfrak{U}'/F \cdot 1) \cong \langle[\bar{\beta}_1]\rangle \oplus \langle[\bar{\beta}_2]\rangle, \tag{4.12}$$

where

$$\bar{\beta}_1(D_j) = \begin{cases} \bar{x}^{(p-1,0)}, & \text{if } j = 1, \\ 0, & \text{if } j = 2, \end{cases} \text{ and } \bar{\beta}_2(D_j) = \begin{cases} 0, & \text{if } j = 1, \\ \bar{x}^{(0,p-1)}, & \text{if } j = 2. \end{cases}$$

Our next task is to compute $H^1(H, H_{[-1]}, \mathfrak{U}'/F \cdot 1)$. Let $\psi \in Z^1(H, H_{[-1]}, \mathfrak{U}'/F \cdot 1)$. Then we have

$$D_i \cdot \psi(l) = \psi([D_i, l]), \text{ for } l \in H \text{ and } i = 1,2. \tag{4.13}$$

If $l \in H_{[0]}$, then by (4.13), we have $D_i\, \psi(l), i = 1,2$. It implies that $\psi(l) \in \langle \bar{x}^{(1,0)}, \bar{x}^{0,1)} \rangle$ for $l \in H_{[0]}$. Since ψ is 1-cocycle, we can verify easily that

$$\begin{aligned} \psi(x^{(1,0)} D_1 - x^{(0,1)} D_2) &= C_1\, \bar{x}^{(1,0)} - C_2\, \bar{x}^{(0,1)}, \\ \psi(x^{(1,0)} D_2) &= C_2\, \bar{x}^{(1,0)}, \\ \psi(x^{(1,0)} D_1) &= C_1\, \bar{x}^{(0,1)}, \end{aligned} \tag{4.14}$$

for some $C_1, C_2 \in F$. Using (4.13), induction on i shows that

$$\begin{aligned} \psi(\mathscr{D}(x^{(i_1,i_2)})) &= \psi(-x^{(i_1,i_2-1)} D_1 + x^{(i_1-1,i_2)} D_2) \\ &= -\, C_1\, \bar{x}^{(i_1,\, i_2-1)} + C_2\, \bar{x}^{(i_1-1,i_2)}, \end{aligned}$$

for $\mathscr{D}(x^{(i_1,i_2)}) \in H_{[i]}$, *i.e.*, $i_1 + i_2 = i-2$. Let

$$\begin{cases} \psi(\mathscr{D}(x^{(i_1,i_2)})) = \bar{x}^{(i_1,i_2-1)}, \\ \psi(\mathscr{D}(x^{(i_1,\, i_2)})) = \bar{x}^{(i_1-1,i_2)}, \end{cases} \tag{4.15}$$

for $(i_1, i_2) \in A(2, \mathbf{m}), (i_1, i_2) \neq \pi$.
It is easy to check that $\psi_1, \psi_2 \in Z^1(H, H_{[-1]}, \mathfrak{U}'/F \cdot 1)$. Thus we have

$$H^1(H, H_{[-1]}, \mathfrak{U}'/F \cdot 1) = \langle [\psi_1] \rangle \oplus \langle [\psi_2] \rangle.$$

Hence by (4.11) and (4.12) we have

$$H^1(H, \mathfrak{U}'/F \cdot 1) \cong \langle [\bar{\beta}_1] \rangle \oplus \langle [\bar{\beta}_2] \rangle \oplus \langle [\psi_1] \rangle \oplus \langle [\psi_2] \rangle. \tag{4.16}$$

Therefore

Corollary 4.1. *Let char $F = p > 3$ and V an irreducible $H(2,(1,1))$-module. Then*

$$H^1(H(2,(1,1)), V) \cong \begin{cases} F, & \text{if } V \cong \tilde{V}(3) \text{ and } p \neq 5, \\ F \oplus F, & \text{if } V \cong \tilde{V}(p-2) \text{ and } p > 5, \\ F \oplus F \oplus \langle [\psi] \rangle, & \text{if } V \cong \tilde{V}(p-2) \text{ and } p = 5, \\ \langle [\bar{\beta}_1] \rangle \oplus \langle [\bar{\beta}_2] \rangle \oplus \langle [\psi_1] \rangle \oplus \langle [\psi_2] \rangle, & \text{if } V \cong (\tilde{N}_0)_{min}, \\ 0, & \text{otherwise} \end{cases}$$

By Lemma 3.4 and 3.5, we have

Theorem 4.2. *Let char $F = p > 3$ and V an irreducible restricted $H(2(1,1))$-module. Then*

$$H^1_*(H(2,(1,1)), V) \cong \begin{cases} F, & \text{if } V \cong \tilde{V}(3) \text{ and } p \neq 5, \\ F \oplus F, & \text{if } V \cong \tilde{V}(p-2) \text{ and } p > 5, \\ F \oplus F \oplus \langle [\psi] \rangle, & \text{if } V \cong \tilde{V}(p-2) \text{ and } p = 5, \\ \langle [\bar{\beta}_1] \rangle \oplus \langle [\bar{\beta}_2] \rangle \oplus \langle [\psi_1] \rangle \oplus \langle [\psi_2] \rangle, & \text{if } V \cong (\tilde{N}_0)_{min}, \\ 0, & \text{otherwise} \end{cases}$$

Note. $W(2, \mathbf{m})$ *and* $H(2, \mathbf{m})$ are the only rank two graded Lie algebras of depth 1 of Cartan type. $W(3,\mathbf{m})$ and $S(3, \mathbf{m})$ are the only rank three graded Lie algebras of depth 1 of Cartan type. We can also use the above methods to determine the first cohomology for $S(3, \mathbf{m})$.

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