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Weierstraß-Type Representation of Affine Spheres

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Abstract. Affine spheres are discussed in the context of loop groups. We show that every affine sphere can be obtained by solving two ordinary differential equations followed by an application of a generalized Birkhoff Decomposition Theorem (which we proof in the Appendix). A geometric interpretation of the coefficients of the ODE is given. Finally the method is applied to construct all ruled surfaces.

Differential geometry begins with surface theory in $\mathbb{R}^3$. The surrounding Euclidean space provides notions like length, angles, normal and covariant derivative. Surfaces that differ only by a rigid motion are considered essentially identical. The basic notions of surface theory are either invariant or transform appropriately under the group of rigid motions. FELIX KLEIN suggested to consider geometry relative to groups of "equiaffine" transformations, i. e. affine transformations for which the linear part has determinant 1. One then investigates surfaces together with basic (generalized) notions which transform appropriately under equiaffine transformations. There is no obvious "first fundamental form". "Affine normal" needs to be defined. The details can be found in [1], [11], [15]. While it is important to have a consistent general theory, special (classes of) examples exhibit the applicability and the beauty of the field.

Certainly one of the most basic example of surfaces are the spheres. It is therefore not surprising that one of the first examples discussed in the context of KLEIN's program are "affine spheres". The study of affine spheres started with TZITZEICA in 1907. His approach was an Euclidean one and affine spheres arose in connection with tetrahedral surfaces. As it turns out, the class of affine spheres is large. Contributions towards a classification have been made since TZITZEICA's initial study, but a classification is still elusive. A discussion of affine spheres can be found in [1], [11], [15], and in the lecture notes [13].

In recent years a new tool, essentially a derivative of soliton theory, has been used to investigate classes of surfaces and to facilitate computer visualization. It has been successfully employed for surfaces of constant mean curvature and surfaces of constant Gauss curvature. The starting point is the observation that into the moving frame equation one can insert a parameter t so that the compatibility condition for

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the moving frame is satisfied for all values of this continuous parameter if and only if the surface is in the class of surfaces under consideration. Integrating the surface equations thus yields a whole family of surfaces in the class. The corresponding moving frame $F(u, v, t)$, u, v certain local coordinates, depends on the parameter as well. In the case of affine spheres u and v are (possibly imaginary) asymptotic line parameters. This extended frame $F(u, v, t) \in SL(3, \mathbb{R})$ thus has values in the group $\Lambda SL(3, \mathbb{R}) = \{t \to g(t) \in SL(3, \mathbb{R})\}$. In the case of surfaces of constant mean curvature, t varies through the unit circle S^1, in the case of constant Gauss curvature, t is a real number. Considering the frame F as an element of a "loop group" provides new tools and techniques. Important in this context are decomposition theorems for loop groups. They allow one to split the frame into two parts. It turns out that all the information about the frame, and thus the surface, is already contained in one of the two factors. Moreover, this factor is the integral of a differential form that does not need to satisfy any constraints: all the constraints implied by the nonlinear defining equations for the surface class disappear in the splitting. Fortunately, this process can be reversed: starting from an unconstrained differential form one obtains one of the two factors of the frame by solving an ODE. A second splitting theorem produces a frame and the surface is produced from the frame.

In the case of surfaces of constant mean curvature the unconstrained differential form consists of a meromorphic function and a holomorphic function. Thus one obtains Weierstraß data as in the case of minimal surfaces.

In the first section we recall the basic notions of affine surface theory. We have followed here in large parts the lecture notes [13] of PAT RYAN.

In Section 3 we have also used different sources. Section 4 introduces the twisted loop groups associated naturally with affine spheres and states the Generalized Birkhoff Decomposition Theorem (Theorem 4.3), which is of fundamental importance for this paper. The generalized Weierstrass data for the associated family of an affine sphere are exhibited in Section 5, and are called "potentials" (associated with an affine sphere). Section 6 shows that a very simple type of λ-dependence for a Maurer - Cartan form characterizes, modulo gauge transformation, the frames of the associated family (which is parametrized by λ) of an affine sphere.

The main results of this paper are contained in Section 7, showing that there is a bijection between all affine spheres and (normalized) potentials. This implies that all affine spheres can be constructed from normalized potentials by the procedure outlined in this paper. In particular, the normalized potentials parametrize all affine spheres. We would like to make it very clear though that the transition from a normalized potential to an affine sphere is technically complicated because of the required group splitting. As a consequence, the relation between properties of the potential and geometric properties of the corresponding affine sphere is difficult to trace through the group splittings used.

In Section 8 we show that potentials essentially determine two transversal asymptotic lines. Thus the splitting procedure employed in this paper serves to fill in the rest of the surface without solving any further differential equation. So the construction of affine spheres is reduced to solving two ordinary differential equations and the largely algebraic procedure of splitting. This makes the method of this paper well suited for the computer aided visulization of affine spheres.

The last section contains examples and the Appendix the proof of Theorem 4.3.

1 Basic notations and results

Let $\mathcal{F}$ be a real two-dimensional manifold and $f : \mathcal{F} \to \mathbb{R}^3$ an immersion. A *transversal vector field* ξ is a mapping $\xi : \mathcal{F} \to \mathbb{R}^3$ such that for each $x \in \mathcal{F}$

$$\mathbb{R}\xi \oplus f_*(T_x\mathcal{F}) = \mathbb{R}^3. \tag{1.1}$$

In Euclidean geometry the unit normal vector field is a "natural" transversal vector field. There is no natural notion of "normal" in the equiaffine geometry, hence no transversal vector field is preferred.

For vector fields X and Y on $\mathcal{F}$ we split the usual directional derivative D according to (1.1) and obtain

$$D_{f_*X} f_*Y = f_*(\nabla_X Y) + h(X, Y)\,\xi. \tag{1.2}$$

This defines a linear connection ∇ on $\mathcal{F}$ and a symmetric $(0, 2)$-tensor field h, the *affine fundamental form.*

Differentiating ξ similarly, we obtain

$$D_{f_*X}\xi = f_*(-SX) + \tau(x)\,\xi. \tag{1.3}$$

This defines the *shape operator* S as a $(1, 1)$-tensor and the *transversal connection* τ as a 1-form on $\mathcal{F}$.

Finally, the standard volume form ω of $\mathbb{R}^3$ induces a volume form θ on $\mathcal{F}$ via

$$\theta(X, Y) = \omega(f_*(X), f_*(Y), \xi). \tag{1.4}$$

Let

$$SA(3) = \{x \to A\,x + b \mid A \in SL(3, \mathbb{R}),\ b \in \mathbb{R}^3\} \tag{1.5}$$

denote the group of equiaffine transformations on $\mathbb{R}^3$. Then the standard volume element of $\mathbb{R}^3$ is up to a non-zero factor the only volume element on $\mathbb{R}^3$ that is invariant under $SA(3)$. Therefore, from an equiaffine point of view θ is essentially unique (also see Section 2). Recall [11], an *equiaffine structure* on $\mathcal{F}$ is a torsion-free connection together with a parallel volume form. Then we note

Proposition 1.1. *For D, ω, ∇, θ and τ as above,*

$$(D, \omega) \textit{ is an equiaffine structure for } \mathbb{R}^3 \textit{ and } \nabla_X\theta = \tau(x)\,\theta. \tag{1.6}$$

Thus, (∇, θ) is an equiaffine structure on $\mathcal{F}$ if and only if $\tau = 0$.

2 The affine normal field

If ξ and $\hat{\xi}$ both are transversal vector fields for the immersion f of $\mathcal{F}$, then

$$\hat{\xi} = f_*(Z) + \varphi\,\xi, \quad Z \in \Gamma(T\mathcal{F}),$$

and φ is a nowhere vanishing real-valued function. Then for vector fields X and Y we have

$$\hat{h}(X,Y) = \frac{1}{\varphi}\,h(x,y), \tag{2.1}$$

$$\hat{\nabla}_X Y = \nabla_X Y - \frac{1}{\varphi}\,h(X,Y)\,Z, \tag{2.2}$$

$$\hat{\tau}(X) = \tau(X) + \frac{1}{\varphi}\,\left(h(Z,X) + X\varphi\right), \tag{2.3}$$

$$\hat{S}(X) = \varphi S(X) - \nabla_x Z + \hat{\tau}(X) Z, \tag{2.4}$$

$$\hat{\theta} = \varphi\,\theta. \tag{2.5}$$

From (2.1) we see that the rank of h is independent of the choice of the transversal vector field. Therefore it makes sense to define: $f(\mathcal{F})$ is called a *non-degenerate affine surface* if h is non-degenerate for some transversal field ξ [1]. In this case h is a semi-Riemannian metric on $\mathcal{F}$ and therefore has its own Levi-Civita connection ∇^h and its own volume form θ^h, given by

$$\theta^h(X_1, X_2) = \pm\left[\left|\det(h(X_i, X_j))_{i,j=1,2}\right|\right]^{1/2}. \tag{2.6}$$

Setting $k = \det(h(X_i, X_j))_{i,j=1,2}$ for any frame X_1, X_2 of $\mathcal{F}$ we obtain

k attains the same value for all frames satisfying $\theta(X_1, X_2) = 1$. (2.7)

If $\hat{k}$ is defined for $\hat{\xi}$ and $\hat{h}$, then $\hat{k} = \varphi^{-4}\,k$. (2.8)

This shows that the quantities defined above transform in a simple way under a change of the transversal field.

Since we are interested in equiaffine surface theory we need to have also natural transformations under $SA(3)$

Proposition 2.1. *Let $f : \mathcal{F} \to \mathbb{R}^3$ be an immersion and $\alpha \in SA(3)$. Let ξ be a transversal vector field for f. Then $\alpha_* f$ is a transversal vector field for the immersion $\alpha \circ f$. With these choices of transversal vector fields, f and $\alpha \circ f$ induce the same ∇, h, S, τ, and θ on $\mathcal{F}$.*

Finally we note that for non-degenerate affine surfaces one can always choose ξ locally so that the corresponding (∇, θ) defines an equiaffine structure on $\mathcal{F}$.

Proposition 2.2. *Let $f : \mathcal{F} \to \mathbb{R}^3$ be a non-degenerate affine surface. Then for each $p_0 \in \mathcal{F}$ there is a transversal vector field ξ defined in some neighbourhood of p_0 such that $\tau = 0$ and $\theta = \pm\theta^h$. Such a ξ is unique up to sign.*

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[1] Take ξ the unit normal vector from the Euclidean geometry. Then the corresponding h is the second fundamental form. Thus an affine surface is non-degenerate iff the second fundamental form is non-degenerate, i.e. if the Gaussian curvature does not vanish.

A transversal vector field satisfying the conditions of the proposition is called *affine normal field* and the corresponding h is called the associated *affine metric*.

We note that computationally the affine normal can be obtained by the following steps:

$$\text{Start with any transversal vector field } \xi. \tag{2.9}$$

$$\text{Pick vector fields } X_i \text{ on } \mathcal{F} \text{ such that } \theta(X_1, X_2) = 1. \tag{2.10}$$

$$\text{Set } k = \det(h(X_i, X_j)_{i,j=1,2}) \text{ and } \varphi = |k|^{\frac{1}{4}}. \tag{2.11}$$

$$\text{Choose } Z \text{ so that } h(Z, X) = -\varphi\, \tau(X) - X\varphi. \tag{2.12}$$

Then $f_*(Z) + \varphi\, \xi$ is the desired affine normal.

It is easy to see that for a non-degenerate affine surface changing the sign of the affine normal changes the signs of θ, h, and S, but not of ∇. Therefore ∇ is globally defined on $\mathcal{F}$. The triple (∇, h, S) associated with an affine normal is called a *Blaschke structure* for $\mathcal{F}$. A non-degenerate affine surface together with a globally defined affine normal is called a *Blaschke surface.*

Of the two possible choices for θ^h we choose the one agreeing with θ.

Theorem 2.1. *For a Blaschke surface the following identities hold:*

$$R(X,Y)Z = h(Y,Z)S(X) - h(X,Z)S(Y) \qquad \textit{(Gauss equation).} \tag{2.13}$$

$$(\nabla_X h)(Y,Z) = \left(\nabla_Y h\right)(X,Z) \qquad \textit{(Codazzi equation).} \tag{2.14}$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X) \qquad \textit{(Ricci equation).} \tag{2.15}$$

$$\nabla\theta = 0 \; (\iff \tau = 0) \qquad \textit{(equiaffine condition).} \tag{2.16}$$

$$\theta = \theta^h \qquad \textit{(volume condition).} \tag{2.17}$$

Here R denotes the curvature tensor of the connection ∇.

In the Riemannian case important classes of surfaces are defined by conditions for the shape operator. Here we have

Proposition 2.3. *Let $\mathcal{F}$ be a Blaschke surface.*

$$S = 0 \textit{ if and only if } R = 0. \tag{2.18}$$

$$\textit{If } S = \lambda \mathrm{I}, \textit{ then } \lambda \textit{ is constant.} \tag{2.19}$$

Since in affine geometry there is no natural "first fundamental form", additional tensors are needed to enable strong structural information. Such a tensor is the $(0,3)$-tensor C defined by

$$C(X,Y,Z) = (\nabla_X h)(Y,Z). \tag{2.20}$$

Proposition 2.4. *Let $\mathcal{F}$ be a Blaschke surface. Then for all vector fields X, Y, Z on $\mathcal{F}$ we have*

$$\nabla_X Y - \nabla^h_X Y = \nabla_Y X - \nabla^h_Y X. \tag{2.21}$$

$$C \quad \text{is symmetric in all three arguments.} \tag{2.22}$$

$$h(Y, \nabla_Z X - \nabla^h_Z X) = h(X, \nabla_Z Y - \nabla^h_Z Y). \tag{2.23}$$

$$C(X, Y, Z) = -2h(\nabla_X Y - \nabla^h_X Y, Z). \tag{2.24}$$

In a sense C measures the difference between $\nabla_X Y$ and $\nabla^h_X Y$. This cubic form, first introduced by Blaschke, is called the *affine cubic form* of $\mathcal{F}$.

Starting with the unit normal vector field $\xi = N$ as transversal vector field one obtains using (2.1) that the affine metric is a scalar multiple of the Euclidean second fundamental form $II_{\text{Euclidean}}$. More precisely one can check directly

$$h = \frac{II_{\text{Euclidean}}}{2\sqrt[4]{|K|}}; \ K \text{ the Gaussian curvature of the induced metric } f^* <\ ,\ >_{\mathbb{R}^3}. \tag{2.25}$$

Consider the "conormal field"

$$\hat{\xi} = \frac{N}{< N, \xi >}; \ N \text{ the Euclidean normal.} \tag{2.26}$$

Then ∇^h is the connection on $\mathcal{F}$ induced by $\hat{\xi}$. Finally the affine normal ξ fulfills

$$\xi = \tfrac{1}{2}\,\Delta_h(f), \text{ where } \Delta_h \text{ denotes the Beltrami operator induced by } h. \tag{2.27}$$

Moreover, by (2.2) it is clear that the affine metric h is the difference between the Levi-Civita connection of the induced metric and the connection induced by the the transversal field $f_*(Z) + \varphi N$.

Proposition 2.5. *Let $f : \mathcal{F} \to \mathbb{R}^3$ be an immersion. Then the following are equivalent*

1.) *The Gaussian curvature of $\mathcal{F}$ never vanishes,*

$$K(p) \neq 0, \ \textit{for all } p \in \mathcal{F}.$$

2.) *The Blaschke metric is non-degenerate.*

3.) *If $K < 0$ then there exist two different asymptotic directions at each point.*

If any of these conditions is satisfied, then the affine normal can be defined as outlined above.

From now on we assume $K < 0$ [2]. It is convenient to use asymptotic line coordinates u, v . In terms of these coordinates one finds from (2.25) and (2.27) that there exists a functions $\omega, A, B : \mathcal{F} \to \mathbb{R}$, such that

$$h = e^{\omega(u,v)}\,du\,dv, \quad \xi = e^{-\omega(u,v)}\,f_{uv}, \quad C = A du^3 + B dv^3. \tag{2.28}$$

[2] The case $K > 0$ is similar to the approach developed here. The asymptotic line coordinates u and v need to be replaced by complex coordinates z and $\bar{z}$, conformal with respect to the second fundamental form.

Consider the moving frame $\Omega = (f_u, f_v, \xi)^t$. Then the evolution of this frame gives the following linear system:

$$\Omega_u = \begin{pmatrix} \omega_u & A\,e^{-\omega} & 0 \\ 0 & 0 & e^{\omega} \\ -H & A_v\,e^{-2\omega} & 0 \end{pmatrix} \Omega, \qquad \Omega_v = \begin{pmatrix} 0 & 0 & e^{\omega} \\ B\,e^{-\omega} & \omega_v & 0 \\ B_u\,e^{-2\omega} & -H & 0 \end{pmatrix} \Omega \quad (2.29)$$

where the affine mean curvature $H = \frac{1}{2}\,\mathrm{tr}(S)$ is given by

$$H = -e^{-\omega}\,\omega_{uv} - A\,B\,e^{-3\omega}. \quad (2.30)$$

The linear system is compatible iff

$$H_u = e^{-3\omega}\,A\,B_u - e^{-\omega}\left(A_v\,e^{-\omega}\right)_v \quad (2.31)$$

$$H_v = e^{-3\omega}\,A_v\,B - e^{-\omega}\left(B_u\,e^{-\omega}\right)_u. \quad (2.32)$$

As for Euclidean geometry here one also has a "Bonnet" theorem:

Theorem 2.2. (BLASCHKE [1], pp. 137) *Consider a manifold $\mathcal{F}$ admitting an affine metric h and a cubic form C, in local coordinates $u, v : \mathcal{F} \to \mathbb{R}$ satisfying*

$$h = e^{\omega(u,v)}\,du dv, \; e^{\omega(u,v)} > 0; \quad C = Adu^3 + Bdv^3.$$

Assume moreover, the coefficients of h and C satisfy (2.30), (2.31), (2.32). *Then there exists an immersion*

$$f : \hat{\mathcal{F}} \to \mathbb{R}^3, \; \hat{\mathcal{F}} \text{ the universal cover of } \mathcal{F}$$

with induced affine metric h and cubic form C. This immersion is unique up to an equiaffine transformation of $\mathbb{R}^3$.

3 Affine Spheres and their associated families

Working with the shape operator S (see 1.3) is more complicated in the affine case than in the Riemannian case, since S may not be diagonalizable and its eigenvalues may not be real. But things should be more tractable if $S = \kappa\,I$. The Blaschke surfaces with this property are called *affine spheres*. If $S = 0$, they are called *improper affine spheres*, if $S \neq 0$ *proper affine spheres*.

Proposition 3.1. *If $\mathcal{F}$ is an affine sphere, then*

$$\xi(p) = -\kappa\,(f(p) - x_0) \text{ for some } x_0 \in \mathbb{R}^3, \text{ if } \mathcal{F} \text{ is a proper sphere,} \quad (3.1)$$

The vectors $\xi(p) \in \mathbb{R}^3$, $p \in \mathcal{F}$ are parallel to each other. (3.2)

This follows from (1.6), since $\tau = 0$ and $S = \kappa\,I$, κ constant.

Note that (3.1) generalizes a well known property of spheres: all normals meet in one point.

We will restrict our attention to proper affine spheres. Therefore we can assume

$$\xi = -H\,f, \quad (3.3)$$

where $H = \frac{1}{2}\mathrm{tr}(S) : \mathcal{F} \to \mathbb{R}$ is called the *affine mean curvature*.

Remark 3.1. If $S \equiv 0$, equation (1.3) yields that the affine normal ξ is a constant vector. The surfaces are special affine minimal surfaces ($H = 0$). The moving frame equation for these surfaces can be integrated explicitly (see [1], p. 216).

By our assumption, the affine mean curvature is constant and different from zero. We need to study the moving frame equations (2.29) in more detail. *We will assume from now on that $\mathcal{F}$ is a proper affine sphere with $K < 0$ and that we have asymptotic line parameters as global coordinates.*

We normalize the factor $-H$ in (3.3) to be $H = -1$: In view of (2.28) we can scale the surface $\mathcal{F}$ in $\mathbb{R}^3$ so that $H = \pm 1$. Changing $u \to -u$, $v \to v$ if necessary we obtain

$$\text{Without restriction of the generality we will assume } H = -1. \tag{3.4}$$

Also note that (3.3) shows in view of (3.4) that the immersion f is an affine normal. Therefore, in the moving frame equations (2.29) we can replace ξ by f. Thus for the moving frame $(f_u, f_v, f)^t$ we obtain the system of equations

$$\begin{pmatrix} f_u \\ f_v \\ f \end{pmatrix}_u = \begin{pmatrix} \omega_u & A\,e^{-\omega} & 0 \\ 0 & 0 & e^{\omega} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_u \\ f_v \\ f \end{pmatrix}, \tag{3.5}$$

$$\begin{pmatrix} f_u \\ f_v \\ f \end{pmatrix}_v = \begin{pmatrix} 0 & 0 & e^{\omega} \\ B\,e^{-\omega} & \omega_v & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_u \\ f_v \\ f \end{pmatrix}. \tag{3.6}$$

The compatibility conditions (2.30), (2.31), (2.32) consequently are

$$\omega_{uv} = e^{\omega} - A\,B\,e^{-2\omega}. \tag{3.7}$$

$$A_v = B_u = 0. \tag{3.8}$$

In this setting we have (see e.g. [1], §46,p.122 together with p. 211)

$$e^{\omega} = \det(f_u, f_v, f). \tag{3.9}$$

The coefficients A, B occurring in the previous section are the coefficients of the cubic form $C = Adu^3 + Bdv^3$ (see [1]) introduced in (2.20). One can show [1], §46,p. 123

$$A^2 = \det(f_u, f_{uu}, f_{uuu}), \tag{3.10}$$

$$B^2 = -\det(f_v, f_{vv}, f_{vvv}). \tag{3.11}$$

This means geometrically that A^2 and $-B^2$ are the "windings" of the curves $v =$ constant and $u =$ constant respectively.

Away from points, where A or B vanish, one can reparametrize the asymptotic coordinate $u \to h(u)$, $v \to g(v)$ so that locally $A = B = 1$. Geometrically this means that the asymptotic lines are parametrized by affine arc length [14].

Points, where A or B vanish, are planar points of the curves $f(u, v_0)$ or $f(u_0, v)$ respectively.

Actually, if $A = B = 1$, then (3.7) becomes

$$\omega_{uv} = e^{\omega} - e^{-2\omega}. \tag{3.12}$$

This is the *Tzitzeica equation*, named after the Rumanian mathematician G. TZITZEICA, who first found and investigated this equation. His goal was a classification of all tetrahedralian surfaces

$$a\,x^{\frac{2}{3}} + b\,y^{\frac{2}{3}} + c\,z^{\frac{2}{3}} = 1. \tag{3.13}$$

For a more detailed discussion of the connection between affine spheres and tetrahedralian surfaces see [8].

TZITZEICA already noticed that (3.12) admits 1-parameter families of solutions, since $u \to \alpha\, u$, $v \to \frac{1}{\alpha}\, v$ leave the equation (3.7) invariant. Thus we obtain moving frame equations, where the coefficient matrices also depend on the additional real parameter α. The set of surfaces f_α associated with this parameter α form the "associated family" of an affine sphere. Any f_α is obviously also an affine spheres.

To make these coefficient matrices particularly simple we "gauge" the moving frame and replace α by $\lambda = \sqrt[3]{\alpha}$. We set

$$\Phi(\lambda, u, v) = \begin{pmatrix} \lambda^{-1}\, e^{-\omega/2} & 0 & 0 \\ 0 & \lambda\, e^{-\omega/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_u \\ f_v \\ f \end{pmatrix} \tag{3.14}$$

and obtain

$$\Phi_u(\lambda, u, v)\, \Phi(\lambda, u, v)^{-1} = \frac{\omega_u}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & A\, e^{-\omega} & 0 \\ 0 & 0 & e^{\omega/2} \\ e^{\omega/2} & 0 & 0 \end{pmatrix}, \tag{3.15}$$

$$\Phi_v(\lambda, u, v)\, \Phi(\lambda, u, v)^{-1} = \frac{\omega_v}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda^{-1} \begin{pmatrix} 0 & 0 & e^{\omega/2} \\ B\, e^{-\omega} & 0 & 0 \\ 0 & e^{\omega/2} & 0 \end{pmatrix}. \tag{3.16}$$

We call further Φ the *modified frame* of the affine sphere f.

Lemma 3.1. *Let ω be a solution to (3.7) smooth at $u = v = 0$ and satisfying (3.8). Then after scaling the coordinates with constants we can assume $\omega(0, 0) = 0$.*

Proof. Define $\delta = e^{\omega(0,0)/2}$. Then in the new coordinates $s = \delta\, u$, $t = \delta\, v$ we obtain

$$e^{\omega(u,v)}\, du dv\big|_{(u,v)=(0,0)} = ds\, dt\big|_{(s,t)=(0,0)},$$

i.e. the coefficient of the affine metric in the new coordinates

$$\tilde{\omega}(s, t) = \omega(u, v) - 2\, \ln(\delta)$$

satisfies $\tilde{\omega}(0, 0) = 0$. □

4 Twisted loop groups associated with affine spheres

In this section we discuss the dependence of the modified frame Φ on λ in more detail. It will turn out to be occasionally useful for our purpose to extend the orginally real parameter λ so that arguments from complex analysis can be applied. This can be easily done, since in the differential equations considered the occuring λ's can be allowed to vary on most of the complex plane. For geometric evaluations we will always restrict to real λ's.

First we note:

Theorem 4.1. *Every solution* Φ *to* (3.15) *and* (3.16) *with initial condition* $\Phi(\lambda, 0, 0) = \mathrm{I}$ *is analytic in* $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. *Moreover*

$$\overline{\Phi(\bar{\lambda}, u, v)} = \Phi(\lambda, u, v), \tag{4.1}$$

$$Q\,\Phi(\varepsilon\,\lambda, u, v)\,Q^{-1} = \Phi(\lambda, u, v), \tag{4.2}$$

$$T\left[\Phi(-\lambda, u, v)^{-1}\right]^t T = \Phi(\lambda, u, v), \tag{4.3}$$

where

$$\varepsilon = e^{\frac{2\pi i}{3}}, \tag{4.4}$$

$$Q = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.5}$$

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.6}$$

Proof. For the first statement we note that the coefficient matrices in (3.15) and (3.16) are obviously analytic in $\lambda \in \mathbb{C}^*$. Therefore also the solution is analytic in $\lambda \in \mathbb{C}^*$, since the initial condition is analytic in $\lambda \in \mathbb{C}^*$. To verify the equations (4.1) to (4.3) it is sufficient to verify the "corresponding equations" for the coefficient matrices, where these corresponding equations are obtained by differentiating Φ for some additional parameter t at $t = 0$ (see Lemma 4.1 below). □

For the purpose of this paper we need to apply "loop group methods". Usually, for such methods $\lambda \in S^1 = \{\mu \in \mathbb{C} \mid |\mu| = 1\}$. Since the modified frames used in this paper are defined for - and uniquely determined by their restriction to - $\lambda \in S^1$, we do not lose any generality by considering Φ as a map from S^1 to $Sl(3, \mathbb{C})$. For applications to affine spheres we will always extend the matrix functions occurring from S^1 to $\mathbb{C}^*$ and restrict to $\mathbb{R}^* = \mathbb{C}^* \cap \mathbb{R} = \mathbb{R} \setminus \{0\}$.

By $G[\lambda]$ we denote the group of continuous maps $\psi : S^1 \to Sl(3, \mathbb{C})$ satisfying for all $\lambda \in S^1$

$$\overline{\psi(\bar{\lambda})} = \psi(\lambda), \tag{4.7}$$

$$Q\,\psi(\varepsilon\,\lambda)\,Q^{-1} = \psi(\lambda), \tag{4.8}$$

$$T\left[\psi(-\lambda)^{-1}\right]^t T = \psi(\lambda), \tag{4.9}$$

For the Fourier expansion $\Psi(\lambda) = \sum_{k \in \mathbb{Z}} \lambda^k\, \psi_k$ we have $\sum_{k \in \mathbb{Z}} \|\psi_k\| < \infty$. (4.10)

For an arbitrary function $h(\lambda)$ we define a norm by setting

$$\|h\| = \sum_{k\in\mathbb{Z}} |h_k|, \text{ if } h(\lambda) = \sum_{k\in\mathbb{Z}} \lambda^k h_k. \tag{4.11}$$

The space $\mathcal{A}$ of all functions for which (4.11) is finite is a Banach space ("Wiener algebra"). For an arbitrary matrix function $A = (a_{ij}(\lambda))$ we set

$$\|A\| = \max_j \Big\{ \sum_i \|a_{ij}\| \Big\} \tag{4.12}$$

It is straightforward to verify that the set of matrix functions for which (4.12) is finite is a Banach algebra. Moreover,

$$\bigwedge Sl(3,\mathbb{C}) = \{A(\lambda) \mid \det(A) = 1\} \text{ is a Banach Lie group,} \tag{4.13}$$

$$\text{The group } G[\lambda] \text{ is a Banach Lie group.} \tag{4.14}$$

For later use we note the straightforward

Lemma 4.1. *The Lie algebra Lie $G[\lambda]$ consists of the matrix functions*

$$\psi = \sum_{k\in\mathbb{Z}} \lambda^k \psi_k$$

such that

$$\overline{\psi(\bar{\lambda})} = \psi(\lambda), \tag{4.15}$$

$$Q\,\psi(\varepsilon\lambda)\,Q^{-1} = \psi(\lambda), \tag{4.16}$$

$$-T\,\psi(-\lambda)^t\,T = \psi(\lambda), \tag{4.17}$$

$$\operatorname{trace}(\psi(\lambda)) = 0. \tag{4.18}$$

More specifically, the ψ_k are the form

$$\begin{aligned}
\psi_{6k} &= \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & -a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, &\quad \psi_{6k+1} &= \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{23} & 0 & 0 \end{pmatrix}, \\
\psi_{6k+2} &= \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & 0 \\ 0 & -a_{13} & 0 \end{pmatrix}, &\quad \psi_{6k+3} &= \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & -2a_{11} \end{pmatrix}, \\
\psi_{6k+4} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ -a_{23} & 0 & 0 \end{pmatrix}, &\quad \psi_{6k+5} &= \begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{13} & 0 \end{pmatrix}.
\end{aligned} \tag{4.19}$$

Proof. Differentiation of the defining equations for $G[\lambda]$ for an additional parameter t at $t = 0$, $\psi(t = 0, \lambda) = \mathrm{I}$, yields (4.15)-(4.18). Conversely, if $\psi(\lambda)$ satisfies these equations, then $\exp(t\,\psi(\lambda)) \in G[\lambda]$ as is easily verified. □

An essential part of our method is the decomposition of a matrix in $G[\lambda]$ into a product of more special matrices. To this end we introduce the following Banach

Lie subgroups

$$G^{(+)}[\lambda] = \{\psi \in G[\lambda] \mid \psi_0 = \mathrm{I} \text{ and } \psi_k = 0 \text{ if } k < 0\}, \tag{4.20}$$

$$G^{(-)}[\lambda] = \{\psi \in G[\lambda] \mid \psi_0 = \mathrm{I} \text{ and } \psi_k = 0 \text{ if } k > 0\}, \tag{4.21}$$

$$G^{(0)} = \{\psi \in G[\lambda] \mid \psi \text{ diagonal}\}. \tag{4.22}$$

The corresponding Banach Lie algebras are

$$Lie\, G^{(+)}[\lambda] = \{\psi \in Lie\, G[\lambda] \mid \Psi_k = 0 \text{ if } k \leq 0\}, \tag{4.23}$$

$$Lie\, G^{(-)}[\lambda] = \{\psi \in Lie\, G[\lambda] \mid \Psi_k = 0 \text{ if } k \geq 0\}, \tag{4.24}$$

$$Lie\, G^{0} = \{\psi \in Lie\, G[\lambda] \mid \psi \text{ diagonal}\}. \tag{4.25}$$

In the following sections we will generalize the Birkhoff Decomposition Theorem to our situation. To this end we introduce somewhat larger groups

$$G_*^{(+)}[\lambda] = \{\psi \in G[\lambda] \mid \psi_k = 0 \text{ if } k < 0\}, \tag{4.26}$$

$$G_*^{(-)}[\lambda] = \{\psi \in G[\lambda] \mid \psi_k = 0 \text{ if } k > 0\}. \tag{4.27}$$

Similarly we define $\bigwedge^{(\pm)} Sl(3,\mathbb{C})$ and $\bigwedge_*^{(\pm)} Sl(3,\mathbb{C})$. Clearly, the $*$-groups only differ in ψ_0 from the un-starred groups.

First we recall the classical

Theorem 4.2. (Birkhoff Decomposition Theorem)

$$\bigwedge Sl(3,\mathbb{C}) = \bigcup_D \bigwedge\nolimits_*^{(-)} Sl(3,\mathbb{C}) \cdot D \cdot \bigwedge\nolimits_*^{(+)} Sl(3,\mathbb{C}) \tag{4.28}$$

where $D = \mathrm{diag}(\lambda^a, \lambda^b, \lambda^c)$, $a \geq b \geq c$.

$$\bigwedge\nolimits^{(-)} Sl(3,\mathbb{C}) \cdot \bigwedge\nolimits_*^{(+)} Sl(3,\mathbb{C}) \tag{4.29}$$

is open and dense in $\bigwedge Sl(3,\mathbb{C})$*. Moreoevr,* $g_- D\, g_+ = \tilde{g}_- \tilde{D}\, \tilde{g}_+$ *implies* $D = \tilde{D}$, $g_- = \tilde{g}_- v_-$, $g_+ = v_+ \tilde{g}_+$, *where* $D^{-1} v_- D = v_+ \in \bigwedge_*^{(+)} Sl(3,\mathbb{C})$.

Note that in (4.28) we can interchange $+$ and $-$ and in (4.29) we can move the "$*$" to either side. This also holds for the Birkhoff Decomposition Theorem stated below. The proof of this theorem will be indicated in the Appendix.

Theorem 4.3. (Generalized Birkhoff Decomposition Theorem)

$$G[\lambda] = \bigcup_D G_*^{(-)}[\lambda]\, D\, G_*^{(+)}[\lambda] \tag{4.30}$$

where D is one of the matrices

$$D = \mathrm{diag}(c_0\, \lambda^{3k}, \lambda^{-3k}, c_0) \tag{4.31}$$

where k is odd if $c_0 = -1$ *and k is even if* $c_0 = 1$.

$$D = \begin{pmatrix} 0 & c_0\, \lambda^{3k+1} & 0 \\ -\lambda^{-3k-1} & 0 & 0 \\ 0 & 0 & c_0 \end{pmatrix} \tag{4.32}$$

where k is even if $c_0 = 1$ *and k is odd if* $c_0 = -1$. *Moreover, the decomposition is unique in the sense that* $g_- D g_+ = g_-' D' g_+'$ *implies* $D = D'$ *and* $g_-' = g_- v_-$, $g_+' = v_+' g_+$, *where* $D^{-1} v_- D = v_+$.

In addition we have

$$G^{(-)}[\lambda] \cdot G_*^{(+)}[\lambda] \text{ is open and dense in } G[\lambda]. \tag{4.33}$$

$$\text{The multiplication map } G^{(-)}[\lambda] \times G_*^{(+)}[\lambda] \to G^{(-)}[\lambda] G_*^{(+)}[\lambda] \text{ is an analytic diffeomorphism.} \tag{4.34}$$

5 Potentials for affine spheres

We return to the discussion of affine spheres. We recall from Theorem 4.1 that the modified frame $\Phi(\lambda, u, v)$ attains values in $G[\lambda]$ and satisfies the modified moving frame equation (3.15) and (3.16). We also recall that we assume w.r.g. $\Phi(\lambda, 0, 0) = I$.

In view of Theorem 4.3 we would expect that $\Phi(\lambda, u, v)$ is for "most (u, v)" in the "open cell"

$$\Omega = G^{(-)}[\lambda] \cdot G^{(0)} \cdot G^{(+)}[\lambda]. \tag{5.1}$$

To make this precise, we consider the "singular set"

$$\mathcal{S} = \{(u, v) \in \mathbb{D} \mid \Phi(\lambda, u, v) \notin \Omega\}. \tag{5.2}$$

and prove

Proposition 5.1. *If* $f : \mathbb{D} \to \mathbb{R}^3$ *is an affine sphere and if* f *is analytic, then the set* $\mathcal{S}$ *is the set of zeros of a non-constant real analytic function and is nowhere dense in* $\mathbb{D}$.

Proof. We follow the procedure of [5] and consider the representation π of $G[\lambda] \subset \bigwedge Sl(3, \mathbb{C})$ in the group of automorphisms of an infinite dimensional Grassmannain Gr. If τ denotes the canonical section of the det* bundle over Gr, then we consider the map $\varphi(u, v) = \tau(\pi(g(\lambda, u, v) \cdot H_+))$. It is known that $g(\lambda, u, v) \in \Omega$ iff $\varphi(u, v) = 0$. Since $\varphi(0, 0) \neq 0$, φ does not vanish identically and $\mathcal{S}$, as the set of zeros of a real analytic function, is of the type stated. □

Remark 5.1. We would like to point out that the immersion is analytic if the affine metric is definite, since then the defining equation is elliptic [1], §76. In our case the metric is not definite and the defining equation is hyperbolic. Using the standard theorem on the solvability of hyperbolic partial differential equations [2], pp. 319, we obtain, that under certain circumstances the affine sphere can be real analytic, but will not be real analytic in general. Namely, considering a Goursat problem to solve equation (3.7) with initial values on the characteristic rays $u > 0$, $v = 0$ and $v > 0$, $u = 0$ which are continuous but not differentiable on all these rays. The uniqueness theorem of the solution (see [2], pp. 319) yields a solution for the affine metric that is continuous by not differentiable at least on the points of the boundary. The same uniqueness theorem can be used to extend the solution in a small strip of the characteristic rays. This solution is clearly not differentiable on all this strip.

The discussion in the rest of the paper only applies to $(u, v) \in \mathbb{D} \setminus \mathcal{S} \cup \mathcal{S}^*$, where

$$\Omega^* = G^{(+)}[\lambda] \cdot G^0 \cdot G^{(-)}[\lambda] \tag{5.3}$$

$$\mathcal{S}^* = \{(u, v) \in \mathbb{D} \mid \Phi(\lambda, u, v) \neq \Omega^*\}. \tag{5.4}$$

The following is obvious now

Lemma 5.1. *Let Φ be the modified frame of an affine sphere and $(u, v) \in \mathbb{D} \setminus \mathcal{S} \cup \mathcal{S}^*$. Then there exist uniquely determined $V^{(\pm)} \in G^{(\pm)}[\lambda]$ and $L^{(\pm)} \in G^0 \cdot G^{(\pm)}[\lambda]$ such that*

$$\Phi = L^{(+)} V^{(-)} = L^{(-)} V^{(+)}. \tag{5.5}$$

We will frequently write

$$L^{(\pm)} = L_0^{(\pm)} \hat{L}^{(\pm)}, \text{ where } L_0^{(\pm)} \in G^0 \text{ and } \hat{L}^{(\pm)} \in G^{(\pm)}[\lambda]. \tag{5.6}$$

The matrices $V^{(\pm)}$ inherit "moving frame equations" from Φ:

Proposition 5.2. *Let Φ be the modified frame of an affine sphere and let $V^{(\pm)}$ be defined by (5.5) for $(u, v) \in \mathbb{D} \setminus \mathcal{S} \cup \mathcal{S}^*$ Then*

a.) *$V^{(+)}$ does not depend on v and satisfies the following linear system of ordinary differential equations*

$$\frac{d}{du} V^{(+)} \left(V^{(+)}\right)^{-1} = \lambda \left(L_0^{(-)}\right)^{-1} \begin{pmatrix} 0 & A e^{-\omega} & 0 \\ 0 & 0 & e^{\omega/2} \\ e^{\omega/2} & 0 & 0 \end{pmatrix} L_0^{(-)} = \lambda T^{(+)} \tag{5.7}$$

with initial condition $V^{(+)}(u = 0) = \mathrm{I}$.

b.) *$V^{(-)}$ does not depend on u and satisfies the following linear system of ordinary differential equations*

$$\frac{d}{dv} V^{(-)} \left(V^{(-)}\right)^{-1} = \lambda^{-1} \left(L_0^{(+)}\right)^{-1} \begin{pmatrix} 0 & 0 & e^{\omega/2} \\ B e^{-\omega} & 0 & 0 \\ 0 & e^{\omega/2} & 0 \end{pmatrix} L_0^{(+)} = \lambda^{-1} T^{(-)} \tag{5.8}$$

with initial condition $V^{(-)}(v = 0) = \mathrm{I}$.

Proof. Differentiating the second equation in (5.5) and sorting terms yields

$$dV^{(+)} \left(V^{(+)}\right)^{-1} = \left(L^{(-)}\right)^{-1} d\Phi\, \Phi^{-1} L^{(-)} - \left(L^{(-)}\right)^{-1} dL^{(-)}. \tag{5.9}$$

In this equation the coefficient matrix of dv on the left side only involve positive powers of λ, while on the right side we obtain in view of (3.16) only non-positive powers of λ. Therefore $\partial_v V^{(+)} = 0$ and $V^{(+)}$ only depends on u. Next we compare the coefficient matrices at du. On the left side only powers λ^k, $k > 0$, occur. On the right, in view of (3.15), only one term with positive power of λ occurs (namely λ^1). This yields (5.7), proving a.). Part b.) is shown analogously. □

We will call every pair of matrices $T^{(+)}$, $T^{(-)}$ of the form (5.7) and (5.8) *potential* (for an affine sphere).

6 Frames of affine spheres modulo gauge

So far in this paper we have started from an affine sphere and have considered differential equations for the modified frame and other associated objects. In the next sections we will start from associated objects ("potentials", see below) and construct affine spheres. For this we prove the crucial observation

Theorem 6.1. *Let* $(0,0) \in \mathbb{D} \subset \mathbb{R}^2$ *be a domain and* $\hat{U}, \hat{V} : \mathbb{D} \to LieG[\lambda]$ *continuous maps of the form* $\hat{U} = \hat{U}_0 + \lambda \hat{U}_1$, $\hat{V} = \hat{V}_0 + \lambda^{-1} \hat{V}_1$. *We assume that the coefficient functions* $(\hat{U}_1)_{23}$ *and* $(\hat{V}_1)_{13}$ *never vanish on* $\mathbb{D}$. *Assume moreover that the differential equation*

$$d\hat{\Phi}\,\hat{\Phi}^{-1} = \left\{\left(\hat{U}_0 + \lambda\,\hat{U}_1\right)du + \left(\hat{V}_0 + \lambda^{-1}\,\hat{V}_1\right)dv\right\} \tag{6.1}$$

with initial condition $\hat{\Phi}(0,0) = \mathrm{I}$ *is solvable, then there exists a "gauge"* $C \in G^{(0)}$, *uniquely determined up to sign, such that* $\Phi = C\,\hat{\Phi}\,C(0,0)^{-1}$ *is the modified frame of an affine sphere satisfying also* $\Phi(\lambda, 0, 0) = \mathrm{I}$.

Proof. We note that due to the assumption that $\hat{U}$ and $\hat{V}$ take values in the Lie algebra $LieG[\lambda]$ we have in $\hat{U}$ and $\hat{V}$ the same distribution of zeros as in (3.15) and (3.16). We want to obtain $(\hat{U}_1)_{23} = (\hat{V}_1)_{13} > 0$. The sign can be adjusted, if necessary, by changing $u \to -u$ and / or $v \to -v$. Introducing a gauge C as stated in the claim yields new maps $U, V : \mathbb{D} \to LieG[\lambda]$ where $U = \partial_u C \cdot C^{-1} + C\,\hat{U}\,C^{-1}$ and $V = \partial_v C \cdot C^{-1} + C\,\hat{V}\,C^{-1}$. If $C = \mathrm{diag}(r, \frac{1}{r}, 1) \in G^{(0)}$, then $(V_1)_{13} = r(\hat{V}_1)_{13}$ and $(U_1)_{23} = r^{-1}(\hat{U}_1)_{23}$, These two coefficients are the same iff $r^2 = (\hat{U}_1)_{23}\left[(\hat{V}_1)_{13}\right]^{-1}$. By our assumption, this equation can be solved with some positive r.

Now we set $e^{\omega/2} = (U_1)_{23}$, $A = (U_1)_{12} \cdot ((U_1)_{23})^2$ and $B = (V_1)_{21} \cdot ((V_1)_{13})^2$. Then U_1 and V_1 are of the form (3.15) and (3.16). Note also that $U, V \in LieG[\lambda]$ implies, that U_0 and V_0 are of the form $\alpha_0 \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$ and $\beta_0 \begin{pmatrix} -1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$ respectively. The compatibility condition for the solvability of (6.1), i.e. for $\partial_u \Phi = U\,\Phi$, $\partial_v \Phi = V\,\Phi$ is $\partial_v U - \partial_u V = -[U, V]$. Collecting terms at like powers of λ this is equivalent with $-\partial_u V_1 = -[U_0, V_1]$, $\partial_v U_0 - \partial_u V_0 = -[U_1, V_1]$ and $\partial_v U_1 = -[U_1, V_0]$. In terms of the matrix coefficients the first and the third equation are equivalent with

$$\alpha_0 = \frac{\omega_u}{2}, \quad \partial_u\left(B\,e^{-\omega}\right) = -2\,\alpha_0 B\,e^{-\omega} \tag{6.2}$$

$$\beta_0 = \frac{\omega_v}{2}, \quad \partial_v\left(A\,e^{-\omega}\right) = -2\,\beta_0 A\,e^{-\omega}. \tag{6.3}$$

Note that this fixes U_0 and V_0 in the form required by (3.15) and (3.16). The remaining conditions are equivalent with $\partial_u B = 0$ and $\partial_v A = 0$. The remaining matrix equation is now equivalent with $\omega_{uv} = -A\,B\,e^{-2\omega} - e^{\omega}$, i.e. with (3.7). □

Remark 6.1. The definition (3.14) of Φ shows that the affine sphere (immersion) is given by the last row of Φ. Since C does not effect this last row, the immersion f can be read of already from $\hat{\Phi}$.

As a corollary to the proof above we obtain

Corollary 6.1. *We retain the assumptions of the Theorem above and denote the rows of $\hat{\Phi}$ by $\hat{\Phi}_j$. Then $C = \mathrm{I}$ iff*

$$\left(\hat{\Phi}_3\right)_u = \lambda\, \alpha\, \hat{\Phi}_1 \text{ and } \left(\hat{\Phi}_3\right)_v = \lambda^{-1}\, \alpha\, \hat{\Phi}_2 \tag{6.4}$$

for some positive function α.

Proof. Taking into account the form of the matrices $\hat{U}_j$, $\hat{V}_j$ we see

$$\left(\hat{\Phi}_3\right)_u = \lambda \left(\hat{U}_1\right)_{23} \hat{\Phi}_1 \quad \text{and} \quad \left(\hat{\Phi}_3\right)_v = \lambda^{-1} \left(\hat{V}_1\right)_{13} \hat{\Phi}_2.$$

This implies the claim. □

Remark 6.2. If the Corollary applies, then $\alpha = e^{\omega/2}$. Moreover, if the coefficients in (6.4) are α and β respectively (instead of α), then the gauge $C = \operatorname{diag}(r, r^{-1}, 1)$ yields $\tilde{\alpha} = \alpha\, r = \beta\, r^{-1} = e^{\omega/2}$, whence

$$\alpha\, \beta = e^{\omega}. \tag{6.5}$$

7 Weierstraß representation for affine spheres

Using the results of Section 6 we are able to reverse the construction outlined in Section 5, i.e. we will construct affine spheres from given potentials. This is the crucial result for this paper.

Theorem 7.1. *Let $\alpha^{(+)}$, $\beta^{(+)}$ be functions of $u \in \mathbb{D}_u$ and $\alpha^{(-)}$, $\beta^{(-)}$ functions of $v \in \mathbb{D}_v$, such that $\alpha^{(+)}$ and $\alpha^{(-)}$ never vanish. We form the matrix functions*

$$T^{(+)}(u) = \begin{pmatrix} 0 & \beta^{(+)}(u) & 0 \\ 0 & 0 & \alpha^{(+)} \\ \alpha^{(+)} & 0 & 0 \end{pmatrix}, \tag{7.1}$$

$$T^{(-)}(v) = \begin{pmatrix} 0 & 0 & \alpha^{(-)} \\ \beta^{(-)} & 0 & 0 \\ 0 & \alpha^{(-)} & 0 \end{pmatrix} \tag{7.2}$$

and consider the systems of ordinary differential equations

$$\tfrac{d}{du} \hat{V}^{(+)} = \lambda\, T^{(+)}\, \hat{V}^{(+)} \tag{7.3}$$

$$\tfrac{d}{dv} \hat{V}^{(-)} = \lambda^{-1}\, T^{(-)}\, \hat{V}^{(-)} \tag{7.4}$$

with initial conditions $\hat{V}^{(+)}(u=0) = \mathrm{I}$ and $\hat{V}^{(-)}(v=0) = \mathrm{I}$.

Then $\hat{V}^{(+)} \in G^{(+)}[\lambda]$, $\hat{V}^{(-)} \in G^{(-)}[\lambda]$ and, after changing $u \to -u$, $v \to -v$ if necessary, on the set

$$\mathbb{D} = \left\{(u, v) \in \mathbb{D}_u \times \mathbb{D}_v \;\middle|\; \hat{V}^{(-)}(v) \left(\hat{V}^{(+)}(u)\right)^{-1} \in G_*^{(+)}[\lambda] \cdot G_*^{(-)}[\lambda]\right\}$$

we can find differentiable solutions $\hat{L}^{(\pm)} \in G_*^{(\pm)}[\lambda]$ such that

$$\left(\hat{L}^{(+)}\right)^{-1} \hat{L}^{(-)} = \hat{V}^{(-)} \left(\hat{V}^{(+)}\right)^{-1} \tag{7.5}$$

$$\hat{\Phi} = \hat{L}^{(-)}\, \hat{V}^{(+)} = \hat{L}^{(+)}\, \hat{V}^{(-)} \tag{7.6}$$

satisfies (6.1). In particular, there exists a gauge $C \in G^{(0)}$*, such that with* $C_0 = C(0,0)$, $\Phi = C\,\hat{\Phi}\,C_0^{-1}$ *is the frame of an affine sphere. The matrices* $L^{(\pm)}$ *and* $V^{(\pm)}$*of (5.5) then are* $L^{(\pm)} = C\,\hat{L}^{(\pm)}\,C_0^{-1}$, $V^{(\pm)} = C_0\,\hat{V}^{(\pm)}\,C^{-1}$.

Proof. On $\mathbb{D}$ we split $\left(\hat{L}^{(+)}\right)^{-1}\hat{L}^{(-)} = V^{(-)}\left(V^{(+)}\right)^{-1}$ with $\hat{L}^{(+)} \in G^{(+)}[\lambda]$. This splitting is unique and produces differentiable $\hat{L}^{(\pm)}$ by Theorem 4.3, (4.34). We set $\hat{\Phi} = \hat{L}^{(-)}\,\hat{V}^{(+)} = \hat{L}^{(+)}\,\hat{V}^{(-)}$ and obtain

$$d\hat{\Phi}\cdot\hat{\Phi}^{-1} = d\hat{L}^{(-)}\cdot\left(\hat{L}^{(-)}\right)^{-1} + \hat{L}^{(-)}\,dV^{(+)}\left(V^{(+)}\right)^{-1}\left(L^{(-)}\right)^{-1} \tag{7.7}$$

$$d\hat{\Phi}\cdot\hat{\Phi}^{-1} = d\hat{L}^{(+)}\cdot\left(\hat{L}^{(+)}\right)^{-1} + \hat{L}^{(+)}\,dV^{(-)}\left(V^{(-)}\right)^{-1}\left(L^{(+)}\right)^{-1}. \tag{7.8}$$

Comparing these two equations we see that (7.7) contains only powers λ^k, $k \le 1$, while (7.8) contains only powers λ^r, $r \ge -1$. Therefore, $d\hat{\Phi}\cdot\hat{\Phi}^{-1}$ is of the form

$$d\hat{\Phi}\cdot\hat{\Phi}^{-1} = \left(\lambda\,\hat{U}_1 + \hat{U}_0\right)du + \left(\lambda^{-1}\,\hat{V}_1 + \hat{V}_0\right)dv \tag{7.9}$$

with $\hat{U} = \lambda\,\hat{U}_1 + \hat{U}_0$, $\hat{V} = \lambda^{-1}\,\hat{V}_1 + \hat{V}_0 \in LieG[\lambda]$.

Moreover, we note that due to our assumptions the coefficient $(\hat{V}_1)_{13}$ and the coefficient $(\hat{U})_{23}$ never vanish on $\mathbb{D}$. Therefore we can apply Theorem 6.1 and after using a gauge $C \in G^{(0)}$, if necessary, $\Phi = C\,\hat{\Phi}\,C(0,0)^{-1}$ is the modified frame of an affine sphere. In view of (7.6) we set $L^{(\pm)} = C\,\hat{L}^{(\pm)}\,C(0,0)^{-1}$ and $V^{(\pm)} = C(0,0)\,\hat{V}^{(\pm)}\,C^{-1}$ for the matrix functions defining Φ as in (5.5). □

Remark 7.1. The proof constructs an affine sphere in a unique fashion from $T^{(\pm)}$. More generally, an affine sphere is produced by two potentials $T^{(\pm)}$ and $\tilde{T}^{(\pm)}$ only, if the corresponding $\hat{\Phi}$ and $\hat{\tilde{\Phi}}$ only differ by a gauge. But this is equivalent with $\tilde{T}^{(\pm)} = C_0\,T^{(\pm)}\,C_0^{-1}$ for some constant diagonal matrix C_0.

Conversely we have

Proposition 7.1. *Let* $T^{(\pm)}$ *and* $\tilde{T}^{(\pm)}$ *satisfy the assumptions of the Theorem above. Assume that there exists some constant matrix* $C_0 \in G^{(0)}$ *such that*

$$\tilde{T}^{(+)} = C_0\,T^{(+)}\,C_0^{-1} \quad \textit{and} \quad \tilde{T}^{(-)} = C_0\,T^{(-)}\,C_0^{-1}.$$

Then the potentials $T^{(\pm)}$ *and* $\tilde{T}^{(\pm)}$ *yield the same affine sphere.*

Proof. We follow the proof of the Theorem above. First we obtain $\hat{\tilde{V}}^{(\pm)} = C_0\,\hat{V}^{(\pm)}\,C_0^{-1}$ and then $R = \hat{V}^{(-)}\left(\hat{V}^{(+)}\right)^{-1}$ relates to the analogous $\tilde{R}$ the same way: $\tilde{R} = C_0\,R\,C_0^{-1}$. Then the choices for the L's yield necessarily $\hat{\tilde{L}}^{(+)} = C_0\,\hat{L}^{(+)}\,C_0^{-1}$ and consequently $\hat{\tilde{L}}^{(-)} = C_0\,\hat{L}^{(-)}\,C_0^{-1}$. Therefore, $\hat{\tilde{\Phi}} = C_0\,\hat{\Phi}\,C_0^{-1}$. The choice of gauge used to obtain the modified frame of an affine sphere is as explained in the proof of Theorem 6.1. This shows that the choices of gauge $\tilde{C}$ and C for $\hat{\tilde{\Phi}}$ and $\hat{\Phi}$ is so that $\tilde{C} = C\,C_0^{-1}$ holds. But then $\tilde{C}(0,0) = C(0,0)\,C_0^{-1}$ and $\tilde{\Phi} = \tilde{C}\,\hat{\tilde{\Phi}}\,\tilde{C}(0,0)^{-1} = C\,C_0^{-1}\,C_0\hat{\Phi}\,C_0^{-1}\,C_0\,C(0,0)^{-1} = C\,\hat{\Phi}\,C(0,0)^{-1} = \Phi$. □

Corollary 7.1. *There is a bijection between the family of affine spheres and the potentials $T^{(+)}$, $T^{(-)}$ for which $\alpha^{(+)}(0) = \alpha^{(-)}(0) > 0$.*

Remark 7.2. Potentials of this type will be called *normalized* potentials.

Proof of Corollary 7.1. Since every normalized potential produces an affine sphere, we have a well-defined map φ from normalized potentials to affine spheres. Let Φ be the modified frame of an affine sphere and $\hat{T}^{(+)}$, $\hat{T}^{(-)}$ potentials for Φ. Then after conjugation with some $C_0 \in G^{(0)}$ we can assume $\alpha^{(+)}(0) = \alpha^{(-)}(0) > 0$. In view of Proposition 7.1 the conjugated potential yields the same affine sphere Φ. Thus φ is surjective. Finally, let $\hat{T}^{(\pm)}$, $\hat{\tilde{T}}^{(\pm)}$ be two normalized potentials producing the same affine sphere. then, as outlined in Remark 7.1, the two potentials are conjugate by some $C_0 \in G^{(0}$. Since $\hat{T}^{(\pm)}$ and $\hat{\tilde{T}}^{(\pm)}$ are normalized, $C_0 = I$ follows. Therefore φ is also injective. □

8 Geometric interpretation for the coefficients of $T^{(\pm)}$

The discussion so far has shown that for every affine sphere we obtain matrices $\lambda^{\pm} T^{(\pm)}$ of the form (5.7) and (5.8). Conversely, Theorem 7.1 shows how from such matrices we can construct an affine sphere.

Theorem 8.1. *Let Φ be the modified frame of an affine sphere defined on $\mathbb{D} = \mathbb{D}_u \times \mathbb{D}_v$, satisfying $\Phi(\lambda, 0, 0) = I$. Then the coefficient matrices $T^{(+)}$ and $T^{(-)}$ in (5.7) and (5.8) respectively, obtained from $V^{(+)}$ and $V^{(-)}$ defined by (5.5) have the form of (7.1) and (7.2), where*

$$\alpha^{(+)}(u) = e^{\omega(u,0)-\frac{1}{2}\omega(0,0)} \tag{8.1}$$

$$\alpha^{(-)}(v) = e^{\omega(0,v)-\frac{1}{2}\omega(0,0)} \tag{8.2}$$

$$\beta^{(+)}(u) = A(u)\, e^{-2\,\omega(u,0)+\omega(0,0)} \tag{8.3}$$

$$\beta^{(-)}(v) = B(v)\, e^{-2\,\omega(0,v)+\omega(0,0)}. \tag{8.4}$$

Proof. Let $L^{(\pm)}$, $V^{(\pm)}$ be defined by (5.5). Then

$$L^{(+)}(u,v)^{-1}\,\Phi(u,v) = V^{(-)}(u,v)$$

and

$$L^{(-)}(u,v)^{-1}\,\Phi(u,v) = V^{(+)}(u,v).$$

Since $\mathbb{D}$ is a product, we can put $u = 0$ and $v = 0$ respectively, obtaining

$$L^{(+)}(0,v)^{-1}\,\Phi(0,v) = V^{(-)}(v) \tag{8.5}$$

$$L^{(-)}(u,0)^{-1}\,\Phi(u,0) = V^{(+)}(u). \tag{8.6}$$

If u is fixed in (3.16), e.g. if $u = 0$, then one obtains an ordinary differential equation for $X(v,\lambda) = \Phi(\lambda, 0, v)$ with initial condition $X(0,\lambda) = I$. Hence $X(v,\lambda)$ contains only the powers λ^k, $k \le 0$. Therefore, in (8.5) the matrix $L^{(+)}(0,v) =$

$L_0^{(+)}(0, v)$ is diagonal and independent of λ. With $D(v) = L^{(+)}(0, v)$ and $X(v, \lambda) = \Phi(\lambda, 0, v)$ we then obtain from (8.5) by differentiation

$$\tfrac{d}{dv}V^{(-)} \cdot \left(V^{(-)}\right)^{-1} = -D^{-1}\, \tfrac{d}{dv}D + D^{-1}\, \tfrac{d}{dv}X \cdot X^{-1}\, D. \tag{8.7}$$

With $D = \mathrm{diag}(\theta, \theta^{-1}, 1)$ this yields

$$\frac{\theta_v}{\theta} = -\frac{\omega_v}{2}, \tag{8.8}$$

$$\beta^{(-)}(v) = \theta(v)^2 \cdot B(v)\, e^{-\omega(0,v)}, \tag{8.9}$$

$$\alpha^{(-)}(v) = \frac{e^{\frac{\omega(0,v)}{2}}}{\theta(v)}. \tag{8.10}$$

Since $L_0^{(+)}(0, 0) = \mathrm{I}$, we integrate $\int_0^v \tfrac{d}{dv}\ln\theta\, dv = -\tfrac{1}{2}\int_0^v \partial_v\omega\, dv$ and obtain

$$\ln\theta(v) = -\tfrac{1}{2}\left(\omega(0, v) - \omega(0, 0)\right).$$

Therefore

$$\theta(v) = e^{-\frac{1}{2}(\omega(0,v)-\omega(0,0))}, \tag{8.11}$$

$$\beta^{(-)}(v) = B(v)\, e^{-2\,\omega(0,v)+\omega(0,0)}, \tag{8.12}$$

$$\alpha^{(-)}(v) = e^{\omega(0,v)-\frac{1}{2}\omega(0,0)}. \tag{8.13}$$

Similarly, from (8.6) we obtain in view of (3.15) $L^{(-)}(u, 0) = L_0^{(-)}(u, 0) = S(u) = \mathrm{diag}(\vartheta, \vartheta^{-1}, 1)$ and the equation

$$\tfrac{d}{du}V^{(+)} \cdot \left(V^{(+)}\right)^{-1} = -S^{-1}\, \tfrac{d}{du}S + S^{-1}\, \tfrac{d}{du}\Phi(u, 0)\, \left(\Phi(u, 0)\right)^{-1}\, S. \tag{8.14}$$

This is equivalent with

$$\frac{\vartheta_u}{\vartheta} = \frac{\omega_u}{2}, \tag{8.15}$$

$$\beta^{(+)}(u) = \vartheta^{-2}\, A(u)\, e^{-\omega(u,0)}, \tag{8.16}$$

$$\alpha^{(+)} = \vartheta(u)\, e^{\frac{\omega(u,0)}{2}}. \tag{8.17}$$

As above this yields

$$\vartheta(u) = e^{\frac{1}{2}(\omega(u,0)-\omega(0,0))}, \tag{8.18}$$

$$\beta^{(+)}(u) = A(u)\, e^{-2\,\omega(u,0)+\omega(0,0)}, \tag{8.19}$$

$$\alpha^{(+)}(u) = e^{\omega(u,0)-\frac{1}{2}\omega(0,0)}. \tag{8.20}$$

This finally gives the claim. □

The definition of a normalized potential together with (8.20) yields

Corollary 8.1. *If $\left(T^{(+)}, T^{(-)}\right)$ is a normalized potential, then $\alpha^{(+)}(0) = \alpha^{(-)}(0) = e^{\frac{1}{2}\omega(0,0)}$ for the associated affine sphere.*

Corollary 8.2. *Let $T^{(+)}$, $T^{(-)}$ be any special potential for an affine sphere. Then $\omega(0,0) = 0$ iff $\alpha^{(+)}(0) = \alpha^{(-)}(0) = 1$.*

The second comment gives some geometric meaning to the special potentials for affine spheres.

Proposition 8.1. *Let $(T^{(+)}, T^{(-)})$ be a normalized potential for an affine sphere and Φ the associated modified frame. Consider the matrices $V^{(\pm)} = (v_{ij}^{(\pm)})$ satisfying $\frac{d}{du}V^{(+)} = \lambda T^{(+)}V^{(+)}$, $\frac{d}{dv}V^{(-)} = \lambda^{-1}T^{(-)}V^{(-)}$. Then*

$$\gamma^{(+)}(u) = \begin{pmatrix} v_{31}^{(+)}(u) \\ v_{32}^{(+)}(u) \\ v_{32}^{(+)}(u) \end{pmatrix}, \tag{8.21}$$

$$\gamma^{(-)}(v) = \begin{pmatrix} v_{31}^{(-)}(v) \\ v_{32}^{(-)}(v) \\ v_{32}^{(-)}(v) \end{pmatrix}, \tag{8.22}$$

are the asymptotic lines on the affine sphere corresponding to $v = 0$ and $u = 0$ respectively.

Proof. Using (8.11) and (8.18) respectively one finds

$$L^{(+)}(0,v) = D = \begin{pmatrix} e^{-\frac{1}{2}(\omega(0,v)-\omega(0,0))} & 0 & \\ 0 & e^{\frac{1}{2}(\omega(0,v)-\omega(0,0))} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$L^{(-)}(u,0) = S = \begin{pmatrix} e^{\frac{1}{2}(\omega(u,0)-\omega(0,0))} & 0 & \\ 0 & e^{-\frac{1}{2}(\omega(u,0)-\omega(0,0))} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now usig $(\Phi_{31}, \Phi_{32}, \Phi_{33})^t = f$ in (8.5) and (8.6) respectily yields the claim. □

Remark 8.1. By the Proposition, giving a potential fixes two transversal asymptotic lines. The splitting procedure leading to Φ then "fills in" the rest of the surface.

From the point of view of the differential equation (3.7) the potential gives essentially the boundary values along the curves $u = 0$ and $v = 0$. This is made more explicit by Theorem 8.1.

9 Examples

Even though the results of the previous sections give a 1-1-relation between special potentials and affine spheres, making this relation explicit is a different matter.

We start with the easiest case from the potential side point of view. We would like to emphasize that all the surfaces considered in this section are assumed to be affine spheres.

9.1 A = B = 0. Assume that for a normalized potential $(T^{(+)}, T^{(-)})$ we have

$$\text{Assume } A = B = 0. \tag{9.1}$$

In addition, by Lemma 3.1, we can assume w. r. g. that $\omega(0,0) = 0$ holds.

Form [14] we know that these are ruled surfaces and are defined by quadratic equations. We will obtain this result by following the procedure outlined in (the proof of) Theorem 7.1.

We start from the normalized potential $(T^{(+)}, T^{(-)})$, which due to Theorem 8.1, is of the form

$$T^{(+)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{\omega(u,0)} \\ e^{\omega(u,0)} & 0 & 0 \end{pmatrix} = e^{\omega(u,0)} A_1, \tag{9.2}$$

$$T^{(-)} = \begin{pmatrix} 0 & 0 & e^{\omega(0,v)} \\ 0 & 0 & 0 \\ 0 & e^{\omega(0,v)} & 0 \end{pmatrix} = e^{\omega(0,v)} B_1. \tag{9.3}$$

We note

$$T^{(+)} \text{ and } T^{(-)} \text{ are nilpotent of order three.} \tag{9.4}$$

Setting

$$\xi(u) = \int_0^u e^{\omega(s,0)}\, ds, \tag{9.5}$$

$$\zeta(v) = \int_0^v e^{\omega(0,t)}\, dt, \tag{9.6}$$

we thus obtain for $\hat{V}^{(\pm)}$

$$\hat{V}^{(+)}(u) = \exp\big(\xi(u)\,\lambda\, A_1\big), \tag{9.7}$$

$$\hat{V}^{(-)}(v) = \exp\big(\zeta(v)\,\lambda^{-1}\, B_1\big). \tag{9.8}$$

Following the construction of Theorem 7.1 we need to consider $\hat{V}^{(-)}\left(\hat{V}^{(+)}\right)^{-1} = L$. Clearly, L does only contain powers λ^k for $-2 \le k \le 2$. Therefore, in

$$\left(\hat{L}^{(+)}\right)^{-1} \hat{L}^{(-)} = L \tag{9.9}$$

the two factors only contain $\lambda^0, \lambda^1, \lambda^2$ and $\lambda^0, \lambda^{-1}, \lambda^{-2}$ respectively. An Ansatz

$$\hat{L}^{(+)} = \mathrm{I} + \lambda \begin{pmatrix} 0 & \alpha_1 & 0 \\ 0 & 0 & \beta_1 \\ \beta_1 & 0 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 0 & 0 \\ \gamma_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and similar for $\hat{L}^{(-)} = D\,(\mathrm{I} + \lambda^{-1}U + \lambda^{-2}V)$ with a diagonal matrix D yields

$$\hat{L}^{(+)} = \exp\left\{\frac{2\,\lambda\,\xi(u)}{2-\xi(u)\,\zeta(v)}\,A_1\right\}, \tag{9.10}$$

$$\hat{L}^{(-)} = \exp\left\{\frac{2\,\lambda^{-1}\,\zeta(v)}{2-\xi(u)\,\zeta(v)}\,B_1\right\}, \tag{9.11}$$

$$D = \mathrm{diag}\big(\tfrac{1}{4}\,(2-\xi(u)\,\zeta(v))^2, 4\,(2-\xi(u)\,\zeta(v))^{-2}, 1\big). \tag{9.12}$$

For $\hat{\Phi} = \hat{L}^{(+)}\,\hat{V}^{(-)} = \hat{L}^{(-)}\,\hat{V}^{(+)}$ we thus obtain

$$\Phi(\lambda, u, v) =$$

$$C\begin{pmatrix} 1 & \dfrac{\zeta^2(v)}{2\,\lambda^2} & \dfrac{\zeta(v)}{\lambda} \\ \dfrac{2\,\xi(u)^2\,\lambda^2}{(2-\xi(u)\,\zeta(v))^2} & \dfrac{4}{(2-\xi(u)\,\zeta(v))^2} & \dfrac{4\,\xi(u)\,\lambda}{(2-\xi(u)\,\zeta(v))^2} \\ \dfrac{2\,\xi(u)\,\lambda}{2-\xi(u)\,\zeta(v)} & \dfrac{2\,\zeta(v)}{\lambda\,(2-\xi(u)\,\zeta(v))} & \dfrac{2+\xi(u)\,\zeta(v)}{2-\xi(u)\,\zeta(v)} \end{pmatrix} C^{-1}(0,0), \tag{9.13}$$

where

$$C = \mathrm{diag}\big(r, \tfrac{1}{r}, 1\big), \quad r = 2\,\frac{e^{\frac{1}{2}(\omega(u,0)-\omega(0.v))}}{2-\xi(u)\,\zeta(v)}. \tag{9.14}$$

Recall (see also Remark 6.1) that the definition (3.14) of Φ implies that the affine sphere (immersion) is the last row of Φ

$$f(u,v) = \frac{1}{2-\xi(u)\,\zeta(v)}\begin{pmatrix} 2\,\lambda\,\xi(u) \\ 2\,\lambda^{-1}\,\zeta(v) \\ 2+\xi(u)\,\zeta(v) \end{pmatrix} \tag{9.15}$$

It turns out that the surface is a quadric, given by the equation

$$z^2 = 2\,x\,y + 1. \tag{9.16}$$

We can use $f(u, v)$ to compute the general solution of the Tzitzeica equation for $A = 0$ or $B = 0$. It is

$$e^{\omega(u,v)} = 4\,\frac{\xi'(u)\,\zeta'(v)}{(2-\xi(u)\,\zeta(v))^2}, \tag{9.17}$$

where ξ' and ζ' denote the derivatives of ξ and ζ respectively.

9.2 General ruled surfaces. General ruled surfaces are defined by

$$A \cdot B \equiv 0 \text{ on all the surface.}$$

In this case the Blaschke metric is still of the form (9.17). We now discuss the case when only one of the differentials Adu^3 or Bdv^3 vanishes identically. W.l.o.g. we assume $A \equiv 0$.

Let $f : M^2 \to \mathbb{R}^3$ be an immersion of such a ruled surface. Then (3.5) imply $e^{-\omega} f_u = \gamma(v)$. Another differentation of this identity with respect to v compared with (3.5) yields the general formula for ruled surfaces

$$f(u,v) = \omega_u(u,v)\,\gamma(v) + \gamma'(v) \tag{9.18}$$

which was first found by RADON [12]. Identity (3.9) implies $\det(\gamma, \gamma', \gamma'') = -1$. Moreover, the second derivative f_{vv} yields by (3.5) the following constraints on γ:

$$\gamma''' = \alpha\,\gamma + \beta\,\gamma'; \quad \alpha = B + \omega_v\,\omega_{vv} - \omega_{vvv}, \quad \beta = \omega_v^2 - 2\,\omega_{vv}. \tag{9.19}$$

Locally we can consider $\xi = u$, $\zeta = v$. In terms of these coordinates one finds

$$f(u,v) = \tfrac{2u}{2-u\,v}\,\gamma(v) + \gamma'(v), \text{ with } \gamma'''(v) = B(v)\,\gamma(v). \tag{9.20}$$

$B \equiv 0$ is included in the above construction. For the curve γ one obtains

$$\gamma(v) = \begin{pmatrix} \dfrac{1}{\zeta(v)} \\ \dfrac{\zeta(v)^2}{2\,\zeta'(v)} \\ \dfrac{\zeta(v)}{\zeta'(v)} \end{pmatrix}. \tag{9.21}$$

In this case the whole surface family (9.15) can be obtained by

$$\xi(u) \to \lambda\,\xi(u), \quad \zeta(v) \to \tfrac{1}{\lambda}\,\zeta(v). \tag{9.22}$$

Remark 9.1. Due to Corollary 7.1 the immersions for ruled surfaces are obtained from certain normalized potentials. In view of (5.7) these are exactly the potentials for which $T_{12}^{(+)} = 0$.

Appendix

The goal of this Appendix is to outline the proof of Theorem 4.3.

We consider the three automorphisms of $\bigwedge Sl(3,\mathbb{C})$

$$(\varphi_1\,g)(\lambda) = T\,\left[g(-\lambda)^t\right]^{-1}\,T, \tag{A.1}$$

$$(\varphi_2\,g)(\lambda) = Q\,g(\varepsilon\,\lambda)\,Q^{-1}, \tag{A.2}$$

$$(\varphi_3\,g)(\lambda) = \overline{g(\bar{\lambda})}, \tag{A.3}$$

where T and Q are given in (4.6) and (4.5) respectively.

On the Lie algebra level we obtain the automorphisms

$$\left(\hat{\varphi}_1\,g\right)(\lambda) = -T\,g(-\lambda)^t\,T, \tag{A.4}$$

$$\left(\hat{\varphi}_2\,g\right)(\lambda) = Q\,g(\varepsilon\,\lambda)\,Q^{-1}, \tag{A.5}$$

$$\left(\hat{\varphi}_3\,g\right)(\lambda) = \overline{g(\bar{\lambda})}. \tag{A.6}$$

Note that $\hat{\varphi}_2$ and $\hat{\varphi}_3$ actually are automorphisms of the associative algebra $\mathrm{Mat}(3, \mathcal{A})$.

It is straightforward to check

$$\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1, \quad \varphi_1 \circ \varphi_3 = \varphi_3 \circ \varphi_1, \quad \varphi_2 \circ \varphi_3 = \varphi_3 \circ \varphi_2^2. \tag{A.7}$$

We are interested in

$$G[\lambda] = \mathrm{Fix}(\varphi_1) \cap \mathrm{Fix}(\varphi_2) \cap \mathrm{Fix}(\varphi_3), \tag{A.8}$$

where $\mathrm{Fix}(\varphi_j) = \{g \in \bigwedge Sl(3, \mathbb{C}) \mid \varphi_j g = g\}$.

$$\varphi_1, \varphi_3 \text{ have order 2 and } \varphi_2 \text{ has order 3} \tag{A.9}$$

$$\varphi = \varphi_1 \circ \varphi_2 \text{ has oder 6 and we have } \mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_1) \cap \mathrm{Fix}(\varphi_2). \tag{A.10}$$

The analogous statements hold on the Lie algebra level.

For the proof of Theorem 4.3 we will use methods and results from Kac-Moody Lie algebras. If $\hat{\varphi}_0$ denotes the restriction of $\hat{\varphi}$ to $g = sl(3, \mathbb{C})$ the "constant" matrices in $\bigwedge sl(3, \mathbb{C})$, then a comparison with [9], Section 8.1 shows

$$\mathrm{Fix}(\hat{\varphi}) = L(g, \hat{\varphi}_0). \tag{A.11}$$

This allows us to exploit [9], 8.1 and [6], 10.5.

It is easy to see that $\hat{\varphi}_0$ is an outer automorphism of $sl(3, \mathbb{C})$. Therefore, [9], Theorem 8.5, [6], Theorem 5.13

$$L(g, \hat{\varphi}_0) \simeq L(g, \mu) \tag{A.12}$$

where μ is a standard outer automorphism of $sl(3, \mathbb{C})$. In particular, $L(g, \hat{\varphi}_0)$ is the loop part of a Kac-Moody algebra of type $\mathbf{A}_2^{(2)}$.

In such an isomorphism, positive roots go to positive roots. Therefore, in loop algebra realizations positive powers of λ are mapped to positive powers of λ.

Now the Birkhoff-Decomposition Theorem for $L(g, \hat{\varphi}_0)$ follows from [10, 4], in the form

$$\mathrm{Fix}(\varphi) = \bigcup_{w \in \mathcal{W}} (\mathrm{Fix}(\varphi))_*^{(-)} \cdot w \cdot (\mathrm{Fix}(\varphi))_*^{(+)}, \tag{A.13}$$

where w denotes a representative in $\mathrm{Fix}(\varphi)$ for $w \in \mathcal{W}$, where $\mathcal{W}$ is the Weyl group of $L(g, \hat{\varphi}_0)$.

$\mathcal{W}$ is finitely generated and the fundamental reflections generating $\mathcal{W}$ have representatives

$$w_i = \exp(f_i) \cdot \exp(-e_i) \cdot \exp(f_i), \tag{A.14}$$

where e_i (resp. f_i) are representatives for the simple positive (resp. negative) root spaces.

In $\mathrm{Fix}(\hat{\varphi})$ we set

$$H_0 = \mathrm{diag}(-1, +1), \tag{A.15}$$

$$E_0 = \begin{pmatrix} 0&0&0\\ 0&0&1\\ 1&0&0 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}, \tag{A.16}$$

$$e_0 = \lambda\, E_0, \qquad e_1 = \lambda\, E_1, \tag{A.17}$$

$$F_0 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 0&1&0 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 0&0&0\\ -1&0&0\\ 0&0&0 \end{pmatrix}, \tag{A.18}$$

$$f_0 = \lambda^{-1}\, F_0, \qquad f_1 = \lambda^{-1}\, F_1. \tag{A.19}$$

Note these are the choices of [9], 8.2 associated with $\hat{\varphi}_0$.

It is important to note that all the matrices listed in (A.15)-(A.19) are actually contained in $g[\lambda] = LieG[\lambda]$. Moreover, since these matrices generate $g[\lambda]$, we obtain naturally a basis of Fix($\hat{\varphi}$) which is also basis for $g[\lambda]$. Moreover, $\mathcal{W}$ permutes these basis elements. Therefore, the proof of [10, 4] carries through without change for $g[\lambda]$ and $G[\lambda]$. This shows

$$G[\lambda] = \bigcup_{w \in \mathcal{W}} G_*^{(-)}[\lambda] \cdot w \cdot G_*^{(+)}[\lambda]. \tag{A.20}$$

We discuss $\mathcal{W}$ in more detail. From [9], Proposition 6.5 we know

$$\mathcal{W} = \dot{\mathcal{W}} \ltimes \mathcal{T} \tag{A.21}$$

where $\dot{\mathcal{W}}$ is the Weyl group of the finite dimensional Lie algebra $\dot{g} \subset g[\lambda]$. Following the construction in [9], 8.2 - see also the warning in [9], 8.3 - we obtain

$$\dot{g} = \begin{pmatrix} \alpha & \beta & 0\\ \gamma & -\alpha & 0\\ 0&0&0 \end{pmatrix} \simeq sl(2\,\mathbb{R}) \tag{A.22}$$

$$\dot{\mathcal{W}} = \{\mathrm{I}, T\}, \tag{A.23}$$

where T is given by (4.6). It remains to determine $\mathcal{T}$.

From [9], 6.5 we know that $\mathcal{T} = \{t_\alpha \mid \alpha \in \frac{1}{2}\,\mathbb{Z}\,\dot{\Delta}_1\}$, where t_α is given by [9], 6.5.3 and $\dot{\Delta}_1$ is the root system for $\dot{g}$ (see [9], 6.2). Therefore, a generator for $\mathcal{T}$ is t_α, where $\alpha = \frac{1}{2}\,\alpha_1$. Since $\alpha_i(c) = 0$, c the center of the Cartan algebra of $g[\lambda]$, [9], 6.5.5 shows $t_\alpha(\alpha_i) = \alpha_i - (\alpha_i, \alpha)\,\delta = \alpha_i - \frac{1}{2}\,\check{a}_i\, a_1^{-1}\, a_{i1}\,\delta$, $i = 0, 1$. Since $a_0 = 2$, $a_1 = 1$, $\check{a}_0 = 1$, $\check{a}_1 = 2$, $A_2^{(2)} = (a_{ij}) = \begin{pmatrix} 2 & -4\\ -1 & 2\end{pmatrix}$, $t_\alpha(\alpha_0) - \alpha_0 + 2\,\delta$, $t_\alpha(\alpha_1) = \alpha_1 - 2\,\delta$. Note $\delta = 2\,\alpha_0 + \alpha_1$ by [9], 6.4. In our realization a straight forward computation shows $[e_0, [e_0, e_1]] = \lambda^3\,\mathrm{diag}(R, R, -2R)$, where R is λ-independent. Thus the action of t_α preserves the position, but increases/decreases the λ-power by six. Therefore, a generator for $\mathcal{T}$ is realized by a matrix of the form $D = \mathrm{diag}(\alpha\,\lambda^{3k}, \beta\,\lambda^{3k}, \gamma)$, $\alpha, \beta, \gamma \in \mathbb{R}$. But $D \in g[\lambda]$ is equivalent with $T\,D^{-1}\,T = D$, whence $(-1)^k\,\beta^{-1} = \alpha$, $\gamma^{-1} = \gamma$. This yields the matrices in (4.31). Using (A.14), (A.17) and (A.18) we obtain for $\dot{\mathcal{W}}$ the explicit realization

$$\dot{\mathcal{W}} = \left\{\mathrm{I}, \begin{pmatrix} 0 & \lambda & 0\\ -\lambda^{-1} & 0 & 0\\ 0 & 0 & 1 \end{pmatrix}\right\}. \tag{A.24}$$

In view of (A.21) this yields the additional matrices of (4.32).

The uniqueness of the double cosets is part of [10, 4].

Next we show that $G^{(-)}[\lambda] \cdot G_*^{(+)}[\lambda]$ is open and dense in $G[\lambda]$. One way of proofing this is to observe that the proof of [4], 3.5 carries through also over the field $\mathbb{R}$ (instead of $\mathbb{C}$). Another proof follows the argument of [3].: One considers the action ρ of $G_*^{(-)}[\lambda] \times G_*^{(+)}[\lambda]$ on $G[\lambda]$, given by $(g_-, g_+) \cdot g = g_- \, g \, g_+^{-1}$. It is easy to show that the image of the differential of ρ has over $w \in \mathcal{W}$ finite codimension $\neq 0$ if $w \neq$ I. Therefore the corresponding orbits through $w \in \mathcal{W}$ are locally closed and nowhere dense in $G[\lambda]$. The Baire Category Theorem now implies the claim.

The final claim follows e.g. from [4], Corollary 3.1.4.

References

[1] W. BLASCHKE, *Vorlesung über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie, II*, Springer, 1923.

[2] R. COURANT and D. HILBERT, *Methoden der Mathematischen Physik*. Springer, 1968.

[3] S. DISNEY, The exponents of loops on the complex general linear group, *Topology* **12** (1973), 297–315.

[4] J. DORFMEISTER, H. GRADL, and J. SZMIGIELSKI, Systems of PDE's obtained from factorization in loop groups, *Acta Appl. Math.* **53** (1998), 1–58.

[5] J. DORFMEISTER, F. PEDIT, and H. WU, Weierstrass type representation of harmonic maps into symmetric spaces, *Comm. Anal. Geom.* **6** (1998), 633–668.

[6] S. HELGASON, *Differential geometry, Lie groups , and symmetric spaces*. Academic Press, 1978.

[7] ALEXANDER ITS, Liouville's theorem and the method of the inverse problem, *J.Sov.Math.* **31** (1985), 3330–3338.

[8] H. JONAS, Geometrische Deutung einer Transformationstheorie, *Sitzungsbericht der Berliner Mathematischen Gesellschaft* **27** (1928), 57–78.

[9] V. KAC, *Infinite dimensional Lie algebras*. Birkhäuser, 1983.

[10] V. KAC and D. PETERSON, Inifinte flag varieties and conjucaty theorem, *Proc. Natl. Acad. Sci. USA* **80** (1983), 1778–1782.

[11] K. NOMIZU and T. SASAKI, *Affine Differential Geometry*. Cambridge University Press, 1994.

[12] J. RADON, *Leipziger Berichte* **70** (1918), p. 153.

[13] P. RYAN, Lectures on Differential Geometry of Affine Hypersurfaces, *Proceedings of the Topology and Geometry Research Center, Kyungpook National University, Taegu, Korea* **6** (1995), 209–235.

[14] P.A. SCHIROKOW and A.P. SCHIROKOW, *Affine Differentialgeometrie*. Leibzig, 1962.

[15] U. SIMON, A. SCHWENK-SCHELLSCHMIDT, and H. VIESEL, *Introduction to the affine differential geometry of hypersurfaces*. Science University of Tokyo, 1991.

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