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Parallel Surfaces in Affine 4-Space

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Abstract. We study affine immersions as introduced by NOMIZU and PINKALL. We classify those affine immersions of a surface in $\mathbb{R}^4$ which are degenerate and have vanishing cubic form (i.e. parallel second fundamental form). This completes the classification of parallel surfaces of which the first results were obtained in the beginning of this century by BLASCHKE and his collaborators.

1 Introduction

We consider the standard affine space $\mathbb{R}^m$ equipped with its standard connection D. Let M^n be a manifold equipped with a torsion free affine connection ∇ and let $x : (M^n, \nabla) \to (\mathbb{R}^m, D)$, $m \geq n$ be an immersion. Following [9], we call x an affine immersion if there exists a transversal $(m-n)$-dimensional bundle σ such that

$$D_X x_*(Y) - x_*(\nabla_X Y) \in \sigma, \tag{1}$$

for all vector fields X and Y which are tangent to M^n. It is immediately clear that if we equip $\mathbb{R}^m$ with a semi-Riemannian metric and take for σ the normal bundle, then isometric immersions provide examples of affine immersions. Also the equiaffine immersions, in the sense of BLASCHKE for hypersurfaces, and in the sense of [16], [18] or [10] for higher codimensions provide examples of affine immersions.

For an affine immersion it is possible to introduce a bilinear form h, called the second fundamental form, which takes values in the tranversal bundle σ by

$$h(X, Y) = D_X x_*(Y) - x_*(\nabla_X Y) \in \sigma. \tag{2}$$

Since ∇ is a torsion free affine connection, h is symmetric in X and Y. Let ξ be a vector field which takes values in σ. Similarly, as for isometric immersions, we can now introduce a normal connection $\nabla^\perp$ and Weingarten operators A_ξ by decomposing $D_X\xi$ into a tangential part and a part in the direction of σ, i.e. we have the Weingarten formula which states that

$$D_X\xi = -x_*(A_\xi X) + \nabla^\perp_X \xi. \tag{3}$$

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Using the Weingarten formula, it is now possible to define the covariant derivative ∇h of the second fundamental form h by

$$(\nabla_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{4}$$

Affine immersions for which ∇h vanishes identically are called parallel immersions. In Riemannian geometry, these immersions and their generalisations have been studied by many people, an overview can be found in [7]. A general classification of the Euclidean parallel submanifolds was obtained in [3]. As far as we know it is still an open problem to classify the semi-Euclidean parallel submanifolds.

In this paper we will focus on surfaces, i. e. the dimension of M^n equals two. All results will be local and valid on a suitable open dense subset of M^2. We say that an affine immersion is linearly full provided that for every point p of M^2 and for every neighborhood U of p, $x(U)$ is not contained in a lower dimensional affine subspace of $\mathbb{R}^n$. Using Lemma 2 of [15] which says that $\sigma = \operatorname{im} h$ if a parallel affine immersion is linearly full, it follows easily that a parallel surface immersion which is linearly full has to be in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$ or $\mathbb{R}^5$. The first case ($m = 2$) clearly implies that M^2 is an affine plane. In the other cases, a nondegeneracy condition can be introduced as follows. Let $u = \{X_1, X_2\}$ be a local basis in a neighborhood of a point p. Then we define for $m = 3$:

$$h_u(X, Y) = \det(X_1, X_2, D_X Y), \tag{5}$$

and for $m = 4$:

$$h_u(X, Y) = \tfrac{1}{2}\left(\det(X_1, X_2, D_{X_1}X, D_{X_2}Y) + \det(X_1, X_2, D_{X_1}Y, D_{X_2}X)\right). \tag{6}$$

It is well known that in both cases the rank of h_u is independent of the choice of the local basis u. We call M^2 nondegenerate if the rank equals 2, 1-degenerate if it equals 1 and 0-degenerate if it equals 0. A surface in $\mathbb{R}^5$ is called nondegenerate if $\det(X_1, X_2, D_{X_1}X_1, D_{X_1}X_2, D_{X_2}X_2) \neq 0$, which again is independent of the choice of basis u. Using Lemma 2 of [15] again, we see that a linearly full affine immersion of M^2 in $\mathbb{R}^5$ is always nondegenerate.

Nondegenerate parallel immersions of a surface in $\mathbb{R}^3$, $\mathbb{R}^4$ and $\mathbb{R}^5$ are considered in respectively [9], [10] and [8]. Therefore, restricting to an open and dense subset if necessary, only the degenerate cases still need considering. If M^2 is contained in $\mathbb{R}^3$, a solution was found in [2]. This leaves only the case that M^2 is linearly full in $\mathbb{R}^4$ and degenerate. If M^2 is 0-degenerate and parallel, the immersion can not be linearly full. Thus we are left with the 1-degenerate parallel surfaces which are linearly full in $\mathbb{R}^4$. We prove the following:

Theorem. *Every 1-degenerate parallel affine surface immersion (x, σ) in $\mathbb{R}^4$ is a ruled surface and can be locally parametrized either by*

I.1. $x(u, v) = \gamma'(u) + v\gamma(u)$, *and*
$\sigma = \operatorname{span}(\xi_1, \xi_2)$ *is given by* (45) *and* (46), *or*

I.2. $x(u, v) = (\varepsilon\gamma(u) + \gamma''(u)) + v\gamma'(u)$, $\varepsilon = \pm 1$, *and*
$\sigma = \operatorname{span}(\xi_1, \xi_2)$ *is given by* (47) *and* (48),

II. $x(u, v) = \alpha(u) + v\beta(u)$, $\beta'' = -\beta$, *and*
$\sigma = \operatorname{span}(\xi_1, \xi_2)$ *is given by* (49) *and* (50).

The paper is organized in two parts. In Section 2 we apply the method of moving frames due to E. CARTAN to an affine immersion of M^2 in $\mathbb{R}^4$. We introduce the affine semiconformal structure (cp. (6)), which was known already to [1]. We end up with a classification of affine surfaces in $\mathbb{R}^4$ with respect to the (non)degeneracy-type of the affine semiconformal structure and normal forms of the second fundamental form h for each type. This part is closely related to [12] and [13].

In Section 3 we restrict to 1-degenerate parallel affine immersions of M^2 in $\mathbb{R}^4$. It turns out that they are ruled surfaces (Lemma 2) and therefore can be parametrized as $x(u, v) = \alpha(u) + v\beta(u)$. We find special frames which simplify the structure equations significantly. A reparametrization finally leads to our main result.

We will use the Euler summation convention.

2 Classification of surfaces in $\mathbb{R}^4$ with respect to their affine semiconformal structure

2.1 The affine frame bundle on $\mathbb{R}^4$. We define a *frame* on $\mathbb{R}^4$ to be an ordered set

$$S_b = \{v_1, v_2, v_3, v_4; b\}, \quad \text{with} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathrm{Gl}(4, \mathbb{R}), b \in \mathbb{R}^4.$$

Let F denote the set of all frames on $\mathbb{R}^4$ and $\pi : F \to \mathbb{R}^4$ the projection map, defined by: $\pi(S_b) = b$. Let $\mathrm{Aff}(\mathbb{R}^4)$ be the Lie group of affine transformations on $\mathbb{R}^4$,

$$\mathrm{Aff}(\mathbb{R}^4) = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline b & 1 \end{array} \right) \;\middle|\; A \in \mathrm{Gl}(4, \mathbb{R}), b \in \mathbb{R}^4 \right\}.$$

Obviously we can identify F with $\mathrm{Aff}(\mathbb{R}^4)$. The local structure of $\mathrm{Aff}(\mathbb{R}^4)$ is encoded in the Lie algebra-valued Maurer-Cartan form $\varpi_{|S} = dSS^{-1} \in \mathrm{aff}(\mathbb{R}^4)$, we use the notation:

$$d \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ b \end{pmatrix} = \left(\begin{array}{c|c} M & 0 \\ \hline n & 0 \end{array} \right) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ b \end{pmatrix}, \qquad M \in M(4 \times 4, \mathbb{R}), n \in \mathbb{R}^4. \tag{7}$$

If we let $\mathrm{Aff}(\mathbb{R}^4)$ act both on F and $\{(b, 1) \in \mathbb{R}^5 \mid b \in \mathbb{R}^4\} \cong \mathbb{R}^4$ by *right multiplication* R_S, $R_S(C) = CS$, then $\pi \circ R_S = R_S \circ \pi$. To obtain the fibers of the bundle $\mathcal{F} := \pi : F \to \mathbb{R}^4$, note that $\pi(S_b) = (0, 0, 0, 0, 1)S_b$, and the isotropy group of $(0, 0, 0, 0, 1)$ is

$$H = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \right\} \subset \mathrm{Aff}(\mathbb{R}^4).$$

We can identify the homogeneous space $\mathrm{Aff}(\mathbb{R}^4)/H \cong \mathbb{R}^4$ and obtain that $\mathcal{F}$ is a principal (right) H-bundle, the *affine frame bundle* on $\mathbb{R}^4$.

2.2 Adaption of the affine frame bundle to an affine surface immersion. Let U be a connected open subset of a two-dimensional oriented manifold M^2 equipped with a torsion free affine connection ∇ and let $x\colon (U, \nabla) \to (\mathbb{R}^4, D)$ be a smooth affine immersion with transversal bundle σ (cp. Sec. 1). We want to adapt the affine frame bundle to the surface by restricting the base manifold to $x(U)$. We define the principal (right) H-bundle $\mathcal{F}^0 = \pi_U\colon F^0 \to U$ as the bundle over U induced by x and the affine frame bundle (cp. [14], vol. V, p. 391f. for the notion of an induced bundle), i. e. $\mathcal{F}^0 = x^*\mathcal{F}$. For the further adaption we take into account the given transversal bundle σ and we use the first order information given by the tangent bundle of x. We call a frame $S_u \in F^0$ a *first order frame* if span$(v_1, v_2) = x_*(T_uM)$ and span$(v_3, v_4) = \sigma$. We denote the set of all first order frames on U by $F^1 \subset F^0$ and use the notation $S_u = \{v_1, v_2, \xi_1, \xi_2, x(u)\} \in F^1$. The subgroup

$$H^1 = \left\{ \left(\begin{array}{cc|c} P & 0 & \\ 0 & Q & 0 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array} \right) \;\middle|\; \det P \neq 0 \right\} \subset H, \tag{8}$$

acts transitively and effectively on F^1. Thus we get a subbundle $\mathcal{F}^1$ of $\mathcal{F}^0$, $\mathcal{F}^1 = \pi_U\colon F^1 \to U$, where we use the same notation for the restriction of π_U to F^1. Obviously $\mathcal{F}^1$ is a principal (right) H^1-bundle, the reduced bundle obtained by reduction of the structure group H of $\mathcal{F}^0$ to H^1 (cp. [6], vol. I, pg. 53). Since the first two legs v_1 and v_2 of a frame $S_u \in F^1$ span the tangent space $x_*(T_uM)$, we get two zero's in the last row of the Maurer-Cartan form $\varpi_{|S_u}$ on F^1 ($\varpi_5^3 = 0$, $\varpi_5^4 = 0$) and the forms ϖ_5^1 and ϖ_5^2 drop down to U (cp. 7). We use the notation:

$$d\begin{pmatrix} v_1 \\ v_2 \\ \xi_1 \\ \xi_2 \\ x \end{pmatrix} = \left(\begin{array}{cc|cc|c} \multicolumn{2}{c|}{\varphi} & \multicolumn{2}{c|}{\psi} & \\ \multicolumn{2}{c|}{\sigma} & \multicolumn{2}{c|}{\tau} & 0 \\ \hline \omega^1 & \omega^2 & 0 & 0 & 0 \end{array} \right) \begin{pmatrix} v_1 \\ v_2 \\ \xi_1 \\ \xi_2 \\ x \end{pmatrix}. \tag{9}$$

For a fixed first order frame field (a smooth cross section of $\mathcal{F}^1$) $S = \{v_1, v_2, \xi_1, \xi_2, x\}$, $v_i = dx(X_i)$, $X_i \in \Gamma(TM)$, the entries of the Maurer-Cartan form define (resp. correspond to) the following quantities ($1 \le i, j, k \le 2$, $3 \le \alpha \le 4$) (cp. (1), (2) and (3)):

$$\nabla_{X_i} X_j = \varphi_j^k(X_i)dx(X_k) \qquad \text{induced connection} \tag{10}$$

$$h^\alpha(X_i, X_j) = \psi_i^\alpha(X_j) \qquad \text{second fundamental forms} \tag{11}$$

$$-dx(A_{\xi_j}(X_i)) = \sigma_j^k(X_i)dx(X_k) \qquad \text{Weingarten operators} \tag{12}$$

$$\nabla^\perp_{X_i}\xi_j = \tau_j^\alpha(X_i)\xi_{(\alpha-2)} \qquad \text{normal connection} \tag{13}$$

It is straightforward to show that $\nabla^\perp$ is a torsion-free affine connection, h^3 and h^4 are symmetric bilinear forms and A_{ξ_1} and A_{ξ_2} are 1-1 tensor fields.

2.3 The affine semiconformal structure. To find the invariants (quantities independent of the choice of frame) we can compute the (infinitesimal) group action (change of frames) either on the Lie group level or on the Lie algebra level (cp. [4])

for the general theory, [12] for the centroaffine case). Since the group of centroaffine transformations is a subgroup of the affine group, the affine invariants are part of the centroaffine ones. A description in detail can be found in [13]. We only will need the action on ψ resp. the second fundamental forms h^3 and h^4. Let $S_u, \tilde{S}_u \in F^1$, then there exists $B \in H^1$ such that $S_u = B\tilde{S}_u$ (cp. (8)). For the Maurer Cartan form we get:

$$\varpi_{|S} B = dSS^{-1}B = d(B\tilde{S})(B\tilde{S})^{-1}B = dB + Bd\tilde{S}\tilde{S}^{-1} = dB + B\varpi_{|\tilde{S}}.$$

For $B = \left(\begin{array}{cc|c} P & 0 & \\ 0 & Q & 0 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right) \in H^1$ an evaluation of this equation and $\psi_i^\alpha = h_{ij}^\alpha \omega^j$ gives:

$$\psi = P\tilde{\psi}Q^{-1}, \tag{14}$$

$$(h^3, h^4) = (P[(Q^{-1})_1^1\tilde{h}^3 + (Q^{-1})_2^1\tilde{h}^4]\,{}^TP,\ P[(Q^{-1})_1^2\tilde{h}^3 + (Q^{-1})_2^2\tilde{h}^4]\,{}^TP). \tag{15}$$

For a frame $S_u = \{v_1, v_2, \xi_1, \xi_2, u\} \in F^1$ we define a symmetric[1] bilinear form ϕ on F^1 by:

$$\phi = \det\psi = \psi_1^3 \odot \psi_2^4 - \psi_2^3 \odot \psi_1^4, \tag{16}$$

i. e. $\phi(X, Y) = \frac{1}{2}\left(\frac{[v_1, v_2, D_X dx(X_1), D_Y dx(X_2)] + [v_1, v_2, D_X dx(Y_1), D_X dx(X_2)]}{[v_1, v_2, \xi_1, \xi_2]}\right)$ for some determinant form $[\ \ ]$ on $\mathbb{R}^4$ (cp. (6)). We can use (14) to determine how ϕ varies along the fibers of $\mathcal{F}^1$:

$$\phi = \det\psi = (\det P)(\det\tilde{\psi})(\det Q^{-1}) = \frac{\det P}{\det Q}\tilde{\phi}\,.$$

Now a *semiconformal structure* compatible with a quadratic form q is defined as the set $\{rq \mid r \in \mathbb{R}\setminus\{0\}\}$ and it makes sense to talk about a semiconformal structure being nondegenerate, definite, etc. (cp. [17], p. 4). The quadratic form associated to ϕ induces a semiconformal structure on the tangent space at each point of U. This structure on U is called the *affine semiconformal structure* induced by x and was known already to [1], p. 375. Depending on the affine semiconformal structure we will call a surface $x(U)$ a *nondegenerate, definite, indefinite* or *1-degenerate surface* if the induced affine semiconformal structure is nondegenerate, definite, indefinite or 1-degenerate. A *0-degenerate surface* is a surface $x(U)$ for which the affine semiconformal structure contains only the zero form.

2.4 Normalization of ψ and classification. We saw that a change of frames induces an action of H^1 on Sym(2) $\times$ Sym(2) (cp. (15)), where Sym(2) denotes the algebra of all symmetric 2×2-matrices:

$$\rho(B)(h^3, h^4) = (P[(Q^{-1})_1^1 h^3 + (Q^{-1})_2^1 h^4]\,{}^TP,\ P[(Q^{-1})_1^2 h^3 + (Q^{-1})_2^2 h^4]\,{}^TP).$$

[1] We denote by $\odot$ the symmetric product of 1-forms: $\omega \odot \eta(X, Y) = \frac{1}{2}(\omega(X)\eta(Y) + \omega(Y)\eta(X))$.

Note that the action can be written as the composition of two actions of the form:

$$\rho_1(P)(h^3, h^4) = (Ph^3\, {}^T\!P, Ph^4\, {}^T\!P), \tag{17}$$

$$\rho_2(Q)(h^3, h^4) = ((Q^{-1})^1_1 h^3 + (Q^{-1})^1_2 h^4, (Q^{-1})^2_1 h^3 + (Q^{-1})^2_2 h^4), \tag{18}$$

namely:

$$\rho\left(\left(\begin{array}{c|c|c} P & 0 & \\ \hline 0 & Q & 0 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right)\right) = \rho_1(P) \circ \rho_2(Q). \tag{19}$$

We want to choose normal forms (representatives of the orbits) in Sym(2) $\times$ Sym(2) under the action of H^1 given by (19). Since the centroaffine situation is very close to the affine one, we will omit some details. A more comprehensive description can be found in [13] (Section 4.1, 4.2). As we just saw the action splits in two parts where span(h^3, h^4) is an invariant of the second part (18). Therefore we want to investigate the orbits of two-pencils[2] under the first part (17) of the action. A first step is to understand the action on a single element $h \in$ Sym(2).

If we restrict ρ_1 to Sl$(2, \mathbb{R})$, we can define an invariant quadratic form q in Sym(2) by

$$q(h) = -\det h. \tag{20}$$

Then Sym(2) with the associated scalar product is isometric to the Minkowski 3-space $\mathbb{R}^3_1$ (see Figure 1, for notations cp. [11]). This is easy to see if we choose

$$E_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),\ E_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right),\ E_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \tag{21}$$

as a basis of Sym(2). Then we get for every $h = aE_0 + bE_1 + cE_2 = \left(\begin{smallmatrix} a+b & c \\ c & a-b \end{smallmatrix}\right)$ $\in$ Sym(2): $q(h) = -\det h = -(a^2 - b^2 - c^2) = -a^2 + b^2 + c^2$.

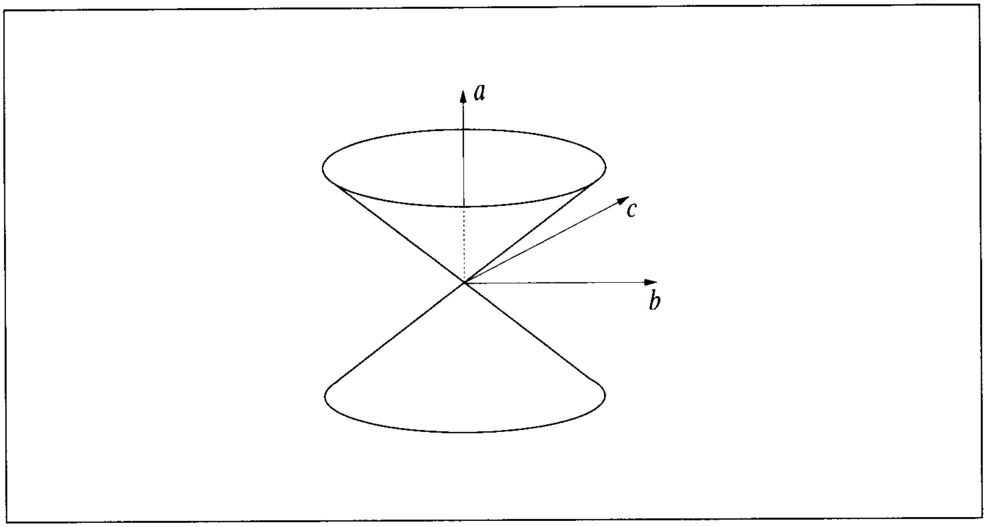
Under this identification ρ_1 defines a representation of Sl$(2, \mathbb{R})$ on $\mathbb{R}^3_1$. The invariance of q means that $\rho_1(P)$ is a linear isometry of $\mathbb{R}^3_1$, i. e. $\rho_1 :$ Sl$(2, \mathbb{R}) \to$ O$_1(3)$. This map is neither 1:1 ($\rho_1(P) = \rho_1(-P)$) nor onto (Sl$(2, \mathbb{R})$ is connected, O$_1(3)$ has four components). However, $\rho_1 :$ PGl$^+(2, \mathbb{R}) \to \mathbb{R}^+\, \mathrm{O}^{++}_1(3)$ is an isomorphism:

Theorem. ([13], Thm. 3) *Let* $\mathrm{O}^{++}_1(3)$ *be the group of all time- and space-orientation preserving isometries of* $\mathbb{R}^3_1$ *and* $\mathrm{PGl}^+(2, \mathbb{R}) = \mathrm{Gl}^+(2, \mathbb{R})/\{\pm \mathrm{Id}\}$, $\mathrm{Gl}^+(2, \mathbb{R}) = \{P \in \mathrm{Gl}(2, \mathbb{R}) \mid \det P > 0\}$. *Identify* $\mathbb{R}^3_1$ *and* Sym(2) *by* ${}^T(a, b, c) \mapsto \left(\begin{smallmatrix} a+b & c \\ c & a-b \end{smallmatrix}\right)$. *Then* $\rho_1 : \mathrm{PGl}^+(2, \mathbb{R}) \to \mathbb{R}^+\, \mathrm{O}^{++}_1(3) = \{rQ \mid r \in \mathbb{R}^+, Q \in \mathrm{O}^{++}_1(3)\}$, *defined by* $\rho_1(P)A = PA\,{}^T\!P$, *is a (Lie group) isomorphism.*

Hence we know that PGl$^+(2, \mathbb{R})$ acts on an element $\left(\begin{smallmatrix} a+b & c \\ c & a-b \end{smallmatrix}\right) \in$ Sym(2) in the same way as $\mathbb{R}^+\, \mathrm{O}^{++}_1(3)$ acts on an element ${}^T(a, b, c) \in \mathbb{R}^3_1$. The later action is well understood. $\mathrm{O}^{++}_1(3)$ acts transitively on (ordered) orthonormal bases which have the same time- and space-orientation. Thus it acts also transitively on two-dimensional space-, time- or lightlike subspaces.

Two-pencils are either space-, time- or lightlike subspaces, either two-dimensional or one-dimensional or just the origin, where dimension and type are invariant

[2] The span of two symmetric bilinear forms is called a two-pencil.



$$\begin{array}{ccccc} h \text{ spacelike} & \Longleftrightarrow & \det h < 0 & \Longleftrightarrow & h \text{ indefinite,} \\ h \text{ timelike} & \Longleftrightarrow & \det h > 0 & \Longleftrightarrow & h \text{ definite,} \\ h \text{ lightlike} & \Longleftrightarrow & \det h = 0 & \Longleftrightarrow & h \text{ degenerate.} \end{array}$$

Figure 1. $(\mathrm{Sym}(2), q) \cong \mathbb{R}^3_1$

under ρ_1. Normal forms for the lower dimensional cases are obvious [5], p. 251. In the two-dimensional case we can choose the following normal forms (cp. (21)):

I.	$\mathrm{span}(h^3, h^4)$ spacelike:	$\mathrm{span}(E_1, E_2)$,
II.	$\mathrm{span}(h^3, h^4)$ lightlike:	$\mathrm{span}(E_2, \frac{1}{2}(E_0 + E_1))$,
III.	$\mathrm{span}(h^3, h^4)$ timelike:	$\mathrm{span}(E_0, E_1)$.

Finally we can still use the second part ρ_2 (18) of the action ρ (19) to map h^3 and h^4 to the suitable basis vectors.

Summarized we obtain the following classes of surfaces in $\mathbb{R}^4$:

		$\mathrm{span}(h^3, h^4)$	ϕ	normal form
I.		spacelike plane	definite	(E_1, E_2)
II.		lightlike plane	1-degenerate	$(E_2, \frac{1}{2}(E_0 + E_1))$
III.		timelike plane	indefinite	(E_0, E_1)
IV.	(a)	spacelike line		$(E_1, 0)$
	(b)	lightlike line	0-degenerate	$(\frac{1}{2}(E_0 + E_1), 0)$
	(c)	timelike line		$(E_0, 0)$
	(d)	$(0, 0)$		$(0, 0)$

3 1-degenerate parallel surfaces in $\mathbb{R}^4$

In the following only 1-degenerate surfaces in $\mathbb{R}^4$ will be considered since only for this class the parallel surfaces are yet not classified (cp. Sec. 1).

3.1 Second order frames. As we have seen before for a 1-degenerate surface (Type II) in $\mathbb{R}^4$ there exists a frame $S \in F^1$ such that $h^3 = E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $h^4 = \frac{1}{2}(E_0 + E_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ resp. $\psi = \begin{pmatrix} \omega^2 & \omega^1 \\ \omega^1 & 0 \end{pmatrix}$. We call such a frame a *second order frame* and denote the set of all second order frames on U by $F^2 \subset F^1$. We can determine the subgroup $H^2 \subset H^1$, which acts transitively and effectively on F^2, by calculating which changes of frames leave the special form of ψ invariant, and we obtain:

$$H^2 = \left\{ \left(\begin{array}{cc|cc|c} a & b & & & \\ 0 & c & \multicolumn{2}{c|}{0} & \\ \hline & & ac & 0 & 0 \\ \multicolumn{2}{c|}{0} & 2ab & a^2 & \\ \hline \multicolumn{4}{c|}{0} & 1 \end{array} \right) \middle| \; ac \neq 0 \right\}. \tag{22}$$

We have constructed a subbundle $\mathcal{F}^2$ of $\mathcal{F}^1$, $\mathcal{F}^2 = \pi_U : F^2 \to U$, which is a principal (right) H^2-bundle, the reduced bundle obtained by reduction of the structure group H^1 of $\mathcal{F}^1$ to H^2. We use the notation $S \in \mathcal{F}^2$ for a second order frame field. The structure equations have the form (cp. (9)):

$$d \begin{pmatrix} v_1 \\ v_2 \\ \xi_1 \\ \xi_2 \\ x \end{pmatrix} = \left(\begin{array}{cc|cc|c} \multicolumn{2}{c|}{\varphi} & \omega^2 & \omega^1 & \\ & & \omega^1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{\sigma} & \multicolumn{2}{c}{\tau} & \\ \hline \omega^1 & \omega^2 & 0 & 0 & 0 \end{array} \right) \begin{pmatrix} v_1 \\ v_2 \\ \xi_1 \\ \xi_2 \\ x \end{pmatrix}. \tag{23}$$

3.2 Parallel surfaces. An affine surface with transversal bundle σ is called *parallel* if the second fundamental form $h = h^3\xi_1 + h^4\xi_2$ is parallel (cp. Sec. 1), i. e.

$$\nabla h = 0. \tag{24}$$

By definition ($\nabla h = C^3\xi_1 + C^4\xi_2$, cp. [10]) this is equivalent to the vanishing of the cubic forms C^3 and C^4. In the following we will use the abbreviation: $\varphi^k_{ji} = \varphi^k_j(X_i)$.

Lemma 1. *If* (x, σ) *is a 1-degenerate parallel surface in* $\mathbb{R}^4$, *then we get for a second order frame field:*

$$\varphi_{21}^1 = 0, \quad \varphi_{22}^1 = 0, \tag{25}$$
$$\nabla_{X_1}^{\perp}\xi_1 = (\varphi_{11}^1 + \varphi_{21}^2)\xi_1, \tag{26}$$
$$\nabla_{X_2}^{\perp}\xi_1 = (\varphi_{12}^1 + \varphi_{22}^2)\xi_1, \tag{27}$$
$$\nabla_{X_1}^{\perp}\xi_2 = 2\varphi_{11}^2\xi_1 + 2\varphi_{11}^1\xi_2, \tag{28}$$
$$\nabla_{X_2}^{\perp}\xi_2 = 2\varphi_{12}^2\xi_1 + 2\varphi_{12}^1\xi_2. \tag{29}$$

Proof. This is a direct consequence of (24), using (4) and $h_{22} = 0$, $h_{12} = \xi_1$ and $h_{11} = \xi_2$ $(h_{ij} := h(X_i, X_j))$. □

Lemma 2. *A 1-degenerate parallel surface in* $\mathbb{R}^4$ *is a ruled surface.*

Proof. $D_{X_2}dx(X_2) = dx(\nabla_{X_2}X_2) + h(X_2, X_2) = \varphi_{22}^2 X_2$. □

Remark. Every ruled surface $x(u, v) = \alpha(u) + v\beta(u)$ in $\mathbb{R}^4$ is k-degenerate ($k \in \{0, 1\}$).

3.3 Further adaption of the frame and the parametrization. We know by now that a 1-degenerate parallel surface in $\mathbb{R}^4$ is a ruled surface where X_2 gives the direction of the ruling. To simplify the computations we would like to find a second order frame field such that $\nabla_{X_2}X_2 = 0$ and such that $\{X_1, X_2\}$ is a Gauss-basis (i. e. $0 = [X_1, X_2]$).

Lemma 3. *For a 1-degenerate parallel surface* (x, σ) *in* $\mathbb{R}^4$ *there exist a frame field* $S = \{v_1, v_2, \xi_1, \xi_2, x\} \in \mathcal{F}^2$ *and local coordinates* (u, v) *such that* $v_1 = dx(\frac{\partial}{\partial u})$, $v_2 = dx(\frac{\partial}{\partial v})$ *and* $D_{\frac{\partial}{\partial v}} dx(\frac{\partial}{\partial v}) = 0$. *Furthermore we can parametrize the surface by* $x(u, v) = \alpha(u) + v\beta(u)$.

Proof. If $S, \tilde{S} \in \mathcal{F}^2$ and $v_i = dx(X_i)$ resp. $\tilde{v}_i = dx(\tilde{X}_i)$, then there exists $A \in H^2$ with $\tilde{X}_1 = aX_1 + bX_2$, $\tilde{X}_2 = cX_2$ and $ac \neq 0$ (cp. (22)). Thus

$$0 = \nabla_{\tilde{X}_2}\tilde{X}_2 \iff X_2(\ln c) = -\varphi_{22}^2$$

$$0 = [\tilde{X}_1, \tilde{X}_2] \iff \begin{cases} X_2(\ln a) = -\varphi_{12}^1, \\ X_1(\ln c) + \frac{b}{a}X_2(\ln c) - \frac{1}{a}X_2(b) = \varphi_{12}^2 - \varphi_{21}^2. \end{cases}$$

□

We call a frame field $S = \{dx(\frac{\partial}{\partial u}), dx(\frac{\partial}{\partial v}), \xi_1, \xi_2, x\} \in \mathcal{F}^2$ with $D_{\frac{\partial}{\partial v}} dx(\frac{\partial}{\partial v}) = 0$, where (u, v) are local coordinates, an *adapted frame field*. Let $x(u, v) = \alpha(u) + v\beta(u)$ be a local parametrization of a 1-degenerate parallel surface in $\mathbb{R}^4$ and $S = \{\alpha' + v\beta', \beta, \xi_1, \xi_2, x\}$ an adapted frame field. (For a function $f(u)$ we write $f' = \frac{\partial}{\partial u} f$.) By Lemma 1 and Lemma 3 the structure equations have the following form:

$$\alpha'' + v\beta'' = D_{\frac{\partial}{\partial u}} dx\big(\tfrac{\partial}{\partial u}\big) = \varphi_{11}^1(\alpha' + v\beta') + \varphi_{11}^2\beta + \xi_2, \tag{30}$$

$$\beta' = D_{\frac{\partial}{\partial v}} dx\big(\tfrac{\partial}{\partial u}\big) = \varphi_{21}^2\beta \qquad +\xi_1, \tag{31}$$

$$D_{\frac{\partial}{\partial v}} dx\big(\tfrac{\partial}{\partial v}\big) = 0, \tag{32}$$

$$D_{\frac{\partial}{\partial u}}\xi_1 = dx(-A_{\xi_1}(\tfrac{\partial}{\partial u})) \qquad +(\varphi_{11}^1 + \varphi_{21}^2)\xi_1, \tag{33}$$

$$D_{\frac{\partial}{\partial v}}\xi_1 = dx\big(-A_{\xi_1}\big(\tfrac{\partial}{\partial v}\big)\big), \tag{34}$$

$$D_{\frac{\partial}{\partial u}}\xi_2 = dx\big(-A_{\xi_2}\big(\tfrac{\partial}{\partial u}\big)\big) \qquad +2\varphi_{11}^2\xi_1 + 2\varphi_{11}^1\xi_2, \tag{35}$$

$$D_{\frac{\partial}{\partial v}}\xi_2 = dx\big(-A_{\xi_2}\big(\tfrac{\partial}{\partial v}\big)\big) \qquad +2\varphi_{21}^2\xi_1. \tag{36}$$

From (31) we get that $\xi_1 = \beta' - \varphi_{21}^2\beta$. Inserted in (33) this gives that $\beta'' - (\varphi_{11}^1 + 2\varphi_{21}^2)\beta'$ must be tangential. Since $dx\big(\frac{\partial}{\partial u}\big)$, $dx\big(\frac{\partial}{\partial v}\big)$ and ξ_1 are linear independent, β'', α', β' and β must be linear dependent. We get two cases:

$$\text{I.}\quad \alpha'(u) = k_1(u)\beta(u) + k_2(u)\beta'(u) + k_3(u)\beta''(u), \tag{37}$$

$$\text{II.}\quad \beta'' \in \operatorname{span}(\beta, \beta'). \tag{38}$$

I. We assume that (37) is true. We will investigate if it is possible to reparametrize the surface such that $\alpha' = \beta''$. If we have coordinates $(\tilde u, \tilde v) \in \tilde U$ and a parametrization $\tilde x(\tilde u, \tilde v) = \tilde\alpha(\tilde u) + \tilde v\tilde\beta(\tilde u)$, then we can reparametrize the surface by a local diffeomorphism $\phi\colon U \to \tilde U$, $\phi(u, v) = (f(u), g(u) + vh(u))$. We get $\tilde x \circ \phi(u, v) =: x(u, v) =: \alpha(u) + v\beta(u)$ with

$$\alpha(u) = \tilde\alpha(f(u)) + g(u)\tilde\beta(f(u)), \tag{39}$$

$$\beta(u) = h(u)\tilde\beta(f(u)), \tag{40}$$

$$h(u)f'(u) \neq 0 \quad \forall u \in U. \tag{41}$$

Obviously the frame $S = (\alpha' + v\beta', \beta, \xi_1, \xi_2)$ is an adapted frame iff $\tilde S = (\tilde\alpha' + \tilde v\tilde\beta', \tilde\beta, \tilde\xi_1, \tilde\xi_2)$ is an adapted frame. (If $J\phi$ is the Jacobi matrix of ϕ, then $S = B\tilde S$ with $B = \left(\begin{array}{cc|c} J\phi & 0 & \\ 0 & \dots & 0 \\ \hline \multicolumn{2}{c|}{0} & 1 \end{array}\right) \in H^2$.)

Using (37) resp. (39), (40) and (37) for $\tilde\alpha'$ $(:= \frac{\partial\tilde\alpha}{\partial\tilde u})$, we obtain for the difference $\beta'' - \alpha'$ the following expression:

$$\begin{aligned}\beta'' - \alpha' &= -k_1\beta - k_2\beta' + (1 - k_3)\beta'' \\ &= (h'' - g' - f'\tilde k_1)\tilde\beta + (2h'f' + hf'' - gf' - f'\tilde k_2)\tilde\beta' + (h(f')^2 - f'\tilde k_3)\tilde\beta''.\end{aligned} \tag{42}$$

Thus $k_3 \equiv 1$ iff $\tilde k_3 \circ f = hf'$. Hence we can find a reparametrization such that $k_3 \equiv 1$ (e. g. $f = \mathrm{id}$, $h = \tilde k_3$) and it stays constant equal one if we restrict to reparametrizations with $h = \frac{1}{f'}$, therefore

$$h' = -\frac{f''}{(f')^2}. \tag{43}$$

Now $k_2 \equiv 0$ iff $\tilde{k}_2 \circ f = -\frac{f''}{(f')^2} - g$. Still we can find such a reparametrization (e. g. $f = \mathrm{id}$, $-\tilde{k}_2 = g$) and k_2 stays constant equal zero if we restrict to reparametrizations with

$$g = -\frac{f''}{(f')^2} = h'. \tag{44}$$

Finally $k_1 \equiv 0$ iff $\tilde{k}_1 \circ f = \frac{1}{f'}(h'' - g') \equiv 0$ (by (44)).

Since our investigations are of local nature, we have to consider two subcases: either $\tilde{k}_1 \equiv 0$ or $\tilde{k}_1(\tilde{u}) \neq 0$ for all $\tilde{u} \in \tilde{U}$. By (42) this is equivalent to either $\beta'' = \alpha'$ or $\beta'' = \alpha' - (f')^2(\tilde{k}_1 \circ f)\beta$. In the second subcase we still can choose f such that $(f')^2(\tilde{k}_1 \circ f) \equiv \pm 1$, thus we have either

$$\alpha' = \beta'' \quad \text{or} \quad \alpha' = \beta'' \pm \beta.$$

Since $x(u, v) = \alpha(u) + v\beta(u)$, we obtain by integration (and if necessary by an affine transformation applied to x) the following two subcases:

$$\text{I.1.} \quad x(u, v) = \gamma'(u) + v\gamma(u) \quad \text{or}$$
$$\text{I.2.} \quad x(u, v) = (\pm\gamma(u) + \gamma''(u)) + v\gamma'(u).$$

II. We assume that $\beta''(u) \in \mathrm{span}(\beta, \beta')(u) \quad \forall u$. Therefore β is a plane curve, contained in the plane spanned by $\beta(u)$ and $\beta'(u)$ for some u. We can reparametrize the surface by a local diffeomorphism $\phi\colon U \to \tilde{U}$, $\phi(u, v) = (f(u), vh(f(u)))$ (cp. the discussion in the first case) such that β is part of an ellipse in $\mathrm{span}(\beta, \beta')$ and, by applying an affine transformation, such that $\beta(u) = (\cos u, \sin u, 0, 0)$, i. e. $\beta'' = -\beta$. We obtain

$$\text{II.} \quad x(u, v) = \alpha(u) + v\beta(u), \quad \beta'' = -\beta.$$

To complete our investigations we need to compute for the three types of ruled surfaces the corresponding transversal bundle σ. This can be done using the structure equations (30) - (36). The computations are lengthy but straightforward. We give only a short outline.

I. Let γ be a smooth function on an open subset of $\mathbb{R}$ such that $G := \det(\gamma, \gamma', \gamma'', \gamma''') \neq 0$. We set

$$L := \ln G \quad \text{and} \quad \gamma^{(4)} =: L'\gamma''' + a\gamma'' + b\gamma' + c\gamma.$$

1. ($x(u, v) = \gamma'(u) + v\gamma(u)$): If we compute the Gauss equations (30) and (31) we obtain

$$\xi_1 = \gamma' - \varphi_{21}^2\gamma, \tag{45}$$

$$\xi_2 = \gamma''' + (v - \varphi_{11}^1)\gamma'' - v\varphi_{11}^1\gamma' - \varphi_{11}^2\gamma. \tag{46}$$

If we differentiate ξ_2 in direction of u and evaluate the Weingarten equation (35), we get:

$$\varphi_{11}^1 = \tfrac{1}{3}(L' + v), \quad \varphi_{11}^2 = \tfrac{1}{3}(b - va + v^2(L' + v)).$$

The Weingarten equation (33) for $\frac{\partial}{\partial u}\xi_1$ finally gives:

$$\varphi_{21}^2 = -\tfrac{1}{6}(L' + 4v).$$

2. $(x(u, v) = (\varepsilon\gamma(u) + \gamma''(u)) + v\gamma'(u), \varepsilon = \pm 1)$: We obtain by (30) and (31) that

$$\xi_1 = \gamma'' - \varphi_{21}^2\gamma', \tag{47}$$

$$\xi_2 = (L' + v - \varphi_{11}^1)\gamma''' + (a + \varepsilon - v\varphi_{11}^1)\gamma'' + (b - \varepsilon\varphi_{11}^1 - \varphi_{11}^2)\gamma' + c\gamma. \tag{48}$$

If we differentiate ξ_2 in direction of u and evaluate the Weingarten equation (35), we get:

$$\begin{aligned}\varphi_{11}^1 &= \tfrac{1}{3}((\ln c + L)' + v),\\ \varphi_{11}^2 &= \tfrac{1}{3}\big\{b + a' - \varepsilon L' - (a + \varepsilon)(\ln c)' + v\big(-L'' + L'(\ln c)' - a - 2\varepsilon\big) + \\ &\qquad + v^2(\ln c + L)' + v^3\big\}.\end{aligned}$$

The Weingarten equation (33) for $\frac{\partial}{\partial u}\xi_1$ finally gives:

$$\varphi_{21}^2 = -\tfrac{1}{6}((\ln c + L)' + 4v).$$

II. Let α, β be smooth functions on an open subset of $\mathbb{R}$ such that $D := \det(\alpha'', \alpha', \beta', \beta) \neq 0$. We set

$$L := \ln D \quad \text{and} \quad \alpha''' =: L'\alpha'' + a\alpha' + b\beta' + c\beta.$$

If we compute the Gauss equations (30) and (31) we obtain

$$\xi_1 = \beta' - \varphi_{21}^2\beta, \tag{49}$$

$$\xi_2 = \alpha'' - \varphi_{11}^1\alpha' - v\varphi_{11}^1\beta' - (v + \varphi_{11}^2)\beta. \tag{50}$$

If we differentiate ξ_2 in direction of u and evaluate the Weingarten equation (35), we get:

$$\varphi_{11}^1 = \tfrac{1}{3}L', \quad \varphi_{11}^2 = \tfrac{1}{3}(b - va - v).$$

The Weingarten equation (33) for $\frac{\partial}{\partial u}\xi_1$ finally gives:

$$\varphi_{21}^2 = -\tfrac{1}{6}L'.$$

Theorem. *Every 1-degenerate parallel affine surface immersion (x, σ) in $\mathbb{R}^4$ is a ruled surface and can be locally parametrized either by*

I.1. $x(u, v) = \gamma'(u) + v\gamma(u)$, *and*
$\sigma = \operatorname{span}(\xi_1, \xi_2)$ *is given by* (45) *and* (46), *or*

I.2. $x(u, v) = (\varepsilon\gamma(u) + \gamma''(u)) + v\gamma'(u)$, $\varepsilon = \pm 1$, *and*
$\sigma = \operatorname{span}(\xi_1, \xi_2)$ *is given by* (47) *and* (48),

II. $x(u, v) = \alpha(u) + v\beta(u)$, $\beta'' = -\beta$, *and*
$\sigma = \operatorname{span}(\xi_1, \xi_2)$ *is given by* (49) *and* (50).

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