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Modular Forms of Rational Weights and Modular Varieties

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Introduction

Recently, BANNAI and others [2], partly motivated by the theory of finite unitary reflection groups, systematically investigated rings of modular forms of one variables which are polynomial rings. In that work, the authors gave a remark that modular forms of rational weights are interesting objects, showing that the ring of modular forms of weights $k/5$ belonging to the principal congruence subgroup of level 5 is generated by two elements of weight 1/5. (This has some connection with KLEIN's work in the 19-th century.) Their work in [2] is a rather "hand-made" case-by-case study. This paper is motivated by their work.

In this paper, for any odd integer $N > 3$, we give some systematic construction of modular forms of one variable of rational weight $(N-3)/2N$ (with a certain multiplier system) belonging to the principal congruence subgroup

$$\Gamma(N) = \left\{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}); a \equiv d \equiv 1 \bmod N,\ b \equiv c \equiv 0 \bmod N\right\}.$$

Here, we shall get $(N-1)/2$ linearly independent forms. Through this, we rewrite their theory of level 5 in a more general context, including the connection between unitary reflection groups and covering groups. We shall also interpret some works of F. KLEIN from the point of view of modular forms of rational weights, determine the rings of modular forms of weight $2k/7$ (k: non-negative integers) for $\Gamma(7)$ and $SL_2(\mathbb{Z})$ exactly, and show their connection with invariant polynomials of unitary reflection group No. 24 in the list in SHEPHARD-TODD [18]. The ring of modular forms of rational weights is also determined for level 9. A similar study seems possible for level 11, judging from KLEIN's work (cf. [5], [6]), but we will give just a very likely candidate for level 11 in this paper, and further investigation will be reported later. By the way, there is no corresponding unitary reflection group in the level 9 or 11 case.

Now, we explain some more technical points of this paper, and also give historical remarks. First, there is no standard automorphy factor for rational weights, so we must explain something about this. For any complex numbers v and s, we take $-\pi < \alpha = \arg(v) \le \pi$ and put $v^s = |v|^s e^{is\alpha}$. Now, the automorphy factor of

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weight $(N-3)/2N$ we consider here for each N is an automorphy factor $J(M,\tau) = v(M)(c\tau+d)^{(N-3)/2N}$ ($M \in \Gamma(N)$, $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$) such that
(1) $v(M)$ is an N-th root of unity, that is, $v(M)^N = 1$ for each $M \in \Gamma(N)$,
(2) $v(M)$ is unramified in the sense of PETERSSON, that is, $v(P) = 1$ for every unipotent element $P \in \Gamma(N)$.

Even under the above conditions (1),(2), the multiplier system is not uniquely determined at all, but the different choices differ only by characters of $\Gamma(N)$. We shall construct $(N-1)/2$ linearly independent modular forms for a certain fixed multiplier system which satisfies the above conditions. More precisely, first we just construct some modular forms on $\Gamma(N)$ having the same weight and the same multiplier system. These are functions obtained from theta constants of some characteristcs in $(2N)^{-1}\mathbb{Z}^2$ divided by some rational power of the Dedekind eta function. Now, although $\Gamma(N)$ is a normal subgroup of $SL_2(\mathbb{Z})$, there is no reason to expect that there exists a natural extension of this multiplier system to $SL_2(\mathbb{Z})$ in general. But, in our case, as a result of our construction, we can see that the multiplier system is naturally prolonged to a multiplier system of $SL_2(\mathbb{Z})$. By virtue of this fact, we can define an action of $SL_2(\mathbb{Z})$ on the ring of modular forms generated by modular forms on $\Gamma(N)$ constructed as above, and we can consider (at least a part of) modular forms on $SL_2(\mathbb{Z})$ of rational weights as invariants of this action. For level 5, we give short simple remarks on the relation between the 5-fold covering of $SL_2(\mathbb{Z})$ and the unitary reflection group G_{600}. This had been essentially known by BANNAI and others, but our explanation is simple and easy. For level 7, we can show that our modular forms generate all the modular forms of weights $2k/7$ with $k \in \mathbb{Z}_{\geq 0}$ with our multiplier system, and for level 7, the invariant polynomials of the unitary reflection group G_{336} (No. 24 of SHEPHARD-TODD [18]) give all modular forms of $SL_2(\mathbb{Z})$ whose weights are in $(4/7)\mathbb{Z}$. Actually, there are three algebraically independent invariant polynomials of G_{336}. One of them is $XY^3 - YZ^3 - ZX^3$, which gives zero after substitution of X, Y, Z by our modular forms. By this, we reprove the famous results of KLEIN that the model of $\Gamma(7)$ is given by the non hyperelliptic quartic curve $XY^3 - YZ^3 - ZX^3 = 0$. Also for $N = 9$, all modular forms of weights $k/3$ with our multiplier systems are generated by our modular forms, and the ring structure is described.

Now, historically, KLEIN treated the theory of modular functions of the principal congruence subgroup. He gave models of modular varieties (at least) for $N = 5$, 7. His results are very close to ours, and he has shown that the abstract quantities which describes the Galois extension of $\mathbb{C}(J)$ are proportional to some theta functions. Also, he considered the action of $SL_2(\mathbb{Z})$ and the invariant form of the action, and his "quantities" can now be regarded as our modular forms of rational weights substantially when $N = 5$ or $N = 7$. But in his time, naturally it seems that he was not conscious at all of modular forms of rational weights, so he did not determine the ring of modular forms in our sense, and his theta functions explained above which come from the "Jacobi equation" seem different from our theta functions and so on. Also, it is far easier to consider the ratio of modular forms or theta functions instead of modular forms themselves, since the subtle behaviour of multiplier systems disappears in the quotient. So we believe that our results and proofs are new and give a new insight to the old work of KLEIN. In a sense, our proof consists of fairly technical calculations and is not trivial at all. Finally, we remark that H. PETERSSON also treated modular forms of complex weights systematically. In particular, he gave a lower bound of the dimension of modular forms of fixed rational weights (of various multiplier systems) by using the Riemann-Roch theorem. The dimension of our space of modular forms is sometimes bigger than this general bound. The exact values of the dimension of this kind of modular forms is not known in general when the weights are small (cf. RANKIN [16] p. 47.)

1 Modular forms of rational weights

1.1 Results. In this section, we state the results on construction of modular forms of rational weights. We denote by H the upper half plane. We take a discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$. For each $M \in \Gamma$, we take a complex number $v(M)$. For a fixed rational number r and any $\tau \in H$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, put $j(M, \tau) = v(M)(c\tau + d)^r$, where we take the principal value for the branch of the rational power. We say that $v(M)$ is a multiplier system, if $j(M_1 M_2, \tau) = j(M_1, M_2\tau) j(M_2, \tau)$ for any $M_1, M_2 \in \Gamma$, and that $j(M, \tau)$ is an automorphy factor of weight r. We say that a holomorphic function $f(\tau)$ on H is a holomorphic modular form of weight r with multiplier system $v(M)$, if $f(M\tau) = f(\tau)v(M)(c\tau + d)^r$ for each $M \in \Gamma$, and f is holomorphic at each cusp of Γ. In some books (e.g. RANKIN [15]), in the definition of multiplier systems, the condition that $j(-1_2, \tau) = 1$ (assuming that $-1_2 \in \Gamma$) is demanded to avoid the case where there exist no non-zero modular forms from the first, but we do not assume this condition for certain reasons. This notion of multiplier systems depends heavily on Γ. For example, if r is an integer, then $(c\tau + d)^r$ is an automorphy factor of the whole group $SL_2(\mathbb{R})$, but for non-integral r, this cannot be true, and it can happen that the multiplier system cannot be extended to any bigger group.

For any complex number z, we put $e(z) = e^{2\pi i z}$. For $\tau \in H$, we put $q = e(\tau/2)$ (which was often used in classical references; and we will not use $q = e(\tau)$ in this paper.) For $m = (m', m'') \in \mathbb{Q}^2$ and $\tau \in H$, $z \in \mathbb{C}$, we define theta functions of characteristic m as usual by

$$\theta_m(\tau, z) = \sum_{p \in \mathbb{Z}} e(\tfrac{1}{2}\tau(p + m')^2 + (p + m')(z + m'')).$$

The theta constants are defined to be $\theta_m(\tau) = \theta_m(\tau, 0)$. The Dedekind eta function $\eta(\tau)$ is by definition

$$\eta(\tau) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n}).$$

Since $\eta(\tau)$ is a nowhere vanishing function on H, we can define $\log \eta(\tau)$ as a single valued function. We fix a branch of $\log \eta(\tau)$ once and for all (for instance, the one which is real for pure imaginary τ). Then we can define $\eta(\tau)^r$ by $e^{r \log(\eta(\tau))}$.

Now we take an odd integer $N > 3$. For each odd r, we put $m = {}^t(\frac{r}{2N}, \frac{1}{2})$. For short, we put $f_r(\tau) = \theta_m(N\tau)$.

Theorem 1.1. *Notation being as above, we have the following results.*
(1) For each odd r with $1 \le r \le N-2$, the functions $f_r(\tau)/\eta(\tau)^{3/N}$ are holomorphic modular forms of $\Gamma(N)$ of rational weight $(N-3)/2N$ with the same multiplier system $v_N(M)$.
(2) These $(N-1)/2$ forms are linearly independent, and any two of them are algebraically independent.
(3) $v_N(M)^N = 1$ for all $M \in \Gamma(N)$, and $f_r(\tau)^N/\eta(\tau)^3$ are usual holomorphic modular forms of $\Gamma(N)$ of integral weight $(N-3)/2$.

Remark. Our multiplier system is unramified in the sense of PETERSSON, that is, $v_N(U) = 1$ for all unipotent elements of $\Gamma(N)$.

We denote by $B_{(N-3)/2N}(\Gamma(N))$ the vector space spanned by the above $(N-1)/2$ modular forms. For the sake of simplicity, we put

$$F_r(\tau) = e\left(\tfrac{(N-1)(r-1)}{4N}\right) f_r(\tau)/\eta(\tau)^{3/N}.$$

For each $M \in SL_2(\mathbb{Z})$, we denote by $v_0(M)$ the multiplier system of $\eta(\tau)^{3/N}$, i.e. $v_0(M)$ is the constant such that $\eta(M\tau)^{3/N} = \eta(\tau)^{3/N} v_0(M)(c\tau + d)^{3/N}$.

Theorem 1.2. *(1) The multiplier system $v_N(M)$ is given by $v_N(M) = v_0(M)^{N^2-1}$ for any $M \in \Gamma(N)$.*
(2) If we put $j(M, \tau) = v_0(M)^{N^2-1}(c\tau + d)^{(N-3)/2N}$ for each $M \in SL_2(\mathbb{Z})$, then this is an automorphy factor of $SL_2(\mathbb{Z})$ which is a prolongation of the automorphy factor of $\Gamma(N)$ considered above. In particular, $j(-1_2, \tau) = (-1)^{(N+1)/2}$.
(3) An action ρ of $SL_2(\mathbb{Z})$ on $B_{(N-3)/2N}(\Gamma(N))$

$$\rho(M)f(\tau) = f(M\tau)j(M, \tau)^{-1}$$

is given on generators of $SL_2(\mathbb{Z})$ by

$$\rho\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)F_r = \sum_{1 \le t \le N-2,\ t\ odd} \frac{1}{\sqrt{N}}\Big(e\left(\tfrac{r-t}{4} + \tfrac{rt}{4N} + \tfrac{3N-1}{8}\right)$$
$$+ (-1)^{(N+1)/2} e\left(-\tfrac{r-t}{4} - \tfrac{rt}{4N} - \tfrac{3N-1}{8}\right)\Big)F_t,$$
$$\rho\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)F_r = e\left(\frac{r^2 - N^2}{8N}\right)F_r,$$
$$\rho(-1_2)F_r = (-1)^{(N+1)/2}F_r.$$

Remark. By virtue of Theorem 1.2 (1), the multiplier system $v_N(M)$ can be expressed in terms of Dedekind sum (cf. e.g. [13]). Namely, for $M \in SL_2(\mathbb{Z})$, put

$$\Phi(M) = \begin{cases} \frac{b}{d} & \text{if } c = 0\,, \\ \frac{a+d}{c} - 12(\operatorname{sgn} c)s(d, |c|) & \text{if } c \neq 0\,, \end{cases}$$

where $s(d, |c|)$ is the Dedekind sum, which is defined by

$$s(h,k) = \sum_{\mu=1}^{k} ((\tfrac{h\mu}{k}))((\tfrac{\mu}{k}))$$

for integers h, k ($k \neq 0$). Here we put

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z}\,, \\ 0 & \text{if } x \in \mathbb{Z}\,. \end{cases}$$

This $\Phi(M)$ is known to be integer valued. For $M \in \Gamma(N)$, we get the following formula

$$v_N(M) = \begin{cases} 1 & \text{if } c = 0\,, \\ e\big(-\frac{(\operatorname{sgn} c)\cdot 3(N^2-1)}{8N}\big)e\big(\frac{N^2-1}{8N}\Phi(M)\big) & \text{if } c \neq 0\,. \end{cases}$$

We will prove the above theorems in the following section.

1.2 Transformation formulas. In this section, we first show that the multiplier systems for $f_r(\tau)/\eta(\tau)^{3/N}$ are all the same, and then we shall show that $v_N(M)^N = 1$ by giving more precise transformation formulas of the functions under $\Gamma(N)$. For these purposes, we review the classical theta transformation formulas. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $m = {}^t(m', m'') \in \mathbb{Q}^2$, we define

$$Mm = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} m' \\ m'' \end{pmatrix} + \frac{1}{2}\begin{pmatrix} cd \\ ab \end{pmatrix}.$$

We also put

$$\phi_m(M) = -\tfrac{1}{2}(m'bdm' + m''acm'' - 2m'bcm'' - ab(dm' - cm'')).$$

Then we get

$$\theta_{Mm}(M(\tau, z)) = \kappa(M)e(\phi_m(M))(c\tau + d)^{1/2}e({}^t z(cz + d)^{-1}cz/2)\theta_m(\tau, z),$$

where $\kappa(M)$ is a constant depending only on M and not on τ, z, or m. We can give a more precise formula for $\kappa(M)$. We use the following notation by PETERSSON. We assume $c, d \in \mathbb{Z}$, $(c, d) = 1$, $c \neq 0$, and d is odd. We put

$$\left(\tfrac{c}{d}\right)^* = \left(\tfrac{c}{|d|}\right),$$
$$\left(\tfrac{c}{d}\right)_* = \left(\tfrac{c}{|d|}\right)(-1)^{(\operatorname{sgn}(c)-1)(\operatorname{sgn}(d)-1)/4}.$$

Here the parenthesis $(*/*)$ of the right hand side is the usual Jacobi symbol. When both c and d are odd, we get

$$\left(\tfrac{d}{|c|}\right) = \left(\tfrac{c}{d}\right)_*(-1)^{(c-1)(d-1)/4}.$$

Also, we put

$$\left(\tfrac{0}{\pm 1}\right)^* = \left(\tfrac{0}{\pm 1}\right)_* = 1.$$

We get

Proposition 1.3. *For each $M \in SL_2(\mathbb{Z})$, $\kappa(M)$ is given as follows.*

$$\kappa(M) = \begin{cases} e\big(\frac{abcd}{4} + \frac{acd^2}{8} - \frac{c}{8}\big)\big(\frac{a}{|c|}\big) & \text{if } c \text{ is odd,} \\ \big(\frac{c}{d}\big)_* e\big(\frac{1}{8}(d-1)\big) = \big(\frac{2c}{d}\big)_* \varepsilon_d^{-1} & \text{if } c \text{ is even.} \end{cases}$$

Here, we put $\varepsilon_d = 1$ or i if $d \equiv 1 \bmod 4$ or $d \equiv 3 \bmod 4$, respectively.

Although most references give the above formula only for M in some smaller (e.g. theta) group, the above proposition should be more or less known classically, and the proof is omitted here. The proof is obtained by using the usual Gaussian sum expression of $\kappa(M)$ and so on.

Lemma 1.4. *We assume N is odd. For any*

$$M' = \begin{pmatrix} a & Nb_0 \\ Nc & d \end{pmatrix} \in \Gamma(N),$$

and for each odd integer r, the number

$$v_1(M') := f_r(M'\tau)/f_r(\tau)(cN\tau + d)^{1/2}$$

is a constant depending only on M' and not on τ and r.

Proof. We put $b = N^2 b_0$. For this b and the same integers a, c, d as above, we put

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

We get

$$M \cdot m = \begin{pmatrix} \frac{dr}{2N} - \frac{c}{2} + \frac{1}{2}cd \\ -\frac{br}{2N} + \frac{a}{2} + \frac{ab}{2} \end{pmatrix} \equiv \begin{pmatrix} \frac{r}{2N} \\ \frac{1}{2} \end{pmatrix} \bmod 1.$$

Indeed, if we put $d = 1 + Nk$, then $\frac{dr}{2N} - \frac{c}{2} + \frac{1}{2}cd = \frac{r}{2N} + \frac{rk}{2} - \frac{c}{2} + \frac{1}{2}cd$. But since N is odd, if $d \equiv 0 \bmod 2$, then $k \equiv 1 \bmod 2$ and hence $kr \equiv 1 \bmod 2$, and vice versa. If d is odd, then $c(1-d)$ is even. But since $ad - bc = 1$, we get that c is even, so $c(1-d)$ is odd. By the above consideration, we always get $rk - c + cd \equiv 0 \bmod 2$. In the same way, if we write $b = Nb_1$, we can show that $-b_1 r + a + ab$ is odd. These imply the above congruence. Hence, we get $\theta_{Mm}(\tau) = e\big(\frac{r}{2N}(-\frac{br}{2N} + \frac{a}{2} + \frac{ab}{2} - \frac{1}{2})\big)\theta_m(\tau)$. Hence, by the theta transformation formula, we get

$$\begin{aligned} f_r(M'\tau) &= \theta_m(M(N\tau)) \\ &= e\big(-\tfrac{r}{2N}\big(-\tfrac{br}{2N} + \tfrac{a}{2} + \tfrac{ab}{2} - \tfrac{1}{2}\big)\big)\theta_{Mm}(M(N\tau)) \\ &= e\big(-\tfrac{r}{2N}\big(-\tfrac{br}{2N} + \tfrac{a}{2} + \tfrac{ab}{2} - \tfrac{1}{2}\big)\big)\kappa(M)e(\phi_m(M))(cN\tau + d)^{1/2} f_r(\tau) \end{aligned}$$

$$\phi_m(M) = -\frac{1}{2}\Big(\frac{bdr^2}{4N^2} + \frac{ac}{4} - \frac{bcr}{2N} - \frac{abdr}{2N} + \frac{abc}{2}\Big).$$

Hence, if we put

$$v_1(M') = f_r(M'\tau)/f_r(\tau)(cN\tau + d)^{1/2},$$

then, writing $a = 1 + kN$, we get

$$v_1(M') = \kappa(M) \times e\Big(\big(-\frac{b_0 d}{8} + \frac{b_0}{4}\big)r^2 + \frac{r}{4}(b_0 N(c + ad - a) - k) - \big(\frac{ac}{8} + \frac{abc}{4}\big)\Big).$$

We can show that $v_1(M')$ does not depend on r. Indeed, since $r^2 \equiv 1 \bmod 8$, we can replace r^2 in the above by 1. The linear terms in r depend only on $r \bmod 4$, so we may assume that $r = 1$ or $r = -1$. Now in order to show that both $r = 1$ and $r = -1$ give the same value in the above, we must show that $b_0 N(c + ad - a) - k \equiv 0 \bmod 2$. This can be shown as follows. If b_0 is even, then a is odd, so k is even, and we get the result. If b_0 is odd, then since we assumed that N is odd, we get $b_0 N(c + ad - a) + k \equiv c + ad - a - k \bmod 2$. If moreover a is odd, then k is even. Since a and b are odd in this case, we get $c + d - 1 \equiv 0 \bmod 2$ by $ad - bc = 1$. If a is even, then k and c are odd. This implies the claim and hence also the lemma. □

We obtain a more precise formula for $v_1(M')$ for $M' \in \Gamma(N)$ as follows.

Lemma 1.5.

$$v_1(M') = \begin{cases} \big(\frac{d}{|c|}\big)e\big(\frac{1}{8}(c(a + d - 3)\big) & \textit{if } c \textit{ is odd,} \\ \big(\frac{c}{d}\big)_* e\big(\frac{(1-2N)(d-1)}{8} + \frac{d(b-c)}{8}\big) & \textit{if } c \textit{ is even.} \end{cases}$$

Proof. Even though $\kappa(M)$ is explicitly known, the proof of this lemma is not trivial. When $c = 0$, the result is easily obtained by direct calculation, so we assume that $c \neq 0$. First we assume that c is odd. We see

$$v_1(M') = \Big(\frac{d}{|c|}\Big)e\Big(-\frac{b_0 d}{8} + \frac{b_0}{4} + \frac{1}{4}b_0 N(c + ad - a) + \frac{1-a}{4N} - \frac{ac}{8} - \frac{ab_0 c}{4} + \frac{abcd}{4} + \frac{acd^2}{8} - \frac{c}{8}\Big).$$

Gathering the parts where the denominator is 8, and noting the relations $N^{-1} \equiv N \bmod 8$, $b = b_0 N^2 \equiv b_0 \bmod 8$, $ad = bc + 1$, and $c^2 \equiv 1 \bmod 8$, we get

$$\tfrac{1}{8}(-bd - ac + acd^2 - c) \equiv \tfrac{1}{8}c(d + a - 3) + \tfrac{1}{4}c(1 - a) \mod 1.$$

So, we see that $v_1(M') \times \big(\frac{d}{|c|}\big)e(-c(a + d - 3)/8)$ is equal to the exponential of the following number

$$\begin{aligned} &\tfrac{1}{4}(b + bN(c + ad - a) + N(1 - a) - abc + abcd + c(1 - a)) \\ &\equiv \tfrac{1}{4}(b(1 + b)(1 + Nc) + (1 - a)(b + 1)(c + N)) \bmod 1. \end{aligned}$$

But, since $b(b + 1) \equiv 1 + Nc \equiv c + N \equiv 0 \bmod 2$, and a and b cannot be odd at the same time, we get $(1 - a)(b + 1) \equiv 0 \bmod 4$. So, the exponential of this number is one, and we have proved the result. Next, we assume that c is even. Noting that $n^2 \equiv 1 \bmod 8$, we get

$$\begin{aligned} v_1(M') = &\big(\tfrac{c}{d}\big)_* e\big(\tfrac{(d-1)}{8}\big) \\ &\times e\big(-\tfrac{bd}{8} + \tfrac{b}{4} + \tfrac{N}{4}b(c + ad - a) + \tfrac{N}{4}(1 - a) - \tfrac{ac}{8} - \tfrac{abc}{4}\big). \end{aligned}$$

Since $ad = bc + 1$,

$$\tfrac{N}{4}b(c+ad-a) = \tfrac{N}{4}b(c+bc+1-a) = \tfrac{Ncb(b+1)}{4} + \tfrac{N}{4}b(1-a)$$
$$\equiv \tfrac{N}{4}b(1-a) \bmod 1.$$

Also by $abc + a = a^2d \equiv d \bmod 8$, we get

$$\tfrac{N}{4}(1-a) + \tfrac{N}{4}b(1-a) = \tfrac{N}{4}(1+b)(1-a) = \tfrac{N}{4}(1+b)(1-d+abc)$$
$$= \tfrac{N(1+b)(1-d)}{4} + \tfrac{Nabc(1+b)}{4},$$
$$-\frac{ac}{8} = \frac{abc^2}{8} - \frac{cd}{8}.$$

Besides, since c is even and hence b is odd, we get easily that $c(2bN+c) \equiv 0 \bmod 8$, so

$$\frac{Nabc(1+b)}{4} + \frac{abc^2}{8} - \frac{abc}{4}$$
$$= \frac{abc(N-1)}{4} + \frac{abc(2bN+c)}{8} \equiv 0 \bmod 1.$$

Hence we get

$$v_1(M') = \left(\tfrac{c}{d}\right)_* e\left(\tfrac{(d-1)}{8}\right)$$
$$\times e\left(-\tfrac{bd}{8} + \tfrac{b}{4} + \tfrac{N(1+b)(1-d)}{4} - \tfrac{cd}{8}\right).$$

Finally we obtain

$$\tfrac{d-1}{8} - \tfrac{bd}{8} + \tfrac{b}{4} + \tfrac{N(1+b)(1-d)}{4} - \tfrac{cd}{8}$$
$$= \tfrac{(1-2N)(d-1)}{8} - \tfrac{d(b-c)}{8}$$
$$= \tfrac{1}{4}b(N-1)(d-1) \equiv 0 \bmod 1.$$

Hence we get the desired result and the lemma is proved. □

If we denote the multiplier system of $\eta(\tau)^{3/N}$ by $v_0(M)$, then we have shown that $F_r(\tau)$ are modular forms of $\Gamma(N)$ of weight $(N-3)/2N$ with the same multiplier system $v_N(M) = v_1(M)/v_0(M)$. The holomorphy at cusps will be shown later. Here, we show the following lemma.

Lemma 1.6. *For any $M \in \Gamma(N)$, we have $v_N(M)^N = 1$.*

To prove this lemma, we need the well known formula of the multiplier system of $\eta(\tau)$ (cf. e.g. [12], [14] p.163, [9] p.51). For any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we have

$$\eta(M\tau) = v(M)(c\tau + d)^{1/2}\eta(\tau),$$

where

$$v(M) = \begin{cases} \left(\frac{d}{c}\right)^* \exp\left(\frac{\pi i}{12}((a+d-bdc-3)c+bd)\right) & \text{if } c \text{ is odd,} \\ \left(\frac{c}{d}\right)_* \exp\left(\frac{\pi i}{12}((a+d-bdc-3d)c+bd+3d-3)\right) & \\ = \left(\frac{c}{d}\right)_* \exp\left(\frac{\pi i}{12}((a-2d-bdc)c+bd+3d-3)\right) & \text{if } c \text{ is even.} \end{cases}$$

By definition, both $v_0(M)^N$ and $v(M)^3$ are the multiplier system of $\eta(\tau)^3$, so we have $v_0(M)^N = v(M)^3$. By the formula for $v(M)$ given above, we can calculate $v_0(M)^N$. As before, we put

$$M = \begin{pmatrix} a & N^2b_0 \\ c & d \end{pmatrix} \qquad M' = \begin{pmatrix} a & Nb_0 \\ Nc & d \end{pmatrix} \in \Gamma(N).$$

In case where c is odd, we get $c^2 \equiv 1 \bmod 8$ and

$$\begin{aligned} v_0(M')^N &= \left(\tfrac{d}{Nc}\right)^* e\left(\tfrac{1}{8}((a+d-N^2b_0dc-3)Nc + Nb_0d\right) \\ &= \left(\tfrac{d}{Nc}\right)^* e\left(\tfrac{1}{8}(a+d-3)Nc\right) = v_0(M)^N \end{aligned}$$

by Lemma 1.5. In case c is even, we note

$$a - 2d - bcd = a - 2d - (ad-1)d = a - 2d - ad^2 + d \equiv -d \bmod 8,$$

and $b = b_0N^2 \equiv b_0 \bmod 8$. Since $d \equiv 1 \bmod N$, we get

$$\begin{aligned} \left(\tfrac{N}{|d|}\right) &= \left(\tfrac{|d|}{N}\right)(-1)^{(N-1)(|d|-1)/4} = \left(\tfrac{\mathrm{sgn}(d)}{N}\right)(-1)^{(N-1)(\mathrm{sgn}(d)d-1)/4} \\ &= (-1)^{(N-1)(d-1)/4}. \end{aligned}$$

Hence we get

$$\begin{aligned} v_0(M')^N &= \left(\tfrac{Nc}{d}\right)_* e\left(\tfrac{3}{8}(d-1) + \tfrac{1}{8}Nb_0d + \tfrac{1}{8}(a-2d-bcd)Nc\right) \\ &= \left(\tfrac{c}{d}\right)_* e\left(\tfrac{1}{8}(N-1)(d-1) + \tfrac{3}{8}(d-1) + \tfrac{1}{8}(b-c)dN\right) \end{aligned}$$

Hence by Lemma 1.5, we get $v_1(M')^N = v_0(M')^N$. □

When $1 \le r \le N-2$, the functions $F_r(\tau)$ are linearly independent. Indeed, for a fixed r with $1 \le r \le N-2$, the minimum in $(p + r/2N)^2$ for $p \in \mathbb{Z}$ is $(r/2N)^2$. So, for each r, the q-expansion starts from non-zero constant times $q^{(r^2-1)/8N}$. In particular, $F_r(\tau)$ is not identically zero. If there exists a linear relation $\sum_{1\le r\le N-2,\ r:odd} c_rF_r = 0$, then the first term of the q-expansion of the left hand side must vanish. The first possible term of the q-expansion is $q^{(r^2-1)/8N}$ for $r=1$ and this term appears only in $F_1(\tau)$. Hence we get $c_1 = 0$. Successively, we get $c_3 = 0$, $c_5 = 0$, and so on. So, the functions $F_r(\tau)$ are linearly independent. Now we take odd numbers r_1, r_2 with $1 \le r_1 < r_2 \le N-2$. Since it is well known that a ring of modular forms is a graded ring (see e.g. [8] p.112 Lemma 12), any algebraic relation between F_{r_1} and F_{r_2} reduces to a homogeneous relation. But a homogeneous relation implies that F_{r_1}/F_{r_2} should be a fixed complex number (a root of a polynomial equation of one variable). This is obviously false, so F_{r_1} and F_{r_2} are algebraically independent. In order to complete the proof of Theorem 1.1, we must show the holomorphy of $F_r(\tau)$ at each cusp of $\Gamma(N)$. This is a corollary of Theorem 1.2 as explained below. The holomorphy at cusps means that $F_r(M\tau)(c\tau + d)^{-(N-3)/2N}$ has a q-expansion only with positive exponents. But Theorem 1.2 implies that the above function is a linear combination of $F_{r'}(\tau)$ which has of course a positive q-expansion.

We shall now prove Theorem 1.2. For $M \in \Gamma(N)$, the multiplier $v_1(M)$ is an 8-th root of unity, as we can see from the explicit formula we gave. We have shown that $v_1(M)^N = v_0(M)^N$. Hence $v_0(M)^{N^2} = v_1(M)^{N^2} = v_1(M)$, and $v_N(M) = v_1(M)/v_0(M) = v_0(M)^{N^2-1}$. Since $v_0(M)$ is the multiplier system attached to $\eta(\tau)^{3/N}$, $v_N(M)$ is the multiplier system attached to $\eta(\tau)^{3(N^2-1)/N}$. So, for any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we see that $v_0(M)^{N^2-1}(c\tau+d)^{3(N^2-1)/2N}$ is an automorphy factor of $SL_2(\mathbb{Z})$, which takes value 1 for $M = -1_2$. But $(c\tau+d)^{3(N^2-1)/2N} = (c\tau+d)^{(N-3)/2N}(c\tau+d)^{(3N-1)/2}$ and $(c\tau+d)^{(3N-1)/2}$ is of course an automorphy factor, so we see that $j(M,\tau) = v_0(M)^{N^2-1}(c\tau+d)^{(N-3)/2N}$ is also a factor of automorphy of $SL_2(\mathbb{Z})$, which is a natural prolongation of the automorphy factor of $F_r(\tau)$ of $\Gamma(N)$. Also, we get $j(-1_2,\tau) = (-1)^{(3N-1)/2} = (-1)^{(N+1)/2}$. So we have proved (2) of the theorem. In particular, if $N \equiv 1 \bmod 4$, there exists no non-zero modular form of $SL_2(\mathbb{Z})$ attached to $j(M,\tau)$. Anyway, the above result implies that $\rho(M)f(\tau) = f(M\tau)j(M,\tau)^{-1}$ is an action of $SL_2(\mathbb{Z})$. For the basis $F_r(\tau)$, we can calculate $\rho\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\right)$ directly from the definition. Now we calculate $\rho(M_0)$ for $M_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. By the theta transformation formula, we get

$$f_r(-\tau^{-1}) = \left(\tfrac{\tau}{Ni}\right)^{1/2} \sum_{p\in\mathbb{Z}} e\left(\tfrac{1}{2N}\left(p-\tfrac{1}{2}\right)^2\tau + \tfrac{pr}{2N}\right).$$

If we put $p = Np_0 + s$, $(p_0 \in \mathbb{Z}, s = 1, 2, \ldots, N)$, we get

$$f_r(-\tau^{-1}) = \left(\tfrac{\tau}{Ni}\right)^{1/2} \sum_{s=1}^{N} \sum_{p_0\in\mathbb{Z}} e\left(\tfrac{N}{2}\left(p_0+\tfrac{2s-1}{2N}\right)^2\tau + \tfrac{(Np_0+s)r}{2N}\right).$$

For $s = (N+1)/2$, we get $p_0 + \frac{2s-1}{2N} = p_0 + \frac{1}{2} = -(-p_0 - 1 + \frac{1}{2})$, and since $e((-p_0-1)/2) = -e(p_0/2)$, the summation over $p_0 \in \mathbb{Z}$ vanishes in this case. For the other s, comparing s with $N-s+1$, we see

$$\begin{aligned} &p_0 + \tfrac{2N-2s+1}{2N} = -\left(-p_0 - 1 + \tfrac{2s-1}{2N}\right) \text{ and} \\ &(Np_0 + N - s + 1)r/2N \equiv (N(-p_0-1)+s)r/2N + r/2N - sr/N \bmod 1, \end{aligned}$$

so we get

$$\begin{aligned} &\sum_{s=(N+3)/2}^{N} \sum_{p_0\in\mathbb{Z}} e\left(\frac{N}{2}\left(p_0+\frac{2s-1}{2N}\right)^2\tau + \frac{(Np_0+s)r}{2N}\right) \\ &= \sum_{s=1}^{(N-1)/2} \sum_{p_0\in\mathbb{Z}} e\left(\frac{N}{2}\left(p_0+\frac{2N-2s+1}{2N}\right)^2\tau + \frac{(Np_0+N-s+1)r}{2N}\right) \\ &= \sum_{s=1}^{(N-1)/2} e\left(\frac{r}{2N} - \frac{sr}{N}\right) \sum_{p_0\in\mathbb{Z}} e\left(\frac{N}{2}\left(p_0+\frac{2s-1}{2N}\right)^2\tau + \frac{(Np_0+s)r}{2N}\right). \end{aligned}$$

Now if we put $t = 2s-1$, we have

$$(Np_0+s)r/2N = \tfrac{1}{2}\left(p_0 + \tfrac{t}{2N}\right) + \tfrac{(r-1)(t+1)}{4N} + \tfrac{1}{4N}.$$

By these calculations, we get

$$f_r(-\tau^{-1}) = \frac{e(-1/8)\sqrt{\tau}}{\sqrt{N}} \times \sum_{\substack{1\le t\le N-2 \\ t\,\mathrm{odd}}} \Big(e\big(\tfrac{(t+1)(r-1)}{4N} + \tfrac{1}{4N}\big) + e\big(-\tfrac{(t+1)(r+1)}{4N} + \tfrac{r}{2N} + \tfrac{1}{4N}\big)\Big) f_t(\tau).$$

On the other hand, we get

$$\eta(-\tau^{-1})^{3/N} = e\big(-\tfrac{3}{8N}\big)(\tau)^{3/N}\eta(\tau)^{3/N}.$$

Hence $v_0(M_0) = e(-3/8N)$ and $v_0(M_0)^{N^2-1} = e(-3/8N)e(3N/8)$. Noting that $e((r-t)/4) = e((t-r)/4)$, and adjusting constants for $F_t(\tau)$, we get an explicit form of $\rho(M_0)$ as in the theorem. Thus we have proved Theorem 1.2 and also Theorem 1.1. □

1.3 Multiplier sytems and dimensions. As is well known, the multiplier system of $\log(\eta(\tau))$ is described by the Dedekind sum $s(d,c)$. By using this, we get the $v_N(M)$ is an unramified multiplier system in the sense of PETERSSON. That is, we get $v_N(U) = 1$ for any unipotent elements of $\Gamma(N)$. By this fact and the Riemann-Roch Theorem, we get the dimension formula for $A_{k(N-3)/2N}(\Gamma(N))$ for sufficiently big k. Namely we get

Lemma 1.7. *For any odd integer $N > 3$ and any integer $k > 4(N-6)/(N-3)$, we have*

$$\dim A_{k(N-3)/2N}(\Gamma(N)) = \frac{kN^2(N-3)}{48}\prod_{p|N}\Big(1-\frac{1}{p^2}\Big) - \frac{N^2(N-6)}{24}\prod_{p|N}\Big(1-\frac{1}{p^2}\Big).$$

The proof is standard and will be omitted here.

Examples.

$$\begin{aligned}
\dim A_{k/5}(\Gamma(5)) &= k+1 && k \ge 0,\\
\dim A_{2k/7}(\Gamma(7)) &= 4k-2 && k \ge 2,\\
\dim A_{k/3}(\Gamma(9)) &= 9k-9 && k \ge 3,\\
\dim A_{4k/11}(\Gamma(11)) &= 20k-25 && k \ge 3,\\
\dim A_{5k/13}(\Gamma(13)) &= 35k-49 && k \ge 3.
\end{aligned}$$

As we will see later, for small values of k, we get

$$\dim A_{2/7}(\Gamma(7)) = 3, \quad \dim A_{1/3}(\Gamma(9)) = 4, \quad \dim A_{2/3}(\Gamma(9)) = 10.$$

2 Examples for levels N = 5, 7, 9, 11.

2.1 The case N = 5: the covering group and G_{600}. When $N = 5$, if we put $\zeta = e(\frac{1}{5})$, we get

$$\rho\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)=\frac{1}{\sqrt{5}}\begin{pmatrix}\zeta^4-\zeta & \zeta^2-\zeta^3\\ \zeta^2-\zeta^3 & \zeta-\zeta^4\end{pmatrix},$$

$$\rho\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right)=\begin{pmatrix}\zeta^2 & 0\\ 0 & \zeta^3\end{pmatrix},$$

$$\rho\left(\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)\right)=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

By these, we get

$$\rho\left(\left(\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right)\right)=\frac{1}{\sqrt{5}}\begin{pmatrix}\zeta^2-\zeta^4 & \zeta^4-1\\ 1-\zeta & \zeta^3-\zeta\end{pmatrix}.$$

So, if we put

$$S_1=\rho\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right),\qquad T_1=\rho\left(\left(\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right)\right),$$

then S_1 and T_1, together with $\lambda 1_2$ (λ is a 10-th primitive root of unity.) generate the unitary reflection group G_{600}, which is No. 16 in SHEPHARD and TODD (cf. [18] p. 282).

We shall see the relation between G_{600} and the (unique non-trivial topological) 5-th covering group $\tilde{G}$ of $SL_2(\mathbb{R})$. The group $\tilde{G}$ is a central extension of $SL_2(\mathbb{R})$.

As a realization of $\tilde{G}$, we take the following one (cf. YOSHIDA [19]). For $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{R})$ and a natural number l with $1\le l\le 5$, we choose x by $c\tau+d=|c\tau+d|e^{ix_l}$, $2\pi(l-1)/5\le x_l\le 2\pi l/5$, and put $\mu_l(g,\tau)=|c\tau+d|^{1/5}e^{ix_l/5}$. Then

$$\tilde{G}=\left\{\left(g,\mu_l(g,\tau)\right);\ g\in SL_2(\mathbb{R}),\ 1\le l\le 5\right\}.$$

Here the group structure is defined by

$$\left(g_1,\mu_{l_1}(g_1,\tau)\right)\left(g_2,\mu_{l_2}(g_2,\tau)\right)=\left(g_1g_2,\mu_{l_1}(g_1,g_2\tau)\mu_{l_2}(g_2,\tau)\right).$$

The projection $pr:\tilde{G}\ni\left(g,\phi(g,z)\right)\to g\in SL_2(\mathbb{R})$ gives the covering map. We take the factor of automorphy $j(M,\tau)=v_5(M)(c\tau+d)^{1/5}$ of $\Gamma(5)$ as in Theorem 1.1. Then $\Gamma(5)\ni M\to(M,j(M,\tau))\in\tilde{G}$ gives an injective isomorphism. This isomorphism is prolonged to $SL_2(\mathbb{Z})$ by virtue of Theorem 1.2. We use the same notation $\Gamma(5)$ and $SL_2(\mathbb{Z})$ for these images in $\tilde{G}$. We denote by $\tilde{\Gamma}$ the pull back of $SL_2(\mathbb{Z})$ in $\tilde{G}$ by the covering map $pr:\tilde{G}\to SL_2(\mathbb{R})$. It is obvious that this is the trivial 5-th covering of $SL_2(\mathbb{Z})$ and $\tilde{\Gamma}\cong SL_2(\mathbb{Z})\times\mathbb{Z}/5\mathbb{Z}$. It is quite obvious from the definition of the group multiplication that the group $\tilde{G}$ acts on the functions $f(\tau)$ on H by $f(\tau)\to f(g\tau)\phi(g,z)^{-1}$ ($(g,\phi(g,z))\in\tilde{G}$.) By virtue of Theorem 1.2, the linear space V over $\mathbb{C}$ spanned by $F_1(\tau)$ and $F_3(\tau)$ is an invariant subspace of the action of $\tilde{\Gamma}$. The group $\Gamma(5)$, regarded as a (normal) subgroup of $\tilde{\Gamma}$, acts trivially on V. Hence $\tilde{\Gamma}/\Gamma(5)$ acts on V. The action is obtained as the action of $SL_2(\mathbb{Z})$ in Theorem 1.2 and the scalar action of 5-th root of unity. By comparing the explicit action with generators of G_{600} given in SHEPHARD-TODD [18] p. 282, we see easily that

Proposition 2.1.

$$G_{600} \cong pr^{-1}(SL_2(\mathbb{Z}))/\Gamma(5).$$

This proposition was known in [2], but our explanation seems clearer. They also observed that modular forms of $SL_2(\mathbb{Z})$ of integral weights are obtained as an invariant polynomial of G_{600} as an obvious application of the theory of unitary reflection group.

By the way, for $N = 5$, BANNAI et al. determined the ring of modular forms of $\Gamma(5)$ of weights $k/5$. Their result is essentially as follows.

Proposition 2.2. ([2]) *Let $v_5(M)$ the multiplier system of $\Gamma(5)$ given in Theorem* 1.1 *and $A^{(1/5)}(\Gamma(5))$ be the ring of holomorphic modular forms of $\Gamma(5)$ of automorphy factor $v_5(M)^k(c\tau + d)^{k/5}$ (k non-negative integers). Then*

$$A_{1/5}(\Gamma(5)) = \mathbb{C}[F_1, F_3],$$

and F_1, F_3 are algebraically independent.

For the details, see [2].

2.2 The case of level N = 7; the graded ring. Here we treat the case of $N = 7$. We put

$$\begin{aligned} x &= e(-1/28)f_1(\tau)\eta(\tau)^{-3/7} = \eta(\tau)^{-3/7}e(-1/28)\theta_{(\frac{1}{14},\frac{1}{2})}(7\tau) \\ &= f_0^{-3/7}\sum_{p\in\mathbb{Z}}(-1)^p q^{7p^2+p}, \\ y &= e(-3/28)f_3(\tau)\eta(\tau)^{-3/7} = \eta(\tau)^{-3/7}e(-3/28)\theta_{(\frac{3}{14},\frac{1}{2})}(7\tau) \\ &= f_0^{-3/7}\sum_{p\in\mathbb{Z}}(-1)^p q^{7p^2+3p}, \\ z &= e(-5/28)f_5(\tau)\eta(\tau)^{-3/7} = \eta(\tau)^{-3/7}e(-5/28)\theta_{(\frac{5}{14},\frac{1}{2})}(7\tau) \\ &= f_0^{-3/7}\sum_{p\in\mathbb{Z}}(-1)^p q^{7p^2+5p} \end{aligned}$$

Here, we set $q = e^{\pi i\tau}$ (and not $e^{2\pi i\tau}$). If we write

$$f_0(\tau) = \prod_{n=1}^{\infty}(1 - q^{2n}),$$

we get

$$\begin{aligned} xf_0^{3/7} &= 1 - q^6 - q^8 + q^{26} + q^{30} - q^{60} - q^{66} + \cdots, \\ yf_0^{3/7} &= q^{2/7}(1 - q^4 - q^{10} + q^{22} + q^{34} - q^{54} - q^{72} + \cdots), \\ zf_0^{3/7} &= q^{6/7}(1 - q^2 - q^{12} + q^{18} + q^{38} - q^{48} - q^{78} + \cdots). \end{aligned}$$

For any non-negative integer k, we denote by $A_{2k/7}(\Gamma(7))$ the vector space of holomorphic modular forms of $\Gamma(7)$ of automorphy factor $v_7(M)^k(c\tau+d)^{2k/7}$. We define a graded ring $A^{(2/7)}(\Gamma(7))$ by

$$A^{(2/7)}(\Gamma(7)) = \bigoplus_{k=0}^{\infty} A_{2k/7}(\Gamma(7)).$$

Proposition 2.3. *Any two among x, y, z are algebraically independent. They satisfy the following algebraic relation*

$$zx^3 + yz^3 = xy^3,$$

and we have

$$\begin{aligned} A^{(2/7)}(\Gamma(7)) &= \mathbb{C}[x,y,z] \\ &= \mathbb{C}[x,z] \oplus y\mathbb{C}[x,z] \oplus y^2\mathbb{C}[x,z] \oplus y^3\mathbb{C}[y,z] \\ &\cong \mathbb{C}[X,Y,Z]/(X^3Z + YZ^3 - XY^3), \end{aligned}$$

where $\oplus$ means the direct sum as modules and X, Y, Z are independent variables. In particular, the subring generated by monomials $x^a y^b z^c$ with $a+b+c \equiv 0 \bmod 7$ is equal to the space of usual modular forms of $\Gamma(7)$ of even integral weights. The generating function of dimensions is given by

$$\sum_{k=0}^{\infty} \dim A_{2k/7}(\Gamma(7)) t^{2k/7} = \frac{1 + t^{2/7} + t^{4/7} + t^{6/7}}{(1 - t^{2/7})^2}$$

Remark. The generating function for even integral weights is well known and given by

$$\sum_{k=0}^{\infty} \dim A_{2k}(\Gamma(7)) t^{2k} = \frac{1 + 24t^2 + 3t^4}{(1 - t^2)^2}.$$

The proof consists of several steps.

First we prove the relation. We can prove this by well known theta relation of WEIERSTRASS as suggested by KLEIN [6], but here we prefer to give a direct elementary proof. We denote by $a_1(l)$, $a_2(l)$, $a_3(l)$ the coefficient at q^l of the q-expansion of each function

$$q^{-6/7}\left(xf_0^{3/7}\right)^3\left(zf_0^{3/7}\right),\ q^{-6/7}\left(yf_0^{3/7}\right)\left(zf_0^{3/7}\right)^3,\ \text{or}\ q^{-6/7}\left(xf_0^{3/7}\right)\left(yf_0^{3/7}\right)^3,$$

respectively. We write

$$\begin{aligned} S_1(l) = \{&(a,b,c,d) \in \mathbb{Z}^4; \\ &2 + 7a^2 + 3a + 7b^2 + 5b + 7c^2 + 5c + 7d^2 + 5d = l\} \\ = \{&(a,b,c,d) \in \mathbb{Z}^4; \\ &(14a+3)^2 + (14b+5)^2 + (14c+5)^2 + (14d+5)^2 = 28(l+1)\}, \end{aligned}$$

$$
\begin{aligned}
S_2(l) &= \{(x,y,z,w) \in \mathbb{Z}^4;\ 7x^2+x+7y^2+y+7z^2+z+7w^2+5w = l\} \\
&= \{(x,y,z,w) \in \mathbb{Z}^4; \\
&\qquad (14x+1)^2+(14y+1)^2+(14z+1)^2+(14w+5)^2 = 28(l+1)\} \\
S_3(l) &= \{(p,q,r,s) \in \mathbb{Z}^4;\ 7p^2+p+7q^2+3q+7r^2+3r+7s^2+3s = l\} \\
&= \{(p,q,r,s) \in \mathbb{Z}^4; \\
&\qquad (14p+1)^2+(14q+3)^2+(14r+3)^2+(14s+3)^2 = 28(l+1)\}.
\end{aligned}
$$

By definition, we get $a_i(l) = \sum_{(a,b,c,d)\in S_i(l)}(-1)^{a+b+c+d}$. and we must show $a_1(l)+a_2(l) = a_3(l)$ for each l . We can show this by comparing elements of S_i directly. (The underlying reason for the equality can be understood by considering the quaternion algebra with discriminant 2∞ and a multiplication by units in the maximal order, but we do not explain the background here.)

Put

$$
\begin{aligned}
S_3^*(l) &= \{(p,q,r,s) \in S_3(l);\ p+q+r+s \equiv 0 \bmod 2\}, \\
S_2^*(l) &= \{(x,y,z,w) \in S_2(l);\ x+y+z+w \equiv 0 \bmod 2\}.
\end{aligned}
$$

Then it is easy to show that the mapping

$$
x = \tfrac{p+q+r+s}{2} \qquad y = \tfrac{-p+q-r+s}{2}
$$
$$
z = \tfrac{-p+q+r-s}{2} \qquad w = \tfrac{-p-q+r+s}{2}
$$

gives a bijection of $S_3^*(l)$ onto $S_2^*(l)$. In the same way, put

$$
\begin{aligned}
S_3^{**}(l) &= \{(p,q,r,s) \in S_3(l);\ p+q+r+s \equiv 1 \bmod 2\}, \\
S_1^{**}(l) &= \{(a,b,c,d) \in S_1(l);\ a+b+c+d \equiv 1 \bmod 2\}.
\end{aligned}
$$

Then the mapping

$$
a = \tfrac{p-q-r-s-1}{2} \qquad b = \tfrac{-p-q-r+s-1}{2}
$$
$$
c = \tfrac{-p+q-r-s-1}{2} \qquad d = \tfrac{-p-q+r-s-1}{2}
$$

gives a bijection of $S_3^{**}(l)$ onto $S_1^{**}(l)$. Finally, we put

$$
\begin{aligned}
S_2^{**}(l) &= \{(x,y,z,w) \in S_2(l);\ x+y+z+w \equiv 1 \bmod 2\}, \\
S_1^{*}(l) &= \{(a,b,c,d) \in S_1(l);\ a+b+c+d \equiv 0 \bmod 2\}.
\end{aligned}
$$

Then the mapping

$$
a = \tfrac{-x-y-z-w-1}{2} \qquad b = \tfrac{-x+y-z+w-1}{2}
$$
$$
c = \tfrac{-x+y+z-w-1}{2} \qquad d = \tfrac{-x-y+z+w-1}{2}
$$

gives a bijection of $S_2^{**}(l)$ onto $S_1^*(l)$. Taking the sign into account, we have proved the relation.

Now, we can show that essentially there are no other relations. Indeed, since $XY^3 - YZ^3 - ZX^3$ is irreducible, this generates a prime ideal $\mathfrak{p}$ in $\mathbb{C}[X,Y,Z]$.

Since the height of $\mathfrak{p}$ is at least one, any prime ideal which properly contains $\mathfrak{p}$ has height greater than 1. Since $\mathbb{C}[x, y, z]$ has transcendental degree two, the ideal of relations of x, y, z must coincide with $\mathfrak{p}$. The module structure is almost a direct consequence of this fact. (These results can be proved also in more elementary direct way, but it is omitted here.) Finally we prove that all modular forms of $\Gamma(7)$ of automorphy factor $v_7(M)^k(c\tau+d)^{2k/7}$ belong to $\mathbb{C}[x, y, z]$. We put

$$A'_{2k/7}(\Gamma(7)) = A_{2k/7}(\Gamma(7)) \cap \mathbb{C}[x, y, z].$$

We would like to show $A'_{2k/7}(\Gamma(7)) = A_{2k/7}(\Gamma(7))$. From our expression of $\mathbb{C}[x, y, z]$ for a direct sum, we easily get

$$\sum_{k=0}^{\infty} \dim A'_{2k/7}(\Gamma(7))t^{2k/7} = \frac{1+t^{2/7}+t^{4/7}+t^{6/7}}{(1-t^{2/7})^2}.$$

Hence we get the dimensions of modular forms of even integral weights in $\mathbb{C}[x, y, z]$ as follows:

$$\sum_{k=0}^{\infty} \dim A'_{2k}(\Gamma(7))t^{2k} = \frac{1+24t^2+3t^4}{(1-t^2)^2}.$$

This coincides with the well-known formula of $\dim A_{2k}(\Gamma)$ and we get $A'_{2k}(\Gamma(7)) = A_{2k}(\Gamma(7))$. So, we assume that $A'_{2k/7}(\Gamma(7)) = A_{2k/7}(\Gamma(7))$ for any k and show that $A'_{(2k-2)/7}(\Gamma(7)) = A_{(2k-2)/7}(\Gamma(7))$. Take $g \in A_{(2k-2)/7}(\Gamma(7))$. Then yg, $zg \in A_{2k/7}(\Gamma(7)) = A'_{2k/7}(\Gamma(7))$, so for some polynomials A_i, B_i ($1 \le i \le 4$) of two variables, we get

$$yg = A_1(x, z) + yA_2(x, z) + y^2A_3(x, z) + y^3A_4(y, z),$$
$$zg = B_1(x, z) + yB_2(x, z) + y^2B_3(x, z) + y^3B_4(y, z).$$

We cancel the left hand side by $z(yg) - y(zg) = 0$. We put $B_3(x, z) = B_5(z) + xB_6(x, z)$. Then we get

$$y^3B_3(x, z) = y^3B_5(z) + (xy^3)B_6(x, z) = y^3B_5(z) + yz^3B_6(x, z) + x^3zB_6(x, z).$$

So we get

$$\begin{aligned} &zA_1(x, z) + yzA_2(x, z) + y^2zA_3(x, z) + y^3zA_4(y, z), \\ &\quad = yB_1(x, z) + y^2B_2(x, z) + y^3B_3(x, z) + y^4B_4(y, z) \\ &\quad = x^3zB_6(x, z) + y\big(B_1(x, z) + z^3B_6(x, z)\big) + y^2B_2(x, z) \\ &\qquad + y^3\big(B_5(z) + yB_4(y, z)\big). \end{aligned}$$

This relation implies

$$\begin{aligned} A_1(x, z) &= x^3B_6(x, z), \\ zA_2(x, z) &= B_1(x, z) + z^3B_6(x, z), \\ zA_3(x, z) &= B_2(x, z), \\ zA_4(y, z) &= B_5(z) + yB_4(y, z). \end{aligned}$$

Since yzg is of positive weight, $B_5(z)$ is a positive power of z. So from the last equality above, we get $B_4(y,z) = zB_7(y,z)$ for some polynomial B_7. In the same way, we can show that $B_2(x,z)$ and $B_1(x,z)$ are divisible by z. We still do not know whether $B_3(x,z)$ is divisible by z. Here we put $xg = C_1(x,z) + yC_2(x,z) + y^2C_3(x,z) + y^3C_4(y,z)$. We have $xzg = xB_1(x,z) + yxB_2(x,z) + y^2xB_3(x,z) + xy^3B_4(y,z)$. The term $xy^3B_4(y,z)$ might be complicated if we want to express it in the direct sum expression in the theorem. But anyway, $B_4(y,z)$ is divisible by z, so we get

$$xy^3B_4(y,z) = zD_1(x,z) + yzD_2(x,z) + y^2zD_3(x,z) + y^3zD_4(y,z)$$

for some polynomials D_i $(1 \le i \le 4)$. Hence, by $z(xg) = x(zg)$, we get

$$xB_3(x,z) + zD_3(x,z) = zC_3(x,z).$$

Since x and z are algebraically independent, $B_3(x,z)$ is divisible by z. Hence we get $zg \in z\mathbb{C}[x,y,z]$ and $g \in \mathbb{C}[x,y,z]$ and $g \in A'_{(2k-2)/7}(\Gamma(7))$. So we get $A'_{2k/7}(\Gamma(7)) = A_{2k/7}(\Gamma(7))$ for all $k \in \mathbb{Z}$ $(k \ge 0)$.

Thus we have proved all the assertions of Proposition 2.3.

2.3 The case of $\mathbf{N = 7}$; modular forms of $\mathbf{SL_2(\mathbb{Z})}$ and $\mathbf{G_{336}}$. We consider the relation between the unitary reflection group G_{336}, the group No. 24 in SHEPHARD-TODD [18] and the action of $SL_2(\mathbb{Z})/\Gamma(7)$ on $A_{2/7}(\Gamma(7))$. We write $a = e(\frac{1}{28})$. Of course we have $a^{14} = -1$. By virtue of Theorem 1.2, the action ρ of $SL_2(\mathbb{Z})$ on $A_{2/7}(\Gamma(7))$ with respect to the basis F_1, F_3, F_5 is given by

$$\rho\left(\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)\right) = \frac{1}{\sqrt{7}}\begin{pmatrix} a^{13}-a & a^3-a^{11} & a^9-a^5\\ a^3-a^{11} & a^5-a^9 & a^{13}-a\\ a^9-a^5 & a^{13}-a & a^{11}-a^3\end{pmatrix}$$

$$\rho\left(\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\right) = \begin{pmatrix} a^4 & 0 & 0\\ 0 & a^8 & 0\\ 0 & 0 & a^{16}\end{pmatrix}$$

$$\rho\left(\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\right) = 1_3.$$

Hence, this defines an action of $PSL_2(\mathbb{F}_7)$. The simple group $PSL_2(\mathbb{F}_7)$ has two irreducible representation of degree 3 which are complex conjugate to one another, and the above ρ gives one of these two. The group generated by the image of ρ and -1_3 is isomorphic to G_{336}. The algebraically independent generators of invariant polynomials of G_{336} are known to be of degree 4, 6 and 14. This means that there should exist holomorphic modular forms of $SL_2(\mathbb{Z})$ of weight 8/7, 12/7 and 4, where the weight $2k/7$ means the automorphy factor $v_0(M)^{48k}(c\tau+d)^{2k/7}$. Although $XY^3 - YZ^3 - ZX^3$ is an invariant polynomial of G_{336}, we proved $xy^3 - yz^3 - zx^3 = 0$, so $A_{8/7}(SL_2(\mathbb{Z})) = 0$. Hence, $A^{(2/7)}(SL_2(\mathbb{Z}))$ is a polynomial ring generated by holomorphic modular forms of weight 12/7 and 4. More explicitly, by noting $F_1 = ax$, $F_3 = a^{15}y = -ay$, $F_5 = az$, a modular form $g \in A_{12/7}(SL_2(\mathbb{Z}))$ is given by

$$g := -F_1^5F_3 + F_3^5F_5 - F_1F_5^5 - 5F_1^2F_3^2F_5^2 = a^6(x^5y - y^5z - xz^5 - 5x^2y^2z^2).$$

This comes from the Hessian of $XY^3 - YZ^3 - ZX^3$ (cf. KLEIN [4]). Since the q-expansion of g starts from $q^{2/7}$, this is a cusp form. Since the dimension of cusp forms of weight 12 on $SL_2(\mathbb{Z})$ is one, we see $g^7 = \Delta(\tau)$, where $\Delta(\tau) = q^2 \prod_{n=1}^{\infty}(1-q^{2n})^{24}$ is the Ramanujan Delta function. We denote by E_4 and E_6 the Eisenstein series of $SL_2(\mathbb{Z})$ of weight 4 and 6:

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n},$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}.$$

Then the invariant of degree 14 is E_4. More explicitly, we get

$$\begin{aligned} E_4 &= x^{14} + y^{14} + z^{14} + 18(x^7y^7 - y^7z^7 - z^7x^7) \\ &\quad - 34(-xy^2z^{11} + yz^2x^{11} - zx^2y^{11}) - 126(x^5y^3z^6 - y^5z^3x^6 - z^5x^3y^6) \\ &\quad - 250(xy^9z^4 - yz^9x^4 - zx^9y^4) + 375(x^4y^8z^2 + y^4z^8x^2 + z^4x^8y^2). \end{aligned}$$

If we put $f = x^3y - y^3z - z^3x$, the above function is obtained by the following covariant

$$\frac{1}{9a^{12}}\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial x\partial z} & \frac{\partial g}{\partial x} \\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y\partial z} & \frac{\partial g}{\partial y} \\ \frac{\partial^2 f}{\partial z\partial x} & \frac{\partial^2 f}{\partial z\partial y} & \frac{\partial^2 f}{\partial z^2} & \frac{\partial g}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} & 0 \end{vmatrix}$$

(cf. KLEIN [4]).

Up to now, we considered invariants of G_{336}, so we demanded that they are invariant by -1_3 and hence treated the case of weight $2k/7$ with k even. If we do not demand the invariance by -1_3, we can consider also modular forms of weight $2k/7$ with odd k.

Proposition 2.4. *The graded ring $A^{(2/7)}(SL_2(\mathbb{Z}))$ of modular forms of weight $2k/7$ (k non-negative integers) is given by*

$$A^{(2/7)}(SL_2(\mathbb{Z})) = \mathbb{C}[g, E_4, E_6].$$

We have a relation $1728g^7 = E_4^3 - E_6^2$ *and*

$$\mathbb{C}[g, E_4, E_6] \cong \mathbb{C}[A, B, C]/(A^3 - B^2 - 1728C^7).$$

Proof. The modular forms of weight $2k/7$ with even k are given by $\mathbb{C}[g, E_4]$ by virtue of the theory of unitary reflection group. So take an odd k and assume that $h \in A_{2k/7}(SL_2(\mathbb{Z}))$. Then $h^2 \in \mathbb{C}[g, E_4]$, so $h^2 = \sum_{3a+7b=k} c_{a,b} g^a E_4^b$ for some $c_{a,b} \in \mathbb{C}$. Denote by a_0 the minimum of a in the above such that $c_{a,b} \neq 0$. If a_0 is odd, then the q-expansion of the right hand side starts from $q^{2a_0/7}$. But by Proposition 2.3, $h \in \mathbb{C}[x, y, z]$, so the q-expansion of h must start from some power of $q^{2/7}$, and that of h^2 starts from a power of $q^{4/7}$. This is a contradiction. So a_0 must be even. Since g is a nowhere vanishing function on H, the function $(h/g^{a_0/2})^2$ is holomorphic on H, but the above result means that it is holomorphic also at the cusp of $SL_2(\mathbb{Z})$. So this is a holomorphic modular form of $SL_2(\mathbb{Z})$ of even weight. Hence $h/g^{a_0/2}$ is also a modular form of $SL_2(\mathbb{Z})$ of integral weight, so $h/g^{a_0/2} \in \mathbb{C}[E_4, E_6]$ and we get $h \in g^{a_0/2}\mathbb{C}[g, E_4]$. □

Remark. E_6 is given as -756^{-1} times the functional determinant of $f = XY^3 - YZ^3 - ZX^3$, the Hessian of f, and E_4 with respect to x, y, z. That is,

$$\begin{aligned}
E_6 = {} & x^{21} + y^{21} - z^{21} - 7(xy^2z^{18} + yz^2x^{18} - zx^2y^{18}) \\
& - 57(x^7y^{14} + y^7z^{14} - z^7x^{14}) + 217(x^4yz^{16} - y^4zx^{16} + z^4xy^{16}) \\
& - 289(x^7z^{14} + y^7x^{14} - z^7y^{14}) - 308(x^4y^{15}z^2 - y^4z^{15}x^2 + z^4x^{15}y^2) \\
& + 637(x^{12}y^3z^6 - y^{12}z^3x^6 + z^{12}x^3y^6) - 1638(x^9y^{11}z + y^9z^{11}x + z^9x^{11}y) \\
& - 4018(x^3y^{13}z^5 + y^3z^{13}x^5 + z^3x^{13}y^5) \\
& - 6279(x^{11}y^8z^2 + y^{11}z^8x^2 - z^{11}x^8y^2) \\
& + 7007(x^6y^5z^{10} - y^6z^5x^{10} + z^6x^5y^{10}) \\
& - 10010(x^8y^9z^4 - y^8z^9x^4 + z^8x^9y^4) - 10296x^7y^7z^7.
\end{aligned}$$

Remark. The four "invariants" of $SL_2(\mathbb{Z})$ above, that is, g, E_4, E_6, and $xy^3 - yz^3 - zx^3$, also appear in KLEIN [4]. But his concern was with modular functions or formal variables, and he did not treat modular forms of rational weight or graded ring of these. In a sense, we have given an easy interpretation of KLEIN's old work from the point of view of the notion of rational weights.

2.4 The case of level 9. When $N = 9$, put

$$x = f_0^{-1/3}\sum_{p=0}^{\infty}(-1)^p q^{9p^2+p},$$

$$y = f_0^{-1/3}q^{2/9}\sum_{p=0}^{\infty}(-1)^p q^{9p^2+3p},$$

$$z = f_0^{-1/3}q^{2/3}\sum_{p=0}^{\infty}(-1)^p q^{9p^2+5p},$$

$$w = f_0^{-1/3}q^{4/3}\sum_{p=0}^{\infty}(-1)^p q^{9p^2+7p}.$$

The multiplier system $v_9(M)$ for these modular forms of weight 1/3 satisfies $v_9(M)^9 = 1$, as shown before, but more strongly we have $v_9(M)^3 = 1$ for any $M \in \Gamma(9)$. Indeed, we have already shown that $v_9(M)^3$ is equal to the multiplier system of η^{80}, and its explicit formula implies $v_9(M)^3 = 1$. We denote by $A_{k/3}(\Gamma(9))$ the space of modular forms of $\Gamma(9)$ of weight $v_9(M)^k(c\tau+d)^{k/3}$ and put $A^{(1/3)}(\Gamma(9)) = \oplus_{k=0}^{\infty} A_{k/3}(\Gamma(9))$. We put

$$F(X, Z, W) = XZ^2 - ZW^2 - WX^2,$$

$$G(X, Y, Z, W) = Y^3 + XW^2 - WZ^2 - ZX^2.$$

Lemma 2.5. *We have*

$$F(x, z, w) = 0, \qquad G(x, y, z, w) = 0.$$

Proof. There are 20 monomials $H(X, Y, Z, W)$ of degree 3 of four variables X, Y, Z, W. The q-expansion of each $H(x, y, z, w)$ can be written as

$$H(x, y, z, w) = \sum_{n=0}^{\infty} a_n q^{l(H)+2n}$$

where $l(H)$ depends on H and $9l(H) \in \{0, 2, 4, 6, \dots, 18, 20, 24, 26, 30, 36\}$. To obtain all the linear relations between $H(x, y, z, w)$, it is sufficient to check the relations between monomials H which have the same $l(H)$ mod 2, that is, between those such that all $l(H)$ belong to one of the following sets $\{0, 2, 4\}$, $\{2/9, 20/9\}$, $\{4/9\}$, $\{8/9, 26/9\}$, $\{10/9\}$, $\{14/9\}$, $\{16/9\}$, $\{2/3, 8/3\}$, $\{4/3, 10/3\}$. Except for the last two sets, it is easily shown that there is no non trivial relation. For example, for $\{0, 2, 4\}$, four modular forms x^3, z^3, xzw, w^3 might have a linear relation, but by comparing the q-expansion of a relation

$$ax^3 + bz^3 + cxzw + dw^3 = 0$$

using

$$x^3 = 1 - q^8 - q^{10} + q^{34} + \cdots,$$

$$z^3 = q^2(1 - q^4 - q^{14} + \cdots),$$

$$xzw = q^2(1 - q^2 - q^4 + q^6 + \cdots),$$

$$w^3 = q^4(1 - 3q^2 + 3q^4 + \cdots),$$

we get $a = b = c = d = 0$ and so on. As for the last two sets, comparing the q-expansion, the only possible relations (up to constants) would be $F(x, z, w) = 0$ and $G(x, y, z, w) = 0$. But by Lemma 1.7, we have $A_1(\Gamma(9)) = 18$, so these really are relations. □

It is also possible to prove these relations by elementary combinatorial argument as in the case $N = 7$. We omit the details.

Theorem 2.6. *(1) The ideal $\mathfrak{a}$ of $\mathbb{C}[X, Y, Z, W]$ generated by F and G is a prime ideal, and we have the following isomorphism.*

$$\mathbb{C}[X, Y, Z, W]/\mathfrak{a} \cong \mathbb{C}[x, y, z, w].$$

(2) Put $B = \mathbb{C}[x, y, z]$. Then

$$B = \mathbb{C}[x, w] \oplus z\mathbb{C}[x, w] \oplus z^2\mathbb{C}[z, w],$$

$$\mathbb{C}[x, y, z, w] = B \oplus yB \oplus y^2B.$$

(3) We have

$$A^{(1/3)}(\Gamma(9)) = \mathbb{C}[x, y, z, w].$$

Incidentally the genus of the modular curve is 10.

Proof. (1) Since $\mathbb{C}[x, y, z, w]$ is an integral domain, there is a prime ideal $\mathfrak{p}$ of $\mathbb{C}[X, Y, Z, W]$ such that $\mathbb{C}[X, Y, Z, W]/\mathfrak{p} \cong \mathbb{C}[x, y, z, w]$. By Lemma 2.5, we get $\mathfrak{a} \subset \mathfrak{p}$. First we show that if $\mathfrak{a}$ is a prime ideal, then $\mathfrak{p} = \mathfrak{a}$. Indeed, since F is an irreducible polynomial, $F(X, Z, W)\mathbb{C}[X, Y, Z, W]$ is a prime ideal properly contained in $\mathfrak{a}$. Hence, if $\mathfrak{a} \neq \mathfrak{p}$ and $\mathfrak{a}$ is a prime ideal, then the height of $\mathfrak{p}$ is at least 3, and the transcendental degree of $\mathbb{C}[X, Y, Z, W]/\mathfrak{p}$ is at most one, which is a contradiction. Next, we show that $\mathfrak{a}$ is a prime ideal. Put $\mathfrak{a}_0 = \mathfrak{a} \cap \mathbb{C}[Y, Z, W]$ and $\mathfrak{p}_0 = \mathfrak{p} \cap \mathbb{C}[Y, Z, W]$. Put also $H = W^2Y^6 - Z^2(Z^3 - W^3)Y^3 + ZW(Z^3 - W^3)^2$. By the relation

$$\begin{aligned} H = &(-X^2Z^2W - (Z^4 - 2ZW^3)X - W^5)F \\ &+ (X^2ZW^2 - XW^4 + W^2Y^3 + 2Z^2W^3 - Z^5)G, \end{aligned}$$

we get $H \in \mathfrak{a}$. Since H is irreducible, $(H) := H\mathbb{C}[Y, Z, W]$ is a prime ideal of $\mathbb{C}[Y, Z, W]$. Since $(H) \subset \mathfrak{a}_0 \subset \mathfrak{p}_0$ and y and z are algebraically independent, we get $\mathfrak{p}_0 = (H)$, and hence $\mathfrak{a}_0 = (H)$. To show that $\mathfrak{a}$ is prime, we take polynomials $P, Q \in \mathbb{C}[X, Y, Z, W]$ such that $PQ \in \mathfrak{a}$. Since $X(Z^3 - W^3) - Y^3W = ZF(X, Z, W) - WG(X, Y, Z, W) \in \mathfrak{a}$, we get $X^m(Z^3 - Y^3)^m \in Y^mZ^m + \mathfrak{a}$ for any natural number m. Hence we can show that there exist some natural numbers m, n and some polynomials $P', Q' \in \mathbb{C}[Y, Z, W]$ such that

$$(Z^3 - W^3)^m P - P' \in \mathfrak{a}, \qquad (Z^3 - W^3)^n Q - Q' \in \mathfrak{a}.$$

Obviously we get $P'Q' \in \mathfrak{a}_0$, and since $\mathfrak{a}_0$ is a prime ideal, we can assume $P' \in \mathfrak{a}_0$, exchanging P' and Q' if necessary. Therefore we get $(Z^3 - W^3)^m P \in \mathfrak{a}$. Next we show that, for any polynomial $P(X, Y, Z, W)$, if $(Z^3 - W^3)P \in \mathfrak{a}$ then $P \in \mathfrak{a}$. That is, if

$$(Z^3 - W^3)P = FQ + GR$$

for polynomials $Q, R \in \mathbb{C}[X, Y, Z, W]$, then there exist polynomials Q_1 and R_1 which are divisible by $(Z^3 - W^3)$ and satisfy $FQ + GR = FQ_1 + GR_1$. Indeed, if we denote by ζ one of the third roots of unity, then by assumption, we get

$$F(X, Z, \zeta Z)Q(X, Y, Z, \zeta Z) + G(X, Y, Z, \zeta Z)R(X, Y, Z, \zeta Z) = 0.$$

But the two polynomials

$$\begin{aligned} F(X, Z, \zeta Z) &= XZ^2 - \zeta^2 Z^3 - \zeta X^2 Z, \\ G(X, Y, Z, \zeta Z) &= Y^3 + \zeta^2 XZ^2 - \zeta Z^3 - \zeta X^2 Z \end{aligned}$$

have no non-trivial common divisors, so there exists a polynomial $P_0(X, Y, Z)$ such that

$$\begin{aligned} Q(X, Y, Z, \zeta Z) &= P_0(X, Y, Z)G(X, Y, Z, \zeta Z), \\ R(X, Y, Z, \zeta Z) &= -P_0(X, Y, Z)F(X, Y, Z, \zeta Z). \end{aligned}$$

If we define Q_1, R_1 by

$$Q_1(X,Y,Z,W)=Q(X,Y,Z,W)-P_0(X,Y,Z)G(X,Y,Z,W),$$
$$R_1(X,Y,Z,W)=R(X,Y,Z,W)+P_0(X,Y,Z)F(X,Z,W),$$

then we get $Q_1(X,Y,Z,\zeta Z)=R_1(X,Y,Z,\zeta Z)=0$. Hence Q_1, R_1 are divisible by $W-\zeta Z$ for any ζ and hence also by Z^3-W^3. We therefore get

$$P=(Z^3-W^3)^{-1}(FQ+GR)=F(Z^3-W^3)^{-1}Q_1+G(Z^3-W^3)^{-1}R_1\in\mathfrak{a}.$$

Hence if $(Z^3-W^3)^m P\in\mathfrak{a}$, then we get $P\in\mathfrak{a}$ by induction. We have thus proved that $\mathfrak{a}$ is a prime ideal. Hence we get $\mathfrak{a}=\mathfrak{p}$ and (1) is proved.
(2) Using the relation $F(x,z,w)=0$ and replacing xz^2 by zw^2+wx^2 as often as possible, we get $B=\mathbb{C}[x,w]+z\mathbb{C}[x,w]+z^2\mathbb{C}[z,w]$. Also by using $G(x,y,z,w)=0$ and replacing y^a with $a\geq 3$ by lower terms of y, we get $\mathbb{C}[x,y,z,w]=B+yB+y^2B$. We show that these sums are direct sums as modules. Since F and G are coprime with each other, if $FQ+GR=0$ then $G\mid Q$ and $F\mid R$. Hence if we put

$$A'_{k/3}(\Gamma(9))=A_{k/3}(\Gamma(9))\cap\mathbb{C}[x,y,z,w],$$

then we get

$$\sum_{k=0}^{\infty}\dim A'_{k/3}(\Gamma(9))t^{k/3}=\frac{(1-t)^2}{(1-t^{1/3})^4}=\frac{(1+t^{1/3}+t^{2/3})^2}{(1-t^{1/3})^2}.$$

This can happen only if the above expressions of B and $\mathbb{C}[x,y,z,w]$ are direct sums. q.e.d.
(3) We show that $A_{k/3}(\Gamma(9))=A'_{k/3}(\Gamma(9))$. First, for integral weights, we get

$$\sum_{k=0}^{\infty}A'_k(\Gamma(9))=\frac{1+16t+10t^2}{(1-t)^2},$$

and we get $\dim A'_k(\Gamma(9))=\dim A_k(\Gamma(9))$ for all $k\in\mathbb{Z}$, $k\geq 0$ by the well known dimension formula (cf. Lemma 1.7.) Now we show that if $A'_{k/3}(\Gamma(9))=A_{k/3}(\Gamma(9))$, then $A'_{(k-1)/3}(\Gamma(9))=A_{(k-1)/3}(\Gamma(9))$. We take $g\in A_{(k-1)/3}(\Gamma(9))$. Then of course we get xg, yg, $wg\in A_{k/3}(\Gamma(9))=A'_{k/3}(\Gamma(9))$. We show that this implies $g\in A'_{k/3}(\Gamma(9))$. By our assumption, we have

$$wg=P_1(x,z,w)+yP_2(x,z,w)+y^2P_3(x,z,w),$$
$$zg=Q_1(x,z,w)+yQ_2(x,z,w)+y^2Q_3(x,z,w),$$
$$xg=R_1(x,z,w)+yR_2(x,z,w)+y^2R_3(x,z,w),$$

for some polynomials P_i, Q_i, R_i $(1\leq i\leq 3)$. By the relation $x(wg)=w(xg)$, we get $xP_i(x,z,w)=wR_i(x,z,w)$ for $i=1,2,3$. Also by $z(wg)=w(zg)$, we get $zP_i(x,z,w)=wQ_i(x,z,w)$ $(1\leq i\leq 3)$. We show in general that if we assume $xP(x,z,w)=wQ(x,z,w)$ and $zP(x,z,w)=wR(x,z,w)$ for some polynomials

Q, $R \in w\mathbb{C}[X, Z, W]$ then $P(x, z, w) \in w\mathbb{C}[x, z, w]$. We can take polynomials P_5, P_6 such that

$$P(x, z, w) = P_4(x, w) + zP_5(x, w) + z^2 P_6(z, w).$$

Since $xz^2 = w(zw + x^2) \in w\mathbb{C}[x, z, w]$, we can write

$$\begin{aligned} &xP(x, z, w) \\ &\quad = xP_4(x, w) + zxP_5(x, w) + wP_7(x, w) + zwP_8(x, w) + z^2 wP_9(z, w) \end{aligned}$$

for some polynomials P_i ($i = 7.\ 8, 9$). We also have

$$xP = wQ = wQ_4(x, w) + wzQ_5(x, w) + z^2 wQ_6(z, w)$$

for some polynomials Q_i ($i = 4, 5, 6$). Hence we get $xP_4(x, w) + wP_7(x, w) = wQ_4(x, w)$, $xP_5(x, w) + wP_8(x, w) = wQ_5(x, w)$, and $wP_9(z, w) = wQ_6(z, w)$. Here, since x, w are algebraically independent and z, w are also, we get $P_4(x, w)$, $P_5(x, w) \in w\mathbb{C}[x, w]$. Now we consider on $P_6(z, w)$. Since

$$zP(x, z, w) = zP_4(x, w) + z^2 P_5(x, w) + z^3 P_6(z, w) = wR(x, z, w),$$

and since $P_4(x, w)$, $P_5(x, w) \in w\mathbb{C}[x, w]$, we get $z^3 P_6(z, w) \in w\mathbb{C}[x, z, w]$. Hence we have

$$z^3 P_6(z, w) = wR_4(x, w) + zwR_5(x, w) + z^2 wR_6(z, w)$$

for some polynomials R_i ($i = 4, 5, 6$). Since the right hand side is a direct sum, we get $R_4 = R_5 = 0$. Hence, taking the algebraic independence of z, w into account, we get $P_6(z, w) \in w\mathbb{C}[z, w]$. Hence, $x(wg), z(wg) \in w\mathbb{C}[x, z, w]$ implies $wg \in w\mathbb{C}[x, z, w]$. and we get $g \in \mathbb{C}[x, z, w]$. Thus we get $A'_{(k-1)/3}(\Gamma(9)) = A_{(k-1)/3}(\Gamma(9))$. □

2.5 The case of level 11. When $N = 11$, put

$$\begin{aligned} x &= e(-1/44)F_1, \qquad & y &= -e(-1/44)F_3, \qquad & z &= e(-1/44)F_5, \\ v &= e(-1/44)F_7, & w &= e(-1/44)F_9. \end{aligned}$$

Then we have

$$\begin{aligned} xf_0^{3/11} &= 1 - q^{10} - q^{12} + q^{42} + q^{46} - q^{96} - q^{102} \cdots \\ yf_0^{3/11} &= q^{2/11}(1 - q^8 - q^{14} + q^{38} + q^{50} \cdots) \\ zf_0^{3/11} &= q^{6/11}(1 - q^6 - q^{16} + q^{34} + q^{54} \cdots) \\ vf_0^{3/11} &= q^{12/11}(1 - q^4 - q^{18} + q^{30} + q^{58} \cdots) \\ wf_0^{3/11} &= q^{20/11}(1 - q^2 - q^{20} + q^{26} + q^{62} \cdots). \end{aligned}$$

As KLEIN [6] has already pointed out, we get the following 15 relations.

$$
\begin{aligned}
x^3v + v^3w - y^3z &= 0,\\
xz^3 - vy^3 - w^3y &= 0,\\
v^3z - z^3y + x^3w &= 0,\\
-y^3x + x^3z + vw^3 &= 0,\\
z^3w + w^3x - v^3y &= 0,\\
z^2vw + x^2yw - y^2zv &= 0,\\
-x^2yv - zvw^2 + xyz^2 &= 0,\\
-w^2yz + v^2yx - x^2wz &= 0,\\
xzv^2 - y^2zw - xvw^2 &= 0,\\
xy^2w - z^2xv + v^2yw &= 0,\\
-xyvw + z^2wy + z^2x^2 - y^3z &= 0,\\
-wvyz - x^2zv + x^2w^2 + xz^3 &= 0,\\
-vyzx + w^2yx + w^2v^2 + x^3w &= 0,\\
-yzxw - v^2zw + v^2y^2 + w^3v &= 0,\\
xzvw - y^2xv + y^2z^2 - yv^3 &= 0.
\end{aligned}
$$

It is plausible that there are essentially no other relations. If this is true, then by using Gröbner bases, we get the following expression.

$$
\begin{aligned}
&\mathbb{C}[x, y, z, v, w]\\
&\quad\cong \mathbb{C}[x, w] \oplus \mathbb{C}[x, w]y \oplus \mathbb{C}[x, w]z \oplus \mathbb{C}[x, w]yz \oplus \mathbb{C}[x, w]xv\\
&\qquad \oplus \mathbb{C}[x, w]zv \oplus \mathbb{C}[x, w]xy^2 \oplus \mathbb{C}[x, w]y^2z \oplus \mathbb{C}[x, w]y^2v \oplus \mathbb{C}[x, w]xyv\\
&\qquad \oplus \mathbb{C}[v, w]v \oplus \mathbb{C}[v, w]xv^2 \oplus \mathbb{C}[v, w]x^2v^2 \oplus \mathbb{C}[y, w]y^2 \oplus \mathbb{C}[z, w]z^2\\
&\qquad \oplus \mathbb{C}[z, w]zv^2 \oplus \mathbb{C}[z, w]zv^3 \oplus \mathbb{C}[z, w]zv^4 \oplus \mathbb{C}[z, w]zv^5 \oplus \mathbb{C}[z, w]z^2v\\
&\qquad \oplus \mathbb{C}[w]yv \oplus \mathbb{C}[w]yvz \oplus \mathbb{C}[w]yvz^2 \oplus \mathbb{C}[w]yv^2 \oplus \mathbb{C}[w]yv^2z\\
&\qquad \oplus \mathbb{C}[w]yv^2z^2 \oplus \mathbb{C}[w]xz^2 \oplus \mathbb{C}[w]yz^2,
\end{aligned}
$$

and then the generating function of dimensions of homogeneous part of $\mathbb{C}[x, y, z, v, w]$ is given by

$$
\sum_{k=0}^{\infty} \dim A_{4k/11} t^{4k/11} = \frac{1 + 3t^{4/11} + 6t^{8/11} + 10t^{12/11}}{(1 - t^{4/11})^2}
$$

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