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On the Spectrum of a Strictly Pseudoconvex CR Manifold

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1 Introduction.

Let M be a compact strictly pseudoconvex $(2n+1)$-dimensional CR manifold and Δ_b the sublaplacian (a subelliptic operator of order $1/2$) corresponding to a fixed choice of contact 1-form θ on M. Let λ_k be the k-th nonzero eigenvalue of Δ_b. Using L^2 methods (i.e. a pseudohermitian analogue of the Bochner formula in Riemannian geometry) A. GREENLEAF has shown (cf. [2]) that the first nonzero eigenvalue λ_1 of Δ_b satisfies:

$$\lambda_1 \geq \tfrac{n}{n+1} C_0 \tag{1}$$

provided that:

$$R_{\alpha\overline{\beta}} Z^\alpha \overline{Z}^\beta + \tfrac{ni}{2}\left(A_{\overline{\alpha}\overline{\beta}} \overline{Z}^\alpha \overline{Z}^\beta - A_{\alpha\beta} Z^\alpha Z^\beta\right) \geq C_0\, h_{\alpha\overline{\beta}} Z^\alpha \overline{Z}^\beta \tag{2}$$

for some $C_0 > 0$ (many notions involved here will be defined through the next section), cf. [2], Theor. 1, p. 192. Our main result consists of the following

Theorem. *Let M be a compact strictly pseudoconvex CR manifold (of CR dimension n). Assume that the problem*

$$\begin{cases} \Delta_b v = \lambda_k v,\ T(v) = 0, \\ \sup v = 1, \\ \inf v = -C,\ 0 < C \leq 1 \end{cases} \tag{3}$$

admits some C^∞ solution v. If

$$\mathrm{Ric}(X - iJX, X + iJX) + 2(n-2) A(X, JX) \geq 0 \tag{4}$$

for any $X \in H(M)$, then

$$\lambda_k \geq \frac{\pi^2}{d_\theta^2}. \tag{5}$$

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Here T is the characteristic direction of (M, θ) and d_θ is the diameter of M with respect to the Webster metric g_θ. In contrast with [2], we employ L^∞ methods. If for instance $M = S^{2n+1}$ (the round sphere carrying the standard pseudohermitian structure) then both (2) and our assumption (4) hold good.

Let M be a strictly pseudoconvex CR manifold of vanishing pseudohermitian torsion. Then our assumption (4) is weaker than (2). However, it must be pointed out that while we work under less restrictive geometric conditions, the proof of (5) requires the existence of a solution of (3) (rather than a solution of (24) alone). As a result, we may estimate terms of the form $u^\alpha(L_2 u_\alpha)$ at a point (where L_2 is a Folland-Stein operator, cf. Section 4). General existence theorems for the solutions of (3) are not known as yet (and this precisely the limitation of our result). An example where (3) may be solved is indicated in Section 6.

If v is a solution of (3) then (by (68) in [1] (a simplification of (6.7) in [2], p. 211)) $\Delta_b v = \Delta v$ (where Δ is the Laplace-Beltrami operator of (M, g_θ)) so that actually $\lambda_k \in \mathrm{Spec}(M, g_\theta)$ and the estimate (5) follows from work by Z. Jiaqing and Y. Hongcang, [4], provided that the metric (here the Webster metric g_θ) has nonnegative Ricci curvature. Nevertheless, this may be seen (cf. Section 6) to be generically stronger than our assumption (4).

2 A reminder of pseudohermitian geometry.

Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex $(2n+1)$-dimensional CR manifold of CR dimension n, and $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ its maximally complex distribution. Here $T_{0,1}(M) = \overline{T_{1,0}(M)}$. Throughout an overbar indicates complex conjugation. Let θ be a contact 1-form on M (i.e. $\mathrm{Ker}(\theta) = H(M)$ and $\theta \wedge (d\theta)^n \neq 0$ everywhere on M) so that the corresponding *Levi form*

$$L_\theta(Z, \overline{W}) = -i\,(d\theta)(Z, \overline{W})$$

is positive definite. Here Z, $W \in T_{1,0}(M)$ and $i = \sqrt{-1}$. Let T be the *characteristic direction* of (M, θ), i.e. the unique nowhere zero tangent vector field transverse to $H(M)$ determined by

$$\theta(T) = 1, \quad T \rfloor d\theta = 0.$$

By a result of S. Webster, [10], (cf. also N. Tanaka, [9]) there is a unique linear connection ∇ on M (the *Webster connection*) so that

i) $H(M)$ is parallel with respect to ∇,
ii) $\nabla J = 0$, $\nabla g_\theta = 0$,
iii) $\pi_+ T_\nabla(Z, W) = 0$

for any $Z \in T_{1,0}(M)$, $W \in T(M) \otimes \mathbb{C}$. Here J is the complex structure of $H(M)$ (given by $J(Z + \overline{Z}) = i(Z - \overline{Z})$ for any $Z \in T_{1,0}(M)$) while g_θ is the *Webster metric* (cf. [10], (2.18), p. 349, and our Appendix). Also T_∇ is the torsion tensor field of ∇ and $\pi_+ : T(M) \otimes \mathbb{C} \to T_{1,0}(M)$ is the natural bundle map (associated with the decomposition $T(M) \otimes \mathbb{C} = T_{1,0}(M) \oplus T_{0,1}(M) \oplus \mathbb{C}T$). As to all local calculations, if $\{T_1, \ldots, T_n\}$ is a (local) frame of $T_{1,0}(M)$

and $\{\theta^1,\ldots,\theta^n\}$ are the (local) complex 1-forms determined by $\theta^\alpha(T_\beta) = \delta^\alpha_\beta$, $\theta^\alpha(T_{\overline{\beta}}) = 0$ and $\theta^\alpha(T) = 0$, then we set $h_{\alpha\overline{\beta}} = L_\theta(T_\alpha, T_\beta)$ and $\nabla T_\beta = \omega^\alpha_\beta \otimes T_\alpha$ (with the usual conventions as to barred indices, e.g. $T_{\overline{\alpha}} = \overline{T_\alpha}$). The following identities hold (cf. also (1.15) and (1.24) in [10], p. 28–29) as a consequence of the axioms i)–iii)

$$d\theta^\alpha = \theta^\beta \wedge \omega^\alpha_\beta + \theta \wedge \tau^\alpha, \tag{6}$$

$$dh_{\alpha\overline{\beta}} = \omega^\mu_\alpha\, h_{\mu\overline{\beta}} + h_{\alpha\overline{\mu}}\, \omega^{\overline{\mu}}_{\overline{\beta}}. \tag{7}$$

Here $\tau^\alpha = A^\alpha_{\overline{\beta}}\theta^{\overline{\beta}}$ and $A^\alpha_{\overline{\beta}}$ are given by $T_\nabla(T, T_{\overline{\beta}}) = A^\alpha_{\overline{\beta}} T_\alpha$. The functions $A^\alpha_{\overline{\beta}}$ are the local manifestation of the *pseudohermitian torsion* τ of ∇ (given by $\tau X = T_\nabla(T, X)$ for any $X \in T(M)$) and enjoy the symmetry property

$$A_{\alpha\beta} = A_{\beta\alpha}, \tag{8}$$

(cf. also (1.23) in [10], p. 28) where $A_{\alpha\beta} = A^{\overline{\sigma}}_\alpha h_{\beta\overline{\sigma}}$ (i.e. τ is self adjoint with respect to g_θ). Throughout we adopt the usual conventions as to the lowering and raising of indices by means of $h_{\alpha\overline{\beta}}$ (respectively its inverse $h^{\alpha\overline{\beta}}$).

Let R be the curvature tensor field of ∇ and set

$$R(T_B, T_C)T_A = R_A{}^D{}_{BC} T_D$$

where $A, B, C, \ldots \in \{1,\ldots,n,\overline{1},\ldots,\overline{n},0\}$ and $T_0 = T$. The following identity holds

$$\begin{aligned} d\omega^\beta_\alpha - \omega^\mu_\alpha \wedge \omega^\beta_\mu = {} & R_\alpha{}^\beta{}_{\lambda\overline{\mu}}\theta^\lambda \wedge \theta^{\overline{\mu}} + W^\beta_{\alpha\lambda}\,\theta^\lambda \\ & - W^\beta_{\alpha\overline{\lambda}}\,\theta^{\overline{\lambda}} \wedge \theta + i\theta^\beta \wedge \tau_\alpha - i\tau^\beta \wedge \theta_\alpha \end{aligned} \tag{9}$$

(cf. also (2.2) in[7], p. 161) where $W^\beta_{\alpha\lambda}$, $W^\beta_{\alpha\overline{\beta}}$ are certain covariant derivatives (with respect to ∇) of the pseudohermitian torsion τ. Cf. the Appendix, where we give a new proof of (9). Next:

$$R_{\lambda\overline{\mu}} = R_\alpha{}^\alpha{}_{\lambda\overline{\mu}}$$

is the *pseudohermitian Ricci tensor* field of (M,θ), cf. e.g. [7], p. 162. If $\mathrm{Ric}(X,Y) = \mathrm{trace}\{Z \mapsto R(Z,X)Y\}$ then $R_{\lambda\overline{\mu}} = \mathrm{Ric}(T_\lambda, T_{\overline{\mu}})$. It should be mentioned however that there are other nonzero components of Ric (besides $R_{\lambda\overline{\mu}}$) which may be computed as certain contractions of covariant derivatives of τ (cf. [1]).

3 The sublaplacian.

Let $0 < \epsilon < 1$. Let N be a Riemannian manifold. A formally self-adjoint differential operator $\mathscr{L} : C^\infty(N) \to C^\infty(N)$ of order 2 on N is *subelliptic of order* ϵ at $x \in N$ if there is an open neighborhood U of x and a constant $C > 0$ so that:

$$\|u\|^2_\epsilon \leq C\Big(|(\mathscr{L}u, u)| + \|u\|^2\Big) \tag{10}$$

for any $u \in C^\infty_0(U)$. Here $\|\cdot\|$ is the L^2 norm and $\|\cdot\|_\epsilon$ is the Sobolev norm of order ϵ, cf. e.g. L. Hörmander, [3].

Let M be a CR manifold (under the assumptions of Section 2). A complex k-form ω on M is a $(0,k)-form$ if $T\rfloor\omega = 0$ and $T_{1,0}(M)\rfloor\omega = 0$. Let $\Lambda^{0,k}(M)$ be the corresponding bundle. The *tangential Cauchy-Riemann operator* is the differential operator $\overline{\partial}_b : \Gamma^\infty(\Lambda^{0,k}(M)) \to \Gamma^\infty(\Lambda^{0,k+1}(M))$ defined as follows. Let ω be a $(0,k)$-form on M. Then $\overline{\partial}_b\omega$ is the unique $(0,k+1)$-form on M which coincides with $d\omega$ when both are restricted to $T_{0,1}(M)\otimes\cdots\otimes T_{0,1}(M)$ ($k+1$ terms). Let g_θ^* be the inner product naturally induced by g_θ on $\Lambda^{0,k}(M)$. Let $\overline{\partial}_b^*$ be the formal adjoint of $\overline{\partial}_b$ with respect to the L^2 inner product:

$$(\alpha,\beta) = \int_M g_\theta^*(\alpha,\beta)\theta\wedge(d\theta)^n$$

for any $(0,k)$-forms α, β on M (at least one of compact support). The *Kohn-Rossi operator* $\Box_b$ is given by

$$\Box_b = 2\left(\overline{\partial}_b^*\,\overline{\partial}_b + \overline{\partial}_b\,\overline{\partial}_b^*\right).$$

The *sublaplacian* Δ_b on M is given by

$$\Delta_b u = \Box_b u - inT(U)$$

for any $u \in C^\infty(M)$. We recall (cf. e.g. J. J. KOHN, [6]) that Δ_b is a subelliptic operator of order $1/2$ at any point of the Riemannian manifold (M, g_θ). Thus (cf. A. MENIKOFF, J. SJÖSTRAND, [8]) Δ_b has a discrete spectrum tending to $+\infty$.

4 Commutation formulae.

Let $u : M \to \mathbb{R}$ be a C^∞ function. The *pseudohermitian Hessian* $\nabla^2 u$ of u is given by

$$(\nabla^2 u)(X,Y) = (\nabla_X du)Y \tag{11}$$

for any $X, Y \in T(M)$. Here ∇ denotes the Webster connection. Unlike the Hessian of a function on a Riemannian manifold $\nabla^2 u$ is not symmetric (as ∇ has nonzero torsion). Let $u_{AB} = (\nabla^2 u)(T_A, T_B)$ for some (local) frame $\{T_1,\ldots,T_n\}$ of $T_{1,0}(M)$. Then

$$\begin{aligned} u_{\alpha\beta} &= u_{\beta\alpha}, \\ u_{\alpha\overline{\beta}} &= u_{\overline{\beta}\alpha} - i\,h_{\alpha\overline{\beta}}\,u_0, \\ u_{\alpha 0} &= u_{0\alpha} + A_\alpha^{\overline{\mu}}\,u_{\overline{\mu}}. \end{aligned} \tag{12}$$

Here $u_A = T_A(u)$. Note that

$$T_\nabla = 2\,\theta\wedge\tau - \Omega_\theta\otimes T$$

(cf. also (5) in [1]). Consequently

$$\begin{aligned} (\nabla^2 u)(X,Y) &= (\nabla^2 u)(Y,X) + \Omega_\theta(X,Y)T(u), \\ (\nabla^2 u)(X,T) &= (\nabla^2 u)(T,X) + (\tau X)(u) \end{aligned}$$

for any $X, Y \in H(M)$ (thus yielding (12)). We define the 3rd order covariant derivative $\nabla^3 u$ by setting

$$(\nabla^3 u)(X, Y, Z) = (\nabla_X \nabla^2 u)(Y, Z).$$

Also set $u_{ABC} = (\nabla^3 u)(T_C, T_A, T_B)$. The *Folland-Stein operators* L_c are given by

$$L_c = \Delta_b - i\, c\, T, \quad c \in \mathbb{R}$$

(so that $L_{-n} = \Box_b$ on functions). We need to show that

$$(Lu)_\beta = -u_{\alpha\beta}{}^{\alpha} - u_{\overline{\alpha}\beta}{}^{\overline{\alpha}} + u^{\overline{\alpha}} R_{\overline{\alpha}\beta} + i\,(n-2)\, u_{\overline{\alpha}}\, A^{\overline{\alpha}}_{\beta} \tag{13}$$

where $L = L_2$. Also $u_{\alpha\beta}{}^{\alpha} = h^{\alpha\overline{\gamma}} u_{\alpha\beta\overline{\gamma}}$, etc. For further use, let us introduce the *Christoffel symbols* $\Gamma^{\alpha}_{A\beta}$ determined by

$$\omega^{\alpha}_{\beta} = \Gamma^{\alpha}_{A\beta} \theta^A$$

where $\theta^0 = \theta$. The sublaplacian Δ_b is also given by

$$\Delta_b u = -u_{\alpha}{}^{\alpha} - u_{\overline{\alpha}}{}^{\overline{\alpha}} \tag{14}$$

where $u_{\alpha}{}^{\alpha} = u_{\alpha\overline{\beta}}\, h^{\alpha\overline{\beta}}$. We have

$$T_\beta(u_\alpha{}^\alpha) = T_\beta(u_{\alpha\overline{\sigma}}) h^{\overline{\sigma}\alpha} - u_\alpha{}^\mu\, T_\beta(h_{\mu\overline{\sigma}}) h^{\overline{\sigma}\alpha}.$$

Using (7) and the identity

$$T_\beta(u_{\alpha\overline{\sigma}}) = u_{\alpha\overline{\sigma}\beta} + \Gamma^{\mu}_{\beta\alpha}\, u_{\mu\overline{\sigma}} + \Gamma^{\overline{\mu}}_{\beta\overline{\sigma}}\, u_{\alpha\overline{\mu}},$$

we obtain

$$T_\beta(u_\alpha{}^\alpha) = h^{\alpha\overline{\sigma}} u_{\alpha\overline{\sigma}\beta} \tag{15}$$

(one replaces the ordinary derivatives by covariant derivatives and observes the cancellation of Christoffel symbols). Similarly

$$T_\beta(u_{\overline{\alpha}}{}^{\overline{\alpha}}) = h^{\alpha\overline{\sigma}} u_{\overline{\sigma}\alpha\beta}. \tag{16}$$

Then (14) – (16) lead to

$$(\Delta_b u)_\beta = -u^{\alpha}{}_{\alpha\beta} - u^{\overline{\alpha}}{}_{\overline{\alpha}\beta}. \tag{17}$$

To prove (13) we shall need the following commutation formulae

$$u_{\overline{\beta}\gamma\alpha} = u_{\alpha\gamma\overline{\beta}} - i\, h_{\alpha\overline{\beta}} u_{0\gamma} - u_\sigma R_\gamma{}^{\sigma}{}_{\alpha\overline{\beta}}, \tag{18}$$

$$u_{\beta\overline{\gamma}\alpha} = u_{\alpha\overline{\gamma}\beta} + i\, u_{\overline{\rho}} (h_{\alpha\overline{\gamma}} A^{\overline{\rho}}_{\beta} - h_{\beta\overline{\gamma}} A^{\overline{\rho}}_{\alpha}), \tag{19}$$

$$u_{\alpha\gamma\overline{\beta}} = u_{\gamma\alpha\overline{\beta}}, \tag{20}$$

$$u_{\alpha\overline{\gamma}\beta} = u_{\overline{\gamma}\alpha\beta} - i\, h_{\alpha\overline{\gamma}} u_{\beta 0}. \tag{21}$$

The identities (20), (21) are straightforward consequences of definitions. The proof of (18), (19) is a rather lengthy calculation based on (9) and on

$$(\nabla_X \omega^{\alpha}_{\beta}) Y - (\nabla_Y \omega^{\alpha}_{\beta}) X = 2\,(d\omega^{\alpha}_{\beta})(X, Y) - \omega^{\alpha}_{\beta}(T_\nabla(X, Y)).$$

We leave the details to the reader. At this point we may use (18)–(21) to rewrite (17) as

$$(\Delta_b u)_\beta = -u_{\alpha\beta}{}^{\alpha} - u_{\overline{\alpha}\beta}{}^{\overline{\alpha}} + 2\,i\,u_{\beta 0} + i\,(n-2)\,u_{\overline{\alpha}}\,A^{\overline{\alpha}}_{\beta} + u_\mu\,h^{\alpha\overline{\gamma}}\,R_\alpha{}^\mu{}_{\beta\overline{\gamma}}\,. \tag{22}$$

Finally, to see that (13) and (22) are equivalent we need a few curvature considerations. Let

$$R_{\beta\overline{\alpha}\rho\overline{\sigma}} = h_{\gamma\overline{\alpha}}\,R_\beta{}^\gamma{}_{\rho\overline{\sigma}}\,,$$

then (cf. also (1.36) in [10], p. 30)

$$\begin{aligned} R_{\beta\overline{\alpha}\rho\overline{\sigma}} &= R_{\rho\overline{\alpha}\beta\overline{\sigma}}\,,\\ R_{\beta\overline{\alpha}\rho\overline{\sigma}} + R_{\overline{\alpha}\beta\rho\overline{\sigma}} &= 0\,,\\ R_{\beta\overline{\alpha}\rho\overline{\sigma}} + R_{\beta\overline{\alpha}\overline{\sigma}\rho} &= 0\,. \end{aligned} \tag{23}$$

Thus (by (23))

$$h^{\alpha\overline{\gamma}}\,R_\alpha{}^\mu{}_{\beta\overline{\gamma}} = h^{\mu\overline{\alpha}}\,R_{\overline{\alpha}\beta}\,.$$

5 Gradient estimates.

Let M be a compact strictly pseudoconvex CR manifold and consider the problem

$$\begin{cases} \Delta_b v = \lambda_k v,\ \lambda_k > 0\,;\\ \sup v = 1\,;\\ \inf v = -C,\ 0 < C \le 1\,. \end{cases} \tag{24}$$

Set

$$u = \tfrac{2}{1+C}\left(v - \tfrac{1-C}{2}\right), \quad a = \tfrac{1-C}{1+C} \tag{25}$$

so that (24) becomes

$$\begin{cases} \Delta_b u = \lambda_k(u+a),\ 0 < a \le 1;\\ \sup u = 1\,;\\ \inf u = -1\,. \end{cases}$$

Let $f\ :\ M \to \mathbb{R}$ be given by

$$f = \frac{\|\pi_+\nabla u\|^2}{1-u^2}\,. \tag{26}$$

Strictly speaking, one should work with $(1+\epsilon)^{-1}u$ instead of u, for some $\epsilon > 0$ (and let $\epsilon \to 0$ in the end). A word on the notation in (26). There ∇u is the gradient of u with respect to the Webster metric g_θ (thought of as an element of $T(M)\otimes\mathbb{C}$) and $\pi_+\nabla u$ is its $(1,0)$ component. Given a (local) frame $\{T_1,\dots,T_n\}$ of $T_{1,0}(M)$ one may write (26) as

$$f = (1-u^2)^{-1}u_\alpha\,u^\alpha$$

where $u^\alpha = h^{\alpha\overline{\beta}}u_{\overline{\beta}}$. Let $\gamma\ :\ M \to (-\pi/2,\pi/2)$ be a C^∞ function so that $u = \sin\gamma$. Then $f = \gamma_\alpha\gamma^\alpha$. Next, consider $F\ :\ (-\pi/2,\pi/2) \to \mathbb{R}$ given by

$$F(t) = \sup\{f(x)\,|\,x \in \gamma^{-1}(t),\ |t| < \tfrac{\pi}{2}\}\,.$$

Clearly, we may (with $F(-\pi/2+0) = 0, F(\pi/2-0) = 0$) regard F as continuous on $[-\pi/2, \pi/2]$. Assume that F attains its maximum at t_0. As M is compact, there is $x_0 \in \gamma^{-1}(t_0)$ so that $f(x_0) = F(t_0)$. Thus

$$\Phi = (f - F(t_0)) \cos^2 \gamma$$

attains its maximum at x_0. Consequently

$$(\nabla\Phi)(x_0) = 0, \tag{27}$$

$$(\nabla^2\Phi)_{x_0}(X, X) \leq 0 \tag{28}$$

for any $X \in T_{x_0}(M)$. Note that $\nabla^2\Phi$ in (28) is the pseudohermitian Hessian of Φ. Yet, it is an elementary matter that the Hessian of Φ (defined by (11)) with respect to *any* linear connection ∇ on M is negative semi-definite at a maximum point of Φ. The sublaplacian Δ_b is also given by

$$\Delta_b u = -\operatorname{trace}\{\pi_H \nabla^2 u\}.$$

Here $(\pi_H B)(X, Y) = B(\pi_H X, \pi_H Y)$ for any $X, Y \in T(M)$ and any bilinear form B. Also $\pi_H : T(M) \to H(M)$ is the natural bundle map (associated with (A.1)). Thus (28) yields

$$(\Delta_b\Phi)(x_0) \geq 0. \tag{29}$$

We wish to prove the following estimate

$$2F(t_0) \leq \lambda_k(1 + a). \tag{30}$$

Let us apply T_β to $\Phi = u_\alpha u^\alpha - F(t_0) \cos^2 \gamma$ so that to yield

$$\Phi_\beta = u^\alpha u_{\beta\alpha} + u^{\overline{\alpha}} u_{\beta\overline{\alpha}} + F(t_0)\gamma_\beta \sin(2\gamma). \tag{31}$$

Next:

$$\begin{aligned}\Phi_{\alpha\overline{\beta}} = {} & u^\sigma_\alpha u_{\overline{\beta}\sigma} + u^{\overline{\sigma}}_\alpha u_{\overline{\beta}\overline{\sigma}} + u^\sigma u_{\overline{\beta}\sigma\alpha} + u^{\overline{\sigma}} u_{\overline{\beta}\overline{\sigma}\alpha} \\ & + F(t_0)\gamma_{\alpha\overline{\beta}} \sin(2\gamma) + 2F(t_0)\gamma_\alpha\gamma_{\overline{\beta}} \cos(2\gamma)\end{aligned} \tag{32}$$

where from (by taking into account (17))

$$\begin{aligned}\Delta_b\Phi = {} & F(t_0)\Delta_b\gamma \cdot \sin(2\gamma) - 4F(t_0)\gamma_\alpha\gamma^\alpha \cos(2\gamma) \\ & - 2\left(u_{\alpha\beta}u^{\alpha\beta} + u^{\overline{\alpha}\beta}_{\alpha\overline{\beta}}\right) - u^\alpha u_{\beta\alpha}{}^\beta - u^{\overline{\alpha}} u_{\beta\overline{\alpha}}{}^\beta - u^\alpha u_{\overline{\beta}\alpha}{}^{\overline{\beta}} - u^{\overline{\alpha}} u_{\overline{\beta}\overline{\alpha}}{}^{\overline{\beta}}.\end{aligned} \tag{33}$$

Let Ric_θ be the Ricci tensor field of (M, g_θ) and set $R^\theta_{\alpha\overline{\beta}} = \mathrm{Ric}_\theta(T_\alpha, T_{\overline{\beta}})$. Then (by (A.3))

$$R^\theta_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}} - \tfrac{1}{2} h_{\alpha\overline{\beta}},$$

so that $R_{\alpha\overline{\beta}} = R_{\overline{\beta}\alpha}$. Then (by (13)) we may rewrite (33) as

$$\begin{aligned}\Delta_b\Phi = {} & F(t_0)\Delta_b\gamma \cdot \sin(2\gamma) - 4F(t_0)\gamma_\alpha\gamma^\alpha \cos(2\gamma) \\ & - 2\left(u_{\alpha\beta}u^{\alpha\beta} + u_{\alpha\overline{\beta}}u^{\alpha\overline{\beta}}\right) + u^\alpha(Lu)_\alpha + u^{\overline{\alpha}}(\overline{L}u)_{\overline{\alpha}} \\ & - 2R_{\alpha\overline{\beta}}u^\alpha u^{\overline{\beta}} + i(n-2)\left(A_{\overline{\alpha}\overline{\beta}}u^{\overline{\alpha}}u^{\overline{\beta}} - A_{\alpha\beta}u^\alpha u^\beta\right).\end{aligned} \tag{34}$$

Assume from now on that

$$\mathrm{Ric}(X - iJX, X + iJX) + 2(n-2)\,A(X,JX) \geq 0 \tag{35}$$

for any $X \in H(M)$. Here $A(X,Y) = g_\theta(\tau X, Y)$. Clearly, if M is five dimensional it suffices to assume $R_{\alpha\bar\beta}$ to be positive semi-definite. Using (29), (34) and (35) we obtain

$$\begin{aligned} u^\alpha (Lu)_\alpha + u^{\bar\alpha}(\bar L u)_{\bar\alpha} \geq 2\left(u_{\alpha\beta}u^{\alpha\beta} + u_{\alpha\bar\beta}u^{\alpha\bar\beta}\right) + 4\,F(t_0)\gamma_\alpha\gamma^\alpha \cos(2\gamma) \\ - F(t_0)\Delta_b\gamma \cdot \sin(2\gamma) \end{aligned} \tag{36}$$

at x_0. Let $V \subset \mathbb{C}^{4n^2}$ be the (real) subspace given by

$$V = \left\{ \left(\begin{smallmatrix} A & B \\ \bar A & \bar B \end{smallmatrix}\right) \,\middle|\, A, B \in \mathcal{M}_n(\mathbb{C}) \right\}.$$

We endow V with the inner product

$$\begin{pmatrix} X_{\alpha\beta} & X_{\alpha\bar\beta} \\ X_{\bar\alpha\beta} & X_{\bar\alpha\bar\beta} \end{pmatrix} \cdot \begin{pmatrix} Y_{\alpha\beta} & Y_{\alpha\bar\beta} \\ Y_{\bar\alpha\beta} & Y_{\bar\alpha\bar\beta} \end{pmatrix} = X_{\alpha\beta}Y^{\alpha\beta} + X_{\alpha\bar\beta}Y^{\alpha\bar\beta}$$

where $Y^{\alpha\beta} = h^{\alpha\bar\lambda}(x_0)\, h^{\beta\bar\mu}(x_0)\, Y_{\bar\lambda\bar\mu}$, etc. The Cauchy-Schwarz inequality on V for the vectors

$$X = \begin{pmatrix} u_{\alpha\beta} & u_{\alpha\bar\beta} \\ u_{\bar\alpha\beta} & u_{\bar\alpha\bar\beta} \end{pmatrix}, \quad Y = \begin{pmatrix} u_\alpha u_\beta & u_\alpha u_{\bar\beta} \\ u_{\bar\alpha}u_\beta & u_{\bar\alpha}u_{\bar\beta} \end{pmatrix}$$

leads to

$$u_{\alpha\beta}u^{\alpha\beta} + u_{\alpha\bar\beta}u^{\alpha\bar\beta} \geq \frac{\left(u_{\alpha\beta}u^\alpha u^\beta + u_{\alpha\bar\beta}u^\alpha u^{\bar\beta}\right)}{2\,(u_\alpha u^\alpha)^2} \tag{37}$$

at x_0. By (27) and (31) we have

$$u^\beta u_{\alpha\beta} + u^{\bar\beta}u_{\alpha\bar\beta} + F(t_0)\gamma_\alpha \sin(2\gamma) = 0 \tag{38}$$

at x_0. Let us contract with u^α in (38) and use the resulting identity to rewrite (37) as

$$u_{\alpha\beta}u^{\alpha\beta} + u_{\alpha\bar\beta}u^{\alpha\bar\beta} \geq 2\,F(t_0)^2 \sin^2\gamma \tag{39}$$

at x_0. Let v be a solution of (3) and u given by (25). The estimates (36) and (39) lead to

$$4\,F(t_0)^2\cos^2\gamma - F(t_0)\Delta_b\gamma \cdot \sin(2\gamma) \leq 2\,\lambda_k u_\alpha u^\alpha. \tag{40}$$

Finally, taking into account the identities

$$u_\alpha u^\alpha = F(t_0)\cos^2\gamma,$$
$$\Delta_b u = \Delta_b\gamma \cdot \cos\gamma + 2\,F(t_0)\sin\gamma$$

(at x_0) we may write (40) as

$$2\,F(t_0)^2\cos^2\gamma - F(t_0)\sin\gamma\left(\lambda_k(u+a) - 2\,F(t_0)\sin\gamma\right) \leq \lambda_k F(t_0)\cos^2\gamma$$

which (after some simplifications) gives

$$2\,F(t_0) \leq \lambda_k(1 + a\sin\gamma) \leq \lambda_k(1+a)$$

and (30) is proved.

6 Proof of the main result.

We define a function $\varphi(t)$ by setting $F(t) = \lambda_k(1 + a\varphi(t))/2$ for any $|t| \leq \pi/2$. Then (by (30)) $\varphi(t) \leq 1$. We shall need the following

Lemma. 1. *Assume $\varphi(t_0) \geq -1$ for some $|t_0| < \pi/2$. Let $y : (-\pi/2, \pi/2) \to \mathbb{R}$ be a C^2 function so that*

i) $y(t) \geq \varphi(t)$ *for any* $|t| < \pi/2$;
ii) $y(t_0) = \varphi(t_0)$;
iii) $y'(t_0) \geq 0$.

Then

$$\varphi(t_0) \leq \sin t_0 - y'(t_0)\sin t_0 \cos t_0 + \tfrac{1}{2}y''(t_0)\cos^2 t_0 .$$

Proof. Let $x_0 \in M$ so that $\gamma(x_0) = t_0$ and $f(x_0) = F(t_0)$. The function $\Phi : M \to \mathbb{R}$ given by

$$\Phi = \left(f - \tfrac{1}{2}\lambda_k(1 + ay \circ \gamma)\right)\cos^2\gamma$$

attains its maximum at x_0 so that

$$\nabla\Phi(x_0) = 0, \quad \Delta_b\Phi(x_0) \geq 0 . \tag{41}$$

Let us apply T_β to $\Phi = u_\alpha u^\alpha - \lambda_k \cos^2\gamma(1 + ay(\gamma))/2$ so that to yield

$$\Phi_\beta = u^\alpha u_{\beta\alpha} + u^{\overline{\alpha}} u_{\beta\overline{\alpha}} + \tfrac{1}{2}\lambda_k\left((1 + ay(\gamma))\sin(2\gamma) - ay'(\gamma)\cos^2\gamma\right)\cdot\gamma_\beta . \tag{42}$$

Next:

$$\begin{aligned}\Phi_{\alpha\overline{\beta}} &= u_\alpha{}^\sigma u_{\overline{\beta}\sigma} + u_\alpha{}^{\overline{\sigma}} u_{\overline{\beta}\overline{\sigma}} + u^\sigma u_{\overline{\beta}\sigma\alpha} + u^{\overline{\sigma}} u_{\overline{\beta}\overline{\sigma}\alpha} \\ &\quad + \tfrac{1}{2}\lambda_k\left((1 + ay)\sin(2\gamma) - ay'\cos^2\gamma\right)\gamma_{\alpha\overline{\beta}} \\ &\quad + \lambda_k\left((1 + ay)\cos(2\gamma) + ay'\sin(2\gamma) - \tfrac{1}{2}ay''\cos^2\gamma\right)\gamma_\alpha\gamma_{\overline{\beta}} .\end{aligned} \tag{43}$$

Consequently (by (14)):

$$\begin{aligned}\Delta_b\Phi &= -u^{\alpha\beta}u_{\alpha\beta} - u^{\alpha\overline{\beta}}u_{\alpha\overline{\beta}} - u^{\overline{\alpha}\beta}u_{\overline{\alpha}\beta} - u^{\overline{\alpha}\overline{\beta}}u_{\overline{\alpha}\overline{\beta}} \\ &\quad - u^\alpha u_{\beta\alpha}{}^\beta - u^{\overline{\alpha}} u_{\beta\overline{\alpha}}{}^\beta - u^\alpha u_{\overline{\beta}\alpha}{}^{\overline{\beta}} - u^{\overline{\alpha}} u_{\overline{\beta}\overline{\alpha}}{}^{\overline{\beta}} \\ &\quad + \lambda_k\left((1 + ay)\sin\gamma - \tfrac{1}{2}ay'\cos\gamma\right)\cos\gamma\cdot\Delta_b\gamma \\ &\quad - 2\,\lambda_k\left((1 + ay)\cos(2\gamma) + ay'\sin(2\gamma) - \tfrac{1}{2}ay''\cos^2\gamma\right)\gamma_\alpha\gamma^\alpha .\end{aligned} \tag{44}$$

Using (13) we may rewrite (44) as

$$\begin{aligned}\Delta_b\Phi &= \lambda_k\left((1 + ay)\sin\gamma - \tfrac{1}{2}ay'\cos\gamma\right)\cos\gamma\cdot\Delta_b\gamma \\ &\quad - 2\,\lambda_k\left((1 + ay)\cos(2\gamma) + ay'\sin(2\gamma) - \tfrac{1}{2}ay''\cos^2\gamma)\right)\gamma_\alpha\gamma^\alpha \\ &\quad - 2\left(u_{\alpha\beta}u^{\alpha\beta} + u_{\alpha\overline{\beta}}u^{\alpha\overline{\beta}}\right) + u^\alpha(Lu)_\alpha + u^{\overline{\alpha}}(\overline{L}u)_{\overline{\alpha}} \\ &\quad - 2R_{\alpha\overline{\beta}}u^\alpha u^{\overline{\beta}} + i\,(n-2)\left(A_{\overline{\alpha}\overline{\beta}}u^{\overline{\alpha}}u^{\overline{\beta}} - A_{\alpha\beta}u^\alpha u^\beta\right) .\end{aligned} \tag{45}$$

Using (41) and (45) we may write (by the assumption (35) on the geometry of (M,θ))

$$\begin{aligned} 2\big(u_{\alpha\beta}u^{\alpha\beta} + u_{\alpha\bar\beta}u^{\alpha\bar\beta}\big) &\le u^{\alpha}(Lu)_{\alpha} + u^{\bar\alpha}(\overline{L}u)_{\bar\alpha} \\ &+ \lambda_k\big((1+ay)\sin\gamma - \tfrac{1}{2}ay'\cos\gamma\big)\cos\gamma\cdot\Delta_b\gamma(x_0) \\ &- 2\,\lambda_k\big((1+ay)\cos(2\gamma) + ay'\sin(2\gamma) - \tfrac{1}{2}ay''\cos^2\gamma\big)F(t_0) \end{aligned} \tag{46}$$

at x_0. By (41) and (42) we have

$$u^{\beta}u_{\alpha\beta} + u^{\bar\beta}u_{\alpha\bar\beta} + \lambda_k\big((1+ay)\sin\gamma - \tfrac{1}{2}ay'\cos\gamma\big)\cos\gamma\cdot\gamma_\alpha = 0 \tag{47}$$

at x_0. Let us contract with u^α in (47) so that to rewrite (37) as

$$2\big(u_{\alpha\beta}u^{\alpha\beta} + u_{\alpha\bar\beta}u^{\alpha\bar\beta}\big) \ge \lambda_k^2\big((1+ay)\sin\gamma - \tfrac{1}{2}ay'\cos\gamma\big)^2 \tag{48}$$

at x_0. Let v be a solution of (3) and u given by (25). The estimates (40) and (48) lead to

$$\begin{aligned} \lambda_k^2\big((1+ay)\sin\gamma - \tfrac{1}{2}ay'\cos\gamma\big)^2 &\le 2\lambda_1 u_\alpha u^\alpha \\ &+ \lambda_k\big((1+ay)\sin\gamma - \tfrac{1}{2}ay'\cos\gamma\big)\cos\gamma\cdot\Delta_b\gamma \\ &- 2\,\lambda_k\big((1+ay)\cos(2\gamma) + ay'\sin(2\gamma) - \tfrac{1}{2}ay''\cos^2\gamma\big)F(t_0)\,. \end{aligned}$$

Since $y'^2\cos^2\gamma \ge 0$ we obtain (after some simplifications)

$$y + \tfrac{1}{2}y'\sin\gamma\cos\gamma \le \sin\gamma - \tfrac{1}{2}y'\tfrac{a+\sin\gamma}{1+ay} + \tfrac{1}{2}y''\cos^2\gamma\,. \tag{49}$$

As $|y| \le 1$ at x_0 we have $a + \sin\gamma \ge ay\sin\gamma + \sin\gamma = (1+ay)\sin\gamma$ so that (49) becomes

$$y - \sin\gamma \le \tfrac{1}{2}y''\cos^2\gamma - y'\sin\gamma\cdot\cos\gamma$$

and Lemma 1 is completely proved. □

We shall need

Lemma. 2. *The function* $\Psi : (-\pi/2, \pi/2) \to \mathbb{R}$ *given by*

$$\Psi(t) = \frac{2\big(2t + \sin(3t)\big)/\pi - 2\sin t}{\cos^2 t}$$

possesses the following properties:

i) $\Psi(-\pi/2+0) = -1$, $\Psi(\pi/2-0) = 1$;
ii) $\Psi \in C^0([-\pi/2,\pi/2]) \cap C^2((-\pi/2,\pi/2))$;
iii) Ψ *satisfies the* ODE

$$y + \sin t\,\cos t\,y' - \tfrac{1}{2}\cos^2 t\,y'' = \sin t\,. \tag{50}$$

Proof. Indeed (50) may be written $(y'/2 - y\tan t - 1/\cos t)' = 0$ so that $y = \big(A(2t + \sin(2t))/2 + 2\sin t + B\big)/(\cos^2 t)$, with A, $B \in \mathbb{R}$, etc. □

Let $h(t) = \varphi(t) - \Psi(t)$ and set $b = \sup h$. We need to show that

$$\varphi(t) \le \Psi(t) \tag{51}$$

for any $|t| \le \pi/2$. The proof is by contradiction. If (51) is false then $b > 0$. Note that $h(\pi/2-0) = -1 - 1/a < 0$ and $h(-\pi/2+0) = 1 - 1/a < 0$ so that b is attained at some $t_0 \in (-\pi/2, \pi/2)$. Next $\varphi(t_0) \geq -1$ (otherwise $\varphi(t_0) < -1$ yields $b < -1 - \Psi(t_0) \leq 0$, a contradiction). Set $y = \Psi + b$. Then y satisfies the hypothesis of Lemma 1 so that (by Lemmae 1 and 2) $\varphi(t_0) \leq \sin t_0 - \sin t_0 \cos t_0 \Psi'(t_0) + \cos^2 t_0 \Psi''(t_0)/2 = \Psi(t_0)$, a contradiction. At this point we may prove (5). To this end, let $x_1, x_2 \in M$ so that $\gamma(x_1) = -\pi/2$ and $\gamma(x_2) = \pi/2$. Let $C : [0,1] \to M$ be a minimizing geodesic of (M, g_θ) so that $C(0) = x_1$ and $C(1) = x_2$ and denote by $\ell(C)$ its (Riemannian) length. By (51) we have

$$2F(t) \leq \lambda_k(1 + a\Psi(t)) \tag{52}$$

for any $|t| \leq \pi/2$. Thus (as $\gamma_0 = 0$)

$$\|\nabla\gamma\|^2 \leq \lambda_k(1 + a\Psi \circ \gamma)$$

everywhere on M. Then we may perform the following estimates

$$\sqrt{\lambda_k}\, d_\theta \geq \sqrt{\lambda_k}\, \ell(C) = \sqrt{\lambda_k} \int_0^1 \|dC/dt\|\, dt$$

$$\geq \int_0^1 \frac{\|dC/dt\|\, \|\nabla\gamma\|}{\sqrt{1 + a\Psi(\gamma(C(t)))}} dt \geq \int_0^1 \frac{g(\frac{dC}{dt}, \nabla\gamma)_{C(t)}}{\sqrt{1 + a\Psi(\gamma(C(t)))}} dt$$

$$= \int_{-\pi/2}^{\pi/2} \frac{dt}{\sqrt{1 + a\Psi(t)}} = \int_0^{\pi/2} \Big(\frac{1}{\sqrt{1 - a\Psi(t)}} + \frac{1}{\sqrt{1 + a\Psi(t)}}\Big) dt$$

$$= \int_0^{\pi/2} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} (a\Psi)^k \big(1 + (-1)^k\big)\, dt$$

where $(2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$ and $(2k)!! = 2^k k!$. Finally

$$\sqrt{\lambda_k}\, d_\theta \geq \pi + \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \big(1 + (-1)^k\big) a^k \int_0^{\pi/2} \Psi(t)^k\, dt \geq \pi$$

and (5) is proved.

Remarks. 1) Let M be a strictly pseudoconvex CR manifold of vanishing pseudohermitian torsion. Suitable contraction of indices in (A.3) furnishes

$$\mathrm{Ric}_\theta(X, Y) = \mathrm{Ric}(X, Y) - \tfrac{1}{2} g_\theta(X, Y) + \tfrac{n+1}{2} \theta(X)\theta(Y) \tag{53}$$

for any $X, Y \in T(M)$. Thus $\mathrm{Ric}(X - iJX, X + iJX) = \mathrm{Ric}_\theta(X, X) + \mathrm{Ric}_\theta(JX, JX) + \|X\|^2$ for any $X \in H(M)$. Consequently, if $\mathrm{Ric}_\theta(X, X) \geq 0$ for any $X \in T(M)$ then $R_{\alpha\bar\beta}$ is positive semi-definite (while the converse does not follow from (53)).

2) The problem of the existence of a solution of (3) is open. If for instance $M = S^{2n+1}$ then (3) has no solution for $k = 1$ (i.e. there is no first degree harmonic polynomial H on $\mathbb{R}^{2(n+1)}$ satisfying $T(H) = 0$). Next, all solutions of

$$\begin{cases} \Delta_b v = \lambda_2 v, \\ T(v) = 0 \end{cases}$$

are given as $v = H_{|S^{2n+1}}$ where $H(x,y) = \sum_{1\le i<j\le n+1} a_{ij}(x_i x_j + y_i y_j)$, $a_{ij} \in \mathbb{R}$ (and $\lambda_2 = 4(n+1)$). Also, for each $(i,j) \in \{1,\dots,n+1\}^2$, $i<j$, the eigenfunction $v_{ij} = H_{ij|S^{2n+1}}$, where $H_{ij} = 2(x_i x_j + y_i y_j)$, has $\sup v_{ij} = 1$ and $\inf v_{ij} = -1$ (i.e. v_{ij} is a solution of (3) with $k=2$ and $C=1$).

3) The estimate (1) may be thought of as an estimate on λ_k, $k \ge 2$. As such, (5) is sharper than (1) provided that

$$d_\theta < \pi \sqrt{\frac{n+1}{nC_0}}\,. \tag{54}$$

However, among the odd dimensional spheres only S^3 and S^5 satisfy (54) (as $M = S^{2n+1}$ yields $C_0 = n+1$, cf. [10]).

Appendix

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold and θ a fixed contact 1-form on M. Let T be the characteristic direction of (M,θ). Define G_θ by setting

$$G_\theta(X,Y) = (d\theta)(X,JY)$$

for any $X, Y \in H(M)$. Since

$$T(M) = H(M) \oplus \mathbb{R}T \tag{A.1}$$

one may extend G_θ to a semi-Riemannian metric g_θ on M by setting

$$g_\theta(X,Y) = G_\theta(X,Y), \quad g_\theta(X,T) = 0, \quad g_\theta(T,T) = 1$$

for any $X, Y \in H(M)$. The Levi-Civita connection ∇^θ of (M, g_θ) and the Webster connection ∇ of (M,θ) are related by

$$\nabla^\theta = \nabla + (\tfrac{1}{2}\Omega_\theta - A) \otimes T + \tau \otimes \theta + \theta \odot J \tag{A.2}$$

where $\Omega_\theta(X,Y) = g_\theta(X,JY)$. Also $\odot$ denotes the symmetric product. Let R^θ, R be the curvature tensor fields of ∇^θ, ∇, respectively. A straightforward calculation based on (A.2) shows that

$$\begin{aligned} R^\theta(X,Y)Z = {} & R(X,Y)Z - (KX \wedge KY)Z + \theta(Z)S(X,Y) \\ & - g_\theta(S(X,Y),Z)T + 2\theta(Z)(\theta \wedge \mathcal{O})(X,Y) \\ & - 2g_\theta((\theta \wedge \mathcal{O})(X,Y),Z)T - \tfrac{1}{2}\Omega_\theta(X,Y)JZ\,. \end{aligned} \tag{A.3}$$

A word on the notation in (A.3). There $S(X,Y) = (\nabla_X \tau)Y - (\nabla_Y \tau)X$. Also $K = \tau + \frac{1}{2}J$ and $\mathcal{O} = \tau^2 + J\tau - \frac{1}{4}I$, where I denotes the identical transformation. Finally $(X \wedge Y)Z = g_\theta(Y,Z)X - g_\theta(X,Z)Y$ is the usual wedge product of two tangent vector fields on (M, g_θ). Using (A.3) and the known symmetries (cf. (d) in [5], vol I, p, 198) of the Riemann-Christoffel tensor field of (M, g_θ)

we obtain

$$\begin{aligned} g_\theta(R(X,Y)Z,W) &= g_\theta(R(W,Z)Y,X) \\ &+ g_\theta((KX \wedge KY)Z,W) - g_\theta((KW \wedge KZ)Y,X) \\ &+ \theta(Y) g_\theta(S(W,Z),X) - \theta(Z) g_\theta(S(X,Y),W) \\ &+ \theta(W) g_\theta(S(X,Y),Z) - \theta(X) g_\theta(S(W,Z),Y) \\ &+ 2\,\theta(Y) g_\theta((\theta \wedge \mathcal{O})(W,Z),X) - 2\,\theta(Z) g_\theta((\theta \wedge \mathcal{O})(X,Y),W) \\ &+ 2\,\theta(W) g_\theta((\theta \wedge \mathcal{O})(X,Y),Z) - 2\,\theta(X) g_\theta((\theta \wedge \mathcal{O})(W,Z),Y) \end{aligned} \tag{A.4}$$

for any tangent vector fields X, Y, Z, W on M. Next (A.4) furnishes

$$\begin{cases} R_\alpha{}^\beta{}_{\lambda\mu} = i(A_{\alpha\mu}\delta_\lambda^\beta - A_{\alpha\lambda}\delta_\mu^\beta), \\ R_\alpha{}^\beta{}_{\bar\lambda\bar\mu} = i(A_{\bar\mu}^\beta h_{\alpha\bar\lambda} - A_{\bar\lambda}^\beta h_{\alpha\bar\mu}), \\ R_\alpha{}^\beta{}_{\lambda 0} = W_{\alpha\lambda}^\beta,\ R_\alpha{}^\beta{}_{\bar\lambda 0} = -W_{\alpha\bar\lambda}^\beta, \end{cases} \tag{A.5}$$

where

$$W_{\alpha\lambda}^\beta = h^{\beta\bar\gamma} A_{\alpha\lambda,\bar\gamma}, \quad W_{\alpha\bar\lambda}^\beta = h^{\beta\bar\gamma} A_{\bar\lambda\bar\gamma,\alpha}$$

and

$$A_{\alpha\beta,\bar\gamma} = (\nabla_{T_{\bar\gamma}} A)(T_\alpha, T_\beta)$$

is the covariant derivative of the pseudohermitian torsion (with respect to the Webster connection). The first two identities in (A.5) lead to

$$R_\alpha{}^\beta{}_{\lambda\mu} \wedge \theta^\mu = 2\,i\,\theta^\beta \wedge \tau_\alpha\,,$$

$$R_\alpha{}^\beta{}_{\bar\lambda\bar\mu} \wedge \theta^{\bar\mu} = 2\,i\,\theta_\alpha \wedge \tau^\beta$$

where $\tau_\alpha = h_{\alpha\bar\beta}\,\tau^{\bar\beta}$, ect. Finally (9) follows from (A.5) and the identity

$$R(X,Y)T_\alpha = 2(d\omega_\alpha^\beta - \omega_\alpha^\mu \wedge \omega_\mu^\beta)(X,Y)T_\beta\,.$$

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