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MARTINGALES WITH GIVEN MAXIMA AND TERMINAL DISTRIBUTIONS

BY

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ABSTRACT

Let μ be any probability measure on $\mathbf{R}$ with $\int |x|\,d\mu(x) < \infty$, and let μ^* denote its associated Hardy and Littlewood maximal p.m. It is shown that for any p.m. ν for which $\mu < \nu < \mu^*$ in the usual stochastic order, there is a martingale $(X_t)_{0 \leqq t \leqq 1}$ for which $\sup_{0 \leqq t \leqq 1} X_t$ and X_1 have respective p.m.'s ν and μ. The proof uses induction and weak convergence arguments; in special cases, explicit martingale constructions are given. These results provide a converse to results of Dubins and Gilat [6]; applications are made to give sharp martingale and 'prophet' inequalities.

0. Introduction

For any martingale $X = (X_t)_{0 \leqq t \leqq 1}$ with integrable right element X_1, let μ and ν be the probability measures associated with X_1 and $M = M(X) := \sup_{0 \leqq t \leqq 1} X_t$ respectively. Blackwell and Dubins [5] and Dubins and Gilat [6] have shown that $\mu < \nu < \mu^*$ (with the usual stochastic order, see (2.1)), where μ^* is the Hardy and Littlewood maximal probability measure associated with μ, and have produced martingales for which $\nu = \mu$ and for which $\nu = \mu^*$. In this paper, this result is sharpened, as a converse question is considered: if μ and ν are probability measures with $\mu < \nu < \mu^*$, is there a martingale $X = (X_t)_{0 \leqq t \leqq 1}$ for which $M \overset{\mathscr{D}}{=} \nu$ and $X_1 \overset{\mathscr{D}}{=} \mu$? The answer is yes, as it is shown that for any p.m. μ on $\mathbf{R}$ with $\int |x|\,d\mu(x) < \infty$, the following collections of probability measures are equal:

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$$\{\nu\colon \text{there is a martingale } X=(X_t)_{0\le t\le 1} \text{ with } M \overset{\mathcal{D}}{=} \nu, \text{ and } X_1 \overset{\mathcal{D}}{=} \mu\} \tag{0.1}$$
$$= \{\nu \text{ is a p.m. on } \mathbf{R} : \mu < \nu < \mu^*\}.$$

The result is given as Theorem 2.1, and is proved from induction and weak convergence arguments.

For given p.m.'s μ_0 and μ_1 on $\mathbf{R}$ with $\int |x|\, d\mu_1(x) < \infty$, the subcollection of the set of p.m.'s of (0.1) given by

$$\{\nu\colon \text{there is a martingale } X=(X_t)_{0\le t\le 1} \text{ with } X_0 \overset{\mathcal{D}}{=} \mu_0,\ M \overset{\mathcal{D}}{=} \nu, \text{ and } X_1 \overset{\mathcal{D}}{=} \mu_1\} \tag{0.2}$$

is characterized in Theorem 3.4. The set in (0.2) is nonempty, for example, if μ_0 and μ_1 are concentrated on some closed interval and $\mu_0 <_c \mu_1$ (i.e., $\int \psi d\mu_0 \leqq \int \psi d\mu_1$ for all continuous convex functions ψ on the interval); see, e.g., Theorem 2 of Strassen [22] or Chapter XI of Meyer [16]. The characterization of the set in (0.2) follows from (0.1) and two intermediate characterizations. One of these states that for any probability measure μ with $\int x d\mu(x) = 0$, the following collections of probability measures are equal:

$$\{\nu\colon \text{there is a martingale } X=(X_t)_{0\le t\le 1} \text{ with } X_0 \equiv 0,\ M \overset{\mathcal{D}}{=} \nu, \text{ and } X_1 \overset{\mathcal{D}}{=} \mu\} \tag{0.3}$$
$$= \{\nu \text{ is a p.m. on } [0,\infty) : \mu < \nu < \mu^*\}.$$

The result (0.3) is easily proved using (0.1). However, a constructive proof of (0.3) is given which uses the martingale (2.4) of Dubins and Gilat [6] or equivalently a time-changed Brownian motion; this approach connects this result to those of Dubins and Gilat [6], Azema and Yor [1, 2], and van der Vecht [23]. Other connections of these martingale questions to embeddings of martingales into Brownian motion, which involve related (but different) issues, have been made by Jacka [12] and Perkins [17].

These stochastic order representations can be used to prove sharp martingale inequalities relating M and X_1. Such inequalities have been given by Dubins and Gilat [6], Gilat [9], and Hardy and Littlewood [10]. In the context of optimal stopping, some of these inequalities have been referred to as 'prophet' inequalities; initial work in this area of 'prophet' inequalities was done by Krengel and Sucheston [14, 15] in the context of independent r.v.'s and sums of independent r.v.'s. In Theorem 4.1, a 'prophet' inequality of Dubins and Pitman [7] (see also Hill and Kertz [11]) is sharpened by use of results from Section 2. We also give an interpretation of our main result in a 'prophet' problem context in Section 4.

1. The set of measures $\mathcal{M}(\mu)$: Definition and properties

In this paper $X = (X_t)_{0 \leqq t \leqq 1}$ is a *martingale* if there is some probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t\}_{0 \leqq t \leqq 1}$ on $(\Omega, \mathcal{F}, P)$ under which (i) $(X_t)_{0 \leqq t \leqq 1}$ is $\{\mathcal{F}_t\}$-adapted, (ii) X_t is integrable for every $0 \leqq t \leqq 1$ and $E(X_t \mid \mathcal{F}_s) = X_s$ a.s. $[P]$ for every $0 \leqq s < t \leqq 1$ and (iii) the paths $t \mapsto X_t$ are right continuous and have left-hand limits for $0 \leqq t \leqq 1$ (RCLL).

We define $\mathcal{M}(\mu)$, the main object of study in this paper. Let $\mathcal{P}(\mathbf{R})$ be the space of probability measures on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, given the topology of convergence-in-distribution, so that under the Prohorov metric, this is a complete, separable metric space (for reference see, e.g., Ethier and Kurtz [8]). For each p.m. μ on $\mathbf{R}$ satisfying $\int |x| \, d\mu(x) < \infty$, let $\mathcal{M}(\mu)$ denote the set in $\mathcal{P}(\mathbf{R})$ given by

$$\mathcal{M}(\mu) = \{\nu \in \mathcal{P}(\mathbf{R}) : \text{there is a martingale } X = (X_t)_{0 \leqq t \leqq 1} \text{ for which } M \stackrel{\mathcal{D}}{=} \nu, \text{ and } X_1 \stackrel{\mathcal{D}}{=} \mu\}, \tag{1.1}$$

where $M = M(X) = \sup_{0 \leqq t \leqq 1} X_t$ and we write $Y \stackrel{\mathcal{D}}{=} \lambda$ if r.v. Y has distribution that of p.m. λ. If p.m. μ has associated r.v. Z, then by letting $X_t = Z$ for all $0 \leqq t \leqq 1$, it is clear that $\mu \in \mathcal{M}(\mu)$, so that $\mathcal{M}(\mu)$ is nonempty.

Our main objective in this section is to show that $\mathcal{M}(\mu)$ is convex and closed; this is done in Propositions 1.2 and 1.6. We will use path properties of martingales. For this purpose, we let $D = D[0, 1]$ denote the space of functions x on $[0, 1]$ that are right continuous and have left-hand limits, given the Skorohod J_1 topology and associated Borel σ-algebra $\mathcal{B}(D)$, and with metric d_0 under which D is a separable complete metric space (for reference, see Billingsley [2] and Ethier and Kurtz [8]). Each martingale $X = (X_t)_{0 \leqq t \leqq 1}$ induces a p.m. on $(D, \mathcal{B}(D))$, denoted by $P_X(\cdot) = P(X \in \cdot)$. We use '$\Rightarrow$' to denote weak convergence of p.m.'s on $(D, \mathcal{B}(D))$ (and of the associated stochastic processes), and write $X^n \Rightarrow X$ for $P_{X^n} \Rightarrow P_X$. We also use $Y_n \Rightarrow Y$ to denote weak convergence (convergence in distribution) of r.v.'s $\{Y_n\}_{n \geqq 1}$ to r.v. Y. The following lemma sets up the proof of convexity of $\mathcal{M}(\mu)$. In the remainder of this section, when discussing $\mathcal{M}(\mu)$, it is implicitly understood that $\mu \in \mathcal{P}(\mathbf{R})$ with $\int |x| \, d\mu(x) < \infty$.

Lemma 1.1. *Given martingales X^1 and X^2, then there is a martingale X for which $X = X^1$ with probability λ, and $= X^2$ with probability $1 - \lambda$.*

Proof. This follows easily by making appropriate definitions. For example, if $X^1 = (X_t^1)_{0 \leqq t \leqq 1}$ and $X^2 = (X_t^2)_{0 \leqq t \leqq 1}$ are martingales with respect to $\{\mathscr{F}_t^1\}$ and $(\Omega^1, \mathscr{F}^1, P^1)$ and $\{\mathscr{F}_t^2\}$ on $(\Omega^2, \mathscr{F}^2, P^2)$ respectively, then define $\Omega = \bigcup_{i=1,2} \Omega^i \times \{i\}$, $\mathscr{F} = \sigma(\Lambda^i \times \{i\}, \Lambda^i \in \mathscr{F}^i, i = 1, 2)$, $\mathscr{F}_t = \sigma(\Lambda^i \times \{i\}, \Lambda^i \in \mathscr{F}_t^i, i = 1, 2)$, $P(A) = \lambda P^1(A^1) + (1 - \lambda)P^2(A^2)$ for $A = A^1 \times \{1\} \cup A^2 \times \{2\}$ in $\mathscr{F}$, and $X_t(\omega, i) = X_t^1(\omega)$ if $(\omega, i) \in \Omega^1 \times \{1\}$, and $= X_t^2(\omega)$ if $(\omega, i) \in \Omega^2 \times \{2\}$. Then $X = (X_t)_{0 \leqq t \leqq 1}$ is a martingale with respect to $\{\mathscr{F}_t\}$ on $(\Omega, \mathscr{F}, P)$ satisfying $P(X \in C) = \lambda P^1(X^1 \in C) + (1 - \lambda)P^2(X^2 \in C)$ for all $C \in \mathscr{B}(D)$; so the conclusion holds for X. □

Proposition 1.2. *$\mathscr{M}(\mu)$ is a convex subset of $\mathscr{P}(\mathbf{R})$.*

Proof. For $i = 1, 2$, let $\nu^i \in \mathscr{M}(\mu)$ with martingale $X^i = (X_t^i)_{0 \leqq t \leqq 1}$ satisfying $M^i \overset{\mathscr{D}}{=} \nu^i$, and $X_1^i \overset{\mathscr{D}}{=} \mu$. Also let $0 < \lambda < 1$. Then the martingale $X = (X_t)_{0 \leqq t \leqq 1}$ of Lemma 1.1 satisfies $M \overset{\mathscr{D}}{=} \lambda\nu^1 + (1 - \lambda)\nu^2$ and $X_1 \overset{\mathscr{D}}{=} \mu$, so that $\lambda\nu^1 + (1 - \lambda)\nu^2 \in \mathscr{M}(\mu)$. □

To set up the proof of closedness of $\mathscr{M}(\mu)$, we give three lemmas.

Lemma 1.3. *Let μ be a p.m. on $(\mathbf{R}, \mathscr{B}(\mathbf{R}))$ with $\int |x| d\mu(x) < \infty$, and assume $X^n = (X_t^n)_{0 \leqq t \leqq 1}$, $n = 1, 2, \ldots$, are martingales satisfying $X_1^n \overset{\mathscr{D}}{=} \mu$ for $n \geqq 1$ and $X^n \Rightarrow X$ in D for some RCLL stochastic process $X = (X_t)_{0 \leqq t \leqq 1}$. Then X is a martingale.*

Proof. We may assume that X^n, $n \geqq 1$, are martingales with respect to the same filtration $\{\mathscr{F}_t\}$ on a probability space $(\Omega, \mathscr{F}, P)$, and X is defined on $(\Omega, \mathscr{F}, P)$ and adapted to $\{\mathscr{F}_t\}$. Use the martingale property of the X^n's and common integrable distribution of X_1^n's to obtain collection $\{X_t^n, 0 \leqq t \leqq 1, n = 1, 2, \ldots\}$ is uniformly integrable. It follows that $\int_C X_t^n dP \to \int_C X_t dP$ as $n \to \infty$ for all $C \in \mathscr{F}$, for each t in $T_X = \{u \in [0, 1] : P(X(u) \neq X(u-)) = 0\}$, and hence that

$$\int_A X_s dP = \int_A X_t dP \qquad (*)$$

$$\text{for all } A \in \mathscr{F}_s, \quad \text{for each } s, t \in T_X \quad \text{with } 0 \leqq s < t \leqq 1$$

(in particular, $0, 1 \in T_X$). By use of right-continuity of X and uniform integrability of $\{X_t : t \in T_X\}$, one obtains (*) for all $0 \leqq s < t \leqq 1$; so X is a martingale. □

Now let a be a fixed small positive number in $(0, 1)$ (e.g., $a = 0.1$). Define set $D_a[0, 1]$ to be that subset of $D[0, 1]$ of functions $x = (x(t))_{0 \leqq t \leqq 1}$ satisfying the following:

(1.2) for the function x there is a constant $b = b(x)$ with $a \leqq b \leqq 1$ for which

(i) $x(t) = x(0)$ if $0 \leqq t < a$; $x(t) - x(0) \geqq (t-a)/(1-t)$ if $a \leqq t < b$; and $x(t) = x(1)$ if $b \leqq t < 1$;

(ii) $x = (x(t))_{0 \leqq t \leqq 1}$ is nondecreasing on $[0, b)$, and $x(b-) - x(0) = (b-a)/(1-b) \geqq x(b) - x(0)$; and

(iii) for $a \leqq t < b$, t is a point of increase of x if and only if $x(t-) - x(0) = (t-a)/(1-t)$.

(Recall t is a point of increase of x iff $x(t+\varepsilon) - x(t-\varepsilon) > 0$ for each $\varepsilon > 0$ small.)

LEMMA 1.4. *Let $X = (X_t)_{0 \leqq t \leqq 1}$ be any martingale. Then there is a martingale $Y = (Y_t)_{0 \leqq t \leqq 1}$ with paths in $D_a[0,1]$ for which $M(Y) \overset{\mathscr{D}}{=} M(X)$ and $Y_1 \overset{\mathscr{D}}{=} X_1$.*

PROOF. Let $X = (X_t)_{0 \leqq t \leqq 1}$ be a martingale with respect to filtration $\{\mathscr{F}_t\}$ on probability space $(\Omega, \mathscr{F}, P)$. For each $x \geqq 0$, let τ_x be the optional stopping time

$\tau_x = \tau(x) = \inf\{0 \leqq t < 1 : X_t - X_0 > x\}$ if this set $\neq \varnothing$, and $= 1$ otherwise.

For the fixed a of (1.2), define the process $Y = (Y_t)_{0 \leqq t \leqq 1}$ by

$$(1.3) \quad Y_t = X_0 \text{ if } 0 \leqq t < a, \quad = X_{\tau((t-a)/(1-t))} \text{ if } a \leqq t < 1, \quad \text{and } = X_1 \text{ if } t = 1;$$

and define filtration $\{\mathscr{G}_t\}$ by $\mathscr{G}_t = \mathscr{F}_0$ if $0 \leqq t < a$, $= \mathscr{F}_{\tau((t-a)/(1-t))+}$ if $a \leqq t < 1$, and $= \mathscr{F}_1$ if $t = 1$. Then $Y = (Y_t)_{0 \leqq t \leqq 1}$ is a martingale with respect to $\{\mathscr{G}_t\}$ (use the Optional Sampling Theorem as given, e.g., in Karatzas and Shreve [13]). It is clear from the construction that $Y_0 \equiv X_0$, $M(X) \overset{\mathscr{D}}{=} M(Y)$, and $X_1 \overset{\mathscr{D}}{=} Y_1$.

Now fix $\omega \in \Omega$. Observe that $M(X)(\omega) < \infty$ (since X is in $D[0,1]$), and let $b = b(\omega)$ satisfy $(b-a)/(1-b) = M(X)(\omega) - X_0(\omega)$. Then from the definition of τ and Y, it follows immediately that $Y_t(\omega) = X_0(\omega)$ if $0 \leqq t < a$, $Y_t(\omega) - Y_0(\omega) \geqq (t-a)/(1-t)$ if $a \leqq t < b$, and $Y_t(\omega) = X_1(\omega)$ if $b \leqq t \leqq 1$; $Y_t(\omega)$ is nondecreasing on $[0, b-)$ and

$$Y_{b-} = (b-a)/(1-b) + X_0(\omega) = M(X)(\omega) \geqq Y_b(\omega) = X_1(\omega);$$

and for $a \leqq t < b$, t is a point of increase of Y if and only if $Y_{t-} - Y_0 = (t-a)/(1-t)$ (i.e., $\lim_{t_n \uparrow t} X_{\tau((t_n-a)/(1-t_n))} = (t-a)/(1-t) + X_0(\omega)$). Thus $Y(\omega) = (Y_t(\omega))_{0 \leqq t \leqq 1}$ is in $D_a[0,1]$. □

Note that the parameter $a > 0$ is used in (1.2) and the subsequent part of the section so that the new process $Y = (Y_t)_{0 \leqq t \leqq 1}$ formed from X in Lemma 1.4 will have both $Y_0 \equiv X_0$ and right continuous paths.

As in Billingsley ([2]: Chapter 3), define the modulus $w'_x(\delta)$ for $0 < \delta < 1$ and $x \in D[0, 1]$ by

$$w'_x(\delta) = \inf_{\{t_i\}} \max_{0 < i \leqq r} w_x[t_{i-1}, t_i),$$

where $w_x[a, b) = \sup\{|x(s) - x(t)| : s, t \in [a, b)\}$ for $a < b$, and the infimum is taken over all finite sets $\{t_i\}$ of points satisfying $0 = t_0 < t_1 < \cdots < t_r = 1$ and $\delta \leqq t_i - t_{i-1}$ for $i = 1, \ldots, r$. Also denote $|x| = \sup_{0 \leqq t \leqq 1} |x(t)|$.

LEMMA 1.5. *If* $x \in D_a[0, 1]$, *then for each* δ *with* $0 < \delta < a$, $w'_x(\delta) \leqq \delta(1 - a)^{-1}(1 + 2|x|)^2$.

PROOF. Let $x \in D_a[0, 1]$ and $0 < \delta < a$. Choose $s_0 \in [0, 1]$ satisfying $x(s_0 -) = |x| = (s_0 - a)/(1 - s_0) + x(0)$ (such an s_0 exists). If $s_0 = a$, then the result holds; suppose $s_0 > a$. Having chosen $s_0 > s_1 > \cdots > s_n \geqq a$, with $x(s_n -) - x(0) = (s_n - a)/(1 - s_n)$, define

$$s_{n+1} = \inf\{s : x(s) = x(s_n - \delta)\}.$$

Then

(i) $s_{n+1} \leqq s_n - \delta$,

(ii) $x(s_{n+1}) \geqq x((s_n - \delta) -) \geqq ((s_n - \delta) - a)/(1 - (s_n - \delta)) + x(0)$, and

(iii) $(1 - s_n)^{-1} \leqq (1 - a)^{-1}(1 + 2|x|)$ (from $(s_n - a)/(1 - s_n) + x(0) = x(s_n -) \leqq |x|$).

It follows that

$$\begin{aligned} &\sup\{|x(s) - x(t)| : s, t \in [s_{n+1}, s_n)\} \\ &\quad \leqq x(s_n -) - x(s_{n+1}) \\ &\quad \leqq ((s_n - a)/(1 - s_n)) - (((s_n - \delta) - a)/(1 - (s_n - \delta))) \qquad (1.4) \\ &\quad \leqq \delta(1 - a)(1 - s_n)^{-2} \\ &\quad \leqq \delta(1 - a)^{-1}(1 + 2|x|)^2. \end{aligned}$$

At this point, either $s_{n+1} \in [0, a]$ or $s_{n+1} \in (a, 1]$. In the first case, stop the procedure; and in the second case, note that $x(s_{n+1} -) - x(0) = (s_{n+1} - a)/(1 - s_{n+1})$ and continue by choosing s_{n+2} by the above procedure. If we label the chosen points obtained in this way by $s_0 > s_1 > \cdots > s_m$, then

$$w'_x(\delta) \leqq \max_{0 \leqq j < m} w_x[s_{j+1}, s_j) \leqq \delta(1-a)^{-1}(1+2|x|)^2$$

by (1.4); and we are done. □

PROPOSITION 1.6. *$\mathcal{M}(\mu)$ is closed subset of $\mathcal{P}(\mathbf{R})$.*

PROOF. Let ν_n, $n = 1, 2, \ldots$, be in $\mathcal{M}(\mu)$ and $\nu_n \Rightarrow \nu$. We show $\nu \in \mathcal{M}(\mu)$. Let $X^n = (X_t^n)_{0 \leqq t \leqq 1}$ be a martingale with $M(X^n) \stackrel{\mathcal{D}}{=} \nu_n$ and $X_1^n \stackrel{\mathcal{D}}{=} \mu$, for $n \geqq 1$. As in the proof of Lemma 1.3 we may assume that the X^n's are martingales with respect to the same filtration on the same probability space. From Lemma 1.4, we may also assume that all paths of the X^n's are in $D_a[0, 1]$. Let P^n denote P_{X^n}, the p.m. induced by X^n on $D[0, 1]$. We claim that $\{P^n\}_{n \geqq 1}$ is tight on $D[0, 1]$.

From Billingsley ([4]: Theorem 15.2), it suffices to show

(1.5)

(i) for each $\eta > 0$, there exists an $\alpha \in \mathbf{R}$ such that

$$P\left(\sup_{0 \leqq t \leqq 1} |X_t^n| > \alpha\right) \leqq \eta \qquad \text{for all } n \geqq 1; \qquad \text{and}$$

(ii) for each ε, $\eta > 0$, there exist δ, with $0 < \delta < 1$, and an integer n_0 such that

$$P(w'_{X^n}(\delta) \geqq \varepsilon) \leqq \eta \qquad \text{for all } n \geqq n_0.$$

Now, (1.5)(i) follows easily from the inequality

$$P\left(\sup_{0 \leqq t \leqq 1} |X_t^n| > \alpha\right) \leqq \alpha^{-1} E|X_1^n| = \alpha^{-1} \int |x| \, d\mu(x),$$

for $\alpha > 0$. For (1.5)(ii), let ε, $\eta > 0$ be given and obtain from Lemma 1.5, for $0 < \delta < a \wedge (\varepsilon(1-a))$, the inequality

$$\begin{aligned} P(w'_{X^n}(\delta) \geqq \varepsilon) &\leqq P(\delta(1-a)^{-1}(1+2|X^n|)^2 \geqq \varepsilon) \\ &\leqq P\left(\sup_{0 \leqq t \leqq 1} |X_t^n| \geqq ((\varepsilon(1-a)\delta^{-1})^{1/2} - 1)/2\right) \\ &\leqq ((\varepsilon(1-a)\delta^{-1})^{1/2} - 1)^{-1} \cdot 2 \int |x| \, d\mu(x). \end{aligned}$$

By choosing δ sufficiently close to 0, (1.5)(ii) follows.

From Prohorov's Theorem ([4]; p. 37), there is a subsequence $\{P^{n_i}\}$ with $P^{n_i} \Rightarrow P^0$ for some p.m. P^0 on $D[0, 1]$. Let $X = (X_t)_{0 \leqq t \leqq 1}$ be a process with paths in $D[0, 1]$ having associated p.m. P^0. We may assume that the martingales X^{n_i}, $i \geqq 1$, and the process X are all defined on the same probability space

$(\Omega, \mathscr{F}, P)$. From the weak convergence, it is immediate that $X_1 \stackrel{\mathscr{D}}{=} \mu$. From Lemma 1.3, we have that X is a martingale. Finally, we have $M(X^{n_i}) \stackrel{\mathscr{D}}{=} \nu_{n_i}$, $\nu_{n_i} \Rightarrow \nu$, and also $M(X^{n_i}) \Rightarrow M(X)$ (from continuity of $Tx = \sup_{0 \leqq t \leqq 1} x(t)$ on $D[0, 1]$ and the Continuous Mapping Theorem as in, e.g., [4]; p. 138). Thus $M(X) \stackrel{\mathscr{D}}{=} \nu$; and it follows that $\nu \in \mathscr{M}(\mu)$. □

2. Characterization of $\mathscr{M}(\mu)$

Our main results are based on use of the usual stochastic ordering for probability measures. For ν_1 and ν_2 *in* $\mathscr{P}(\mathbf{R})$, we write $\nu_1 < \nu_2$ if

$$\int \phi d\nu_1 \leqq \int \phi d\nu_2 \qquad \text{for every nondecreasing function } \phi \text{ on } \mathbf{R}. \tag{2.1}$$

It is straightforward to show (2.1) is equivalent to

$$\nu_1[x, \infty) \leqq \nu_2[x, \infty) \qquad \text{for every } x \in \mathbf{R} \tag{2.2}$$

(for a reference on stochastic orderings, see, e.g., Stoyan [21]).

As in Section 1, μ denotes any p.m. on $\mathbf{R}$ satisfying $\int |x| d\mu(x) < \infty$, and $\mathscr{M}(\mu)$ is the set defined in (1.1). It is known that under the partial ordering of (2.1),

(2.3)
(i) there exists a least upper bound of $\mathscr{M}(\mu)$, denoted μ^*, and $\mu^* \in \mathscr{M}(\mu)$; and
(ii) there exists a greatest lower bound of $\mathscr{M}(\mu)$, the p.m. μ, and $\mu \in \mathscr{M}(\mu)$.

The result (2.3)(ii) is immediate from the remarks after (1.1). For future reference, we discuss result (2.3)(i) and a representation of μ^*; see Blackwell and Dubins [5] or Dubins and Gilat [6] for details.

Let F denote the distribution function associated with p.m. μ. Let F^{-1} denote the left continuous inverse of F defined on $(0, 1)$ by $F^{-1}(w) = \inf\{z : F(z) \geqq w\}$ and extended to $[0, 1]$ by setting $F^{-1}(0) = F^{-1}(0+)$ and $F^{-1}(1) = F^{-1}(1-)$ (for references on F^{-1}, see, e.g., [19] of [20]). On the probability space $([0, 1], \mathscr{B}([0, 1]), \lambda)$, where λ denotes Lebesgue measure, the r.v. F^{-1} has d.f. F and associated p.m. μ. Define the filtration $\{\mathscr{F}_t\}$ by $\mathscr{F}_t = \sigma\{\mathscr{B}([0, t]), (t, 1]\}$ for $0 < t \leqq 1$ and $\mathscr{F}_0 = \{\phi, [0, 1]\}$. Then the stochastic process $(Z_t)_{0 \leqq t \leqq 1}$, defined by

$$Z_t = E(F^{-1} \mid \mathscr{F}_t), \tag{2.4}$$

is a martingale with respect to $\{\mathscr{F}_t\}$, satisfying $Z_0 = \int x d\mu(x)$ and $Z_1 = F^{-1}$, with associated p.m. μ. Now, it was shown in [5] and [6] that the Hardy and Littlewood maximal function h associated with F, defined by $h(w) = (1-w)^{-1}\int_{[w,1)} F^{-1}(u)du$ (with $h(1) = F^{-1}(1)$), as an r.v. on this probability space has associated p.m. μ^*. This function h is continuous and nondecreasing on [0, 1]; $h(0) = \int x d\mu(x)$ and $h(1) = F^{-1}(1-) = x_F$, the right endpoint of the support of F; and $F^{-1} \leqq h$. Thus it follows from the representation

$$(2.5)\quad Z_t(w) = F^{-1}(w) \text{ if } 0 < w \leqq t \leqq 1, \quad \text{and} \quad = h(t) \text{ if } 0 \leqq t < w < 1$$

for $0 < w < 1$, that $M(Z)) = h$ a.e. and has associated p.m. μ^*. This martingale demonstrates that $\mu^* \in \mathscr{M}(\mu)$.

THEOREM 2.1. $\mathscr{M}(\mu) = \{\nu \in \mathscr{P}(\mathbf{R}) : \mu < \nu < \mu^*\}$.

From (2.2) it is clear that $\mathscr{M}(\mu) \subset \{\nu \in \mathscr{P}(\mathbf{R}) : \mu < \nu < \mu^*\}$; to establish Theorem 2.1, we must show that

(2.6)
$$\begin{gathered}\text{given p.m.'s } \mu \text{ and } \nu \text{ on } \mathbf{R} \text{ with } \mu < \nu < \mu^*,\\ \text{there is a martingale } X = (X_t)_{0 \leqq t \leqq 1} \text{ for which } M \stackrel{\mathscr{D}}{=} \nu \text{ and } X_1 \stackrel{\mathscr{D}}{=} \mu.\end{gathered}$$

We prove (2.6) after Proposition 2.3.

LEMMA 2.2. *Fix* $N \in \{1, 2, \ldots\}$ *and let* μ *be a p.m. on* $\mathbf{R}$ *with* $\int |x| d\mu(x) < \infty$. *Let* $x_1 < \cdots < x_N$ *and* $0 \leqq a_N \leqq \cdots \leqq a_2 \leqq a_1$ *be numbers satisfying* $\mu[x_i, \infty) \leqq a_i \leqq \mu^*[x_i, \infty)$ *for* $i = 1, \ldots, N$.

Then there exist p.m.'s μ_1 *and* μ_2 *on* $\mathbf{R}$ *satisfying* $\int |x| d\mu_1(x) < \infty$ *and* $\int |x| d\mu_2(x) < \infty$; $\mu_2[x_i, \infty)$ *is constant in* $i = 1, \ldots, N$; *and there is a number* $\lambda \in [0, 1]$ *such that* $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ *and* $\lambda\mu_1^*[x_i, \infty) + (1-\lambda)\mu_2[x_i, \infty) \geqq a_i$ *for every* $i = 1, \ldots, N$, *with equality holding for at least one* $i = 1, \ldots, N$.

PROOF. *Case* 1. In this case assume that p.m. μ has no point mass. Let $\bar{z} = \alpha(z)$ be any continuous function on $[x_N, \infty)$ taking values in $(-\infty, x_1)$ with $\bar{z} = \alpha(z)$ decreasing to $-\infty$ as z increases to $+\infty$. Without loss of generality, assume that $\mu([\bar{z}, z]) > 0$ for all z in $[x_N, \infty)$. (If $\mu([\bar{z}, z]) = 0$ for some z in $[x_N, \infty)$, then the conclusion follows by taking $\mu_1 = \mu = \mu_2$ and $\lambda = \max_{0 \leqq i \leqq N} c_i$, where $c_i = (a_i - \mu[x_N, \infty))/(\mu^*[x_i, \infty) - \mu[x_N, \infty))$ if $\mu[x_N, \infty) < \mu^*[x_N, \infty)$ and $c_i = 0$ if $\mu[x_N, \infty) = \mu^*[x_N, \infty)$). Let $\lambda(z)$ be the strictly positive function defined on $[x_N, \infty)$ by $\lambda(z) = \mu([\bar{z}, z])$; observe that $\lambda(z)$ is a continuous, increasing function with $\lim_{z \uparrow \infty} \lambda(z) = 1$.

For each z in $[x_N, \infty)$, define probability measures $\mu_z(\cdot)$ and $\bar{\mu}_z(\cdot)$ on $\mathscr{B}(\mathbf{R})$ by

$$\mu_z(A) = (\lambda(z))^{-1}\mu(A \cap [\bar{z}, z]); \qquad \text{and}$$

(2.7)

$$\bar{\mu}_z(A) = (1-\lambda(z))^{-1}\mu(A \cap [\bar{z}, z]^c) \text{ if } \lambda(z) < 1, \quad \text{and} \quad = \varepsilon_y(A) \text{ if } \lambda(z) = 1$$

for some $y \in (\bar{z}, x_1)$. As at the beginning of this section, associated with p.m. μ are the functions F^{-1} and h on $[0, 1]$; and analogously, associated with p.m. μ_z are functions F_z^{-1} and h_z, and p.m. μ_z^*. From the continuity of $F(x)$ and from representations of F_z^{-1} and h_z in terms of F, F^{-1}, and h, one obtains that for each $i = 1, \ldots, N$, $\mu_z^*[x_i, \infty)$ and $\bar{\mu}_z[x_i, \infty)$ are continuous in z over $[x_N, \infty)$ with $\lim_{z \uparrow \infty} \mu_z^*[x_i, \infty) = \mu^*[x_i, \infty)$ and $\lim_{z \uparrow \infty} (1-\lambda(z))\bar{\mu}_z[x_i, \infty) = 0$.

Finally, define $\hat{z}$ in $[x_N, \infty]$ by

$$\hat{z} = \inf\{z : \lambda(z)\mu_z^*[x_i, \infty) + (1-\lambda(z))\bar{\mu}_z[x_i, \infty) \geqq a_i \quad \text{for all } i = 1, \ldots, N\}.$$

We may assume that $\hat{z} < \infty$ (otherwise, it must follow that $\mu^*[x_i, \infty) = a_i$ for some $i \in \{1, \ldots, N\}$, and the conclusion follows by letting $\lambda = 1$, $\mu_1 = \mu$, and $\mu_2 = \varepsilon_y$). If we let $\lambda = \lambda(\hat{z})$, $\mu_1 = \mu_{\hat{z}}$, and $\mu_2 = \bar{\mu}_{\hat{z}}$, then μ_1 and μ_2 are probability measures on $\mathbf{R}$ and λ is a number in $[0, 1]$ satisfying the desired conclusions.

Case 2. In this case μ is any p.m. on $\mathbf{R}$ satisfying the hypotheses of this lemma for numbers $x_1 < \cdots < x_N$ and $0 \leqq a_N \leqq \cdots \leqq a_1$. The proof in this case is similar to, but technically more complicated than, the proof in Case 1. The key new notion is that portions of the weights of atoms are taken into account. Letting x_F denote the right endpoint of the support of F, one associates with pairs (z, ρ) with $x_N < z \leqq x_F$ and $0 \leqq \rho \leqq 1$, pairs $(\bar{z}, \bar{\rho})$ with $-\infty \leqq \bar{z} < x_1$ and $0 \leqq \bar{\rho} \leqq 1$; constants

$$\Delta_{z,\rho} = (1-\bar{\rho})\mu\{\bar{z}\} + \mu(\bar{z}, z) + (1-\rho)\mu\{z\};$$

and p.m.'s $\mu_{z,\rho}^1$ and $\mu_{z,\rho}^2$ on $\mathscr{B}(\mathbf{R})$ defined by

$$\mu_{z,\rho}^1(A) := \Delta_{z,\rho}^{-1}\{(1-\bar{\rho})\mu\{\bar{z}\}\varepsilon_{\bar{z}}(A) + \mu(A \cap (\bar{z}, z)) + (1-\rho)\mu\{z\}\varepsilon_z(A)\},$$

and for $\Delta_{z,\rho} < 1$,

$$\mu_{z,\rho}^2(A) := (1-\Delta_{z,\rho})^{-1}\{\mu(A \cap (-\infty, \bar{z})) + \bar{\rho}\mu\{\bar{z}\}\varepsilon_{\bar{z}}(A) + \rho\mu\{z\}\varepsilon_z(A) + \mu(A \cap (z, \infty))\}$$

and for $\Delta_{z,\rho} = 1$, $\mu_{z,\rho}^2(A) = \varepsilon_y(A)$, where $\bar{z} < y < x_1$, so that continuity and limit arguments analogous to those of Case 1 can be applied. □

PROPOSITION 2.3. *Fix* $N \in \{1, 2, \dots\}$ *and let* μ *be a p.m. on* $\mathbf{R}$ *with* $\int |x| \, d\mu(x) < \infty$. *Let* $x_1 < \cdots < x_N$ *and* $0 \leqq a_N \leqq \cdots \leqq a_1 \leqq 1$ *be numbers satisfying* $\mu[x_i, \infty) \leqq a_i \leqq \mu^*[x_i, \infty)$ *for* $i = 1, \dots, N$. *Then there exists a p.m.* $\nu \in \mathcal{M}(\mu)$ *satisfying* $\nu[x_i, \infty) = a_i$ *for* $i = 1, \dots, N$.

PROOF. The proof is by induction on N. For $N = 1$, we assume $\mu[x_1, \infty) < \mu^*[x_1, \infty)$ (the case of equality is trivial), and define $\nu = \lambda\mu + (1 - \lambda)\mu^*$, where μ and μ^* are the p.m.'s in $\mathcal{M}(\mu)$ in (2.3) and $\lambda := (\mu^*[x_1, \infty) - a_1)/(\mu^*[x_1, \infty) - \mu[x_1, \infty))$. Then $\nu \in \mathcal{M}(\mu)$ from Proposition 1.2, and $\nu[x_1, \infty) = a_1$ by definition. We assume the result holds for N, and will show the result holds for $N + 1$. So assume there are numbers

$$0 \leqq a_{N+1} \leqq a_N \leqq \cdots \leqq a_1 \leqq 1 \quad \text{and} \quad x_1 < \cdots < x_N < x_{N+1}$$

such that $\mu[x_i, \infty) \leqq a_i \leqq \mu^*[x_i, \infty)$ for $i = 1, \dots, N + 1$.

We claim that it suffices to show the induction result holds if

$$\text{(2.8)} \qquad \mu^*[x_{i_0}, \infty) = a_{i_0} \qquad \text{for some } i_0 \in \{1, \dots, N + 1\}.$$

Indeed, suppose the induction result is true under condition (2.8). Let μ_1 and μ_2 be the p.m.'s in the conclusion of Lemma 2.2. If $\lambda = 0$ in the conclusion of Lemma 2.2, then $\mu[x_i, \infty) = a_i$ for $i = 1, \dots, N + 1$ and the result follows by letting $\nu = \mu$. Suppose $\lambda > 0$, and define $\tilde{a}_i = \lambda^{-1}(a_i - (1 - \lambda)\mu_2[x_i, \infty))$ for $i = 1, \dots, N + 1$. Then $\mu_1[x_i, \infty) \leqq \tilde{a}_i \leqq \mu_1^*[x_i, \infty)$ for all $i = 1, \dots, N + 1$, with $\tilde{a}_i = \mu_1^*[x_i, \infty)$ for some i; here the left inequality follows from $\mu[x_i, \infty) \leqq a_i$ for all i and $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$, and the right inequality and equality follows from Lemma 2.2 and the definition of $\tilde{a}_i$. Also $0 \leqq \tilde{a}_{N+1} \leqq \cdots \leqq \tilde{a}_1 \leqq 1$, from the property of μ_2 that $\mu_2[x_i, \infty)$ is constant in $i = 1, \dots, N + 1$. Since the induction result is assumed to hold under condition (2.8), we obtain a p.m. $\nu_1 \in \mathcal{M}(\mu_1)$ satisfying $\nu_1[x_i, \infty) = \tilde{a}_i$ for every $i = 1, \dots, N + 1$. Let $U = (U_t)_{0 \leqq t \leqq 1}$ be a martingale with $M(U) \stackrel{\mathcal{D}}{=} \nu_1$ and $U_1 \stackrel{\mathcal{D}}{=} \mu_1$. We know that $\mu_2 \in \mathcal{M}(\mu_2)$; so there is a martingale $W = (W_t)_{0 \leqq t \leqq 1}$ with $M(W) \stackrel{\mathcal{D}}{=} \mu_2$ and $W_1 \stackrel{\mathcal{D}}{=} \mu_2$. Finally, define p.m. $\nu := \lambda\nu_1 + (1 - \lambda)\mu_2$ and observe that $\nu[x_i, \infty) = a_i$ for $i = 1, \dots, N + 1$. If we define $X = (X_t)_{0 \leqq t \leqq 1}$ as a martingale mixture of U and W as in Lemma 1.1, then $M(X) \stackrel{\mathcal{D}}{=} \lambda\nu_1 + (1 - \lambda)\mu_2 = \nu$ and $X_1 \stackrel{\mathcal{D}}{=} \lambda\mu_1 + (1 - \lambda)\mu_2 = \mu$, so that $\nu \in \mathcal{M}(\mu)$.

Assume that (2.8) is satisfied. We prove the reduction result in two cases. We give the proof for $1 < i_0 < N + 1$; the proofs for $i_0 = 1$ and $i_0 = N + 1$ are slight modifications of this argument.

Case 1. In this case assume that μ has no point mass. Let y be the number in $[0, x_{i_0}]$ satisfying $x_{i_0} = (\mu[y,\infty))^{-1}\int_{[y,\infty)} z d\mu(z)$, so that $\mu[y,\infty) = \mu^*[x_{i_0},\infty) = a_{i_0}$ from use of the maximal function h associated with μ and (2.6). Define p.m.'s μ_1 and μ_2 on $\mathbf{R}$ by, for $A \in \mathscr{B}(\mathbf{R})$,

$$\mu_1(A) = \mu\{A \cap (-\infty, y)\} + \mu[y,\infty)\varepsilon_{x_{i_0}}(A),$$

$$\mu_2(A) = (\mu[y,\infty))^{-1}\mu\{A \cap [y,\infty)\}. \tag{2.9}$$

First observe that μ_1 is a p.m. satisfying $\int x d\mu_1(x) = \int x d\mu(x)$ and

$$\mu_1[x_i,\infty) \leqq a_i \leqq \mu_1^*[x_i,\infty) \qquad \text{for } i = 1,\ldots,i_0 - 1. \tag{2.10}$$

The first inequality in (2.10) follows from $\mu_1[x_i,\infty) = \mu[x_i,\infty) \leqq a_i$ for $x_i \leqq y$, and from $\mu_1[x_i,\infty) = \mu[y,\infty) = \mu^*[x_{i_0},\infty) = a_{i_0} \leqq a_i$ for $y \leqq x_i \leqq x_{i_0}$; the second inequality in (2.10) follows from $\mu_1^*[x_i,\infty) = \mu^*[x_i,\infty)$ for $i = 1,\ldots,i_0$ (e.g., compare μ_1, F_1, F_1^{-1}, h_1, and μ_1^* to μ, F, F^{-1}, h, and μ^* of (2.4)). From the induction hypothesis, there is a p.m. ν_1 in $\mathscr{M}(\mu_1)$ satisfying $\nu_1[x_i,\infty) = a_i$ for $i = 1,\ldots,i_0 - 1$, and hence there is also a martingale $Y = (Y_t)_{0\leqq t\leqq 1}$ with $M(Y) \overset{\mathscr{D}}{=} \nu_1$ and $Y_1 \overset{\mathscr{D}}{=} \mu_1$ (we also have $\nu_1[x_{i_0},\infty) = a_{i_0}$ since $\nu_1[x_{i_0},\infty) = P(M(Y) \geqq x_{i_0}) = P(Y_1 = x_{i_0}) = \mu_1[x_{i_0},\infty) = \mu[y,\infty) = a_{i_0}$). Second, observe that μ_2 is a p.m. satisfying $\int x d\mu_2(x) = x_{i_0}$ and

$$\mu_2[x_i,\infty) \leqq a_i/\mu[y,\infty) \leqq \mu_2^*[x_i,\infty) \qquad \text{for } i = i_0 + 1,\ldots,N+1 \tag{2.11}$$

(use that $\mu_2[x_i,\infty) = \mu[x_i,\infty)/\mu[y,\infty)$ and $\mu_2^*[x_i,\infty) = \mu^*[x_i,\infty)/\mu[y,\infty)$ for $i = i_0 + 1,\ldots,N$). From the induction hypothesis, we can obtain a p.m. ν_2 and a martingale $Z = (Z_t)_{0\leqq t\leqq 1}$ satisfying $Z_0 \equiv x_{i_0}$, $M(Z) \overset{\mathscr{D}}{=} \nu_2$, $Z_1 \overset{\mathscr{D}}{=} \mu_2$, and $\nu_2[x_i,\infty) = a_i/\mu[y,\infty)$ for $i = i_0 + 1,\ldots,N+1$.

We may assume martingales Y and Z and their respective filtrations $\{\mathscr{G}_t\}$ and $\{\mathscr{H}_t\}$ are defined and independent on the same probability space. Define filtration $\{\mathscr{F}_t\}$ by $\mathscr{F}_t = \mathscr{G}_{2t}$ if $0 \leqq t < \frac{1}{2}$, and $= \sigma\{\mathscr{G}_1, \mathscr{H}_{2(t-1/2)}\}$ if $\frac{1}{2} \leqq t \leqq 1$; and define stochastic process $X = (X_t)_{0\leqq t\leqq 1}$ by $X_t = Y_{2t}$ if $0 \leqq t \leqq \frac{1}{2}$; and $= Y_1$ if $\frac{1}{2} \leqq t \leqq 1$ and $Y_1 < x_{i_0}$; and $= Z_{2(t-1/2)}$ if $\frac{1}{2} \leqq t \leqq 1$ and $Y_1 = x_{i_0}$. Then $(X_t)_{0\leqq t\leqq 1}$ is a martingale w.r.t. $\{\mathscr{F}_t\}$. Also $X_1 \overset{\mathscr{D}}{=} \mu$, since

$$\begin{aligned} P(X_1 \in A) &= P(Y_1 \in A, Y_1 < x_{i_0}) + P(Y_1 \in A, X_1 = x_{i_0}) \\ &= \mu(A \cap (-\infty, y)) + \mu_2(A)\mu_1\{x_{i_0}\} \\ &= \mu(A \cap (-\infty, y)) + ((\mu[y,\infty))^{-1}\mu(A \cap [y,\infty)))\mu[y,\infty) = \mu(A); \end{aligned}$$

and $M(X)$ has associated p.m. ν satisfying $\nu[x_i, \infty) = a_i$ for all $i = 1, \ldots, N+1$ since

$$\nu[x_i, \infty) = P(M(X) \geqq x_i) = P(M(Y) \geqq x_i) = \nu_1[x_i, \infty) = a_i \quad \text{for } i = 1, \ldots, i_0,$$

and

$$\begin{aligned} \nu[x_i, \infty) &= P(M(X) \geqq x_i) = P(M(Z) \geqq x_i, X_1 = x_{i_0}) \\ &= P(M(Z) \geqq x_i) P(X_1 = x_{i_0}) \\ &= \nu_2[x_i, \infty)\mu[y, \infty) = a_i \qquad \text{for } i = i_0 + 1, \ldots, N+1. \end{aligned}$$

This completes the induction result in this case.

Case 2. In this case μ is any p.m. on $\mathbf{R}$ with $\int |x| \, d\mu(x) < \infty$ satisfying the hypotheses of this proposition with respect to the numbers $x_1 < \cdots < x_{N+1}$ and $0 \leqq a_{N+1} \leqq \cdots \leqq a_1 \leqq 1$, and satisfying (2.8) for index i_0. As in Lemma 2.2, the new idea in this case is that of 'splitting atoms'. Recall the objects μ, F, F^{-1}, h, and μ^*, and define numbers u_0, a_0, b_0, and y, by

$$x_{i_0} = h(u_0); (a_0, b_0] = \{u : F^{-1}(u) = F^{-1}(u_0)\}; \quad \text{and} \quad y = F^{-1}(u_0). \tag{2.12}$$

In particular, this gives $\mu^*[x_{i_0}, \infty) = 1 - u_0 = \mu[y, \infty) - (u_0 - a_0)$ and

$$\begin{aligned} x_{i_0} &= (1 - u_0)^{-1} \int_{[u_0, 1)} F^{-1}(u) du \\ &= (1 - u_0)^{-1} \left\{ \int_{[a_0, 1)} F^{-1}(u) du - F^{-1}(u_0)(u_0 - a_0) \right\}. \end{aligned}$$

For notational convenience, we denote $c_0 := \mu[y, \infty) - (u_0 - a_0)$. Analogous to (2.9), define p.m.'s μ_1 and μ_2 on $\mathbf{R}$ by, for $A \in \mathscr{B}(\mathbf{R})$,

$$\begin{aligned} &\mu_1(A) = \mu((-\infty, y) \cap A) + (u_0 - a_0)\varepsilon_y(A) + c_0 \varepsilon_{x_{i_0}}(A) \qquad \text{and} \\ &\mu_2(A) = c_0^{-1}\{(b_0 - u_0)\varepsilon_y(A) + \mu(A \cap (y, \infty))\}. \end{aligned} \tag{2.13}$$

Analogous to Case 1, we have μ_1 is a p.m. satisfying $\mu_1[x_i, \infty) \leqq a_i \leqq \mu_1^*[x_i, \infty)$ for $i = 1, \ldots, i_0 - 1$; and we may apply the induction hypothesis to obtain p.m. ν_1 and martingale $Y = (Y_t)_{0 \leqq t \leqq 1}$ satisfying $M(Y) \overset{\mathscr{D}}{=} \nu_1$, $Y_1 \overset{\mathscr{D}}{=} \mu_1$ and $\nu_1[x_i, \infty) = a_i$ for $i = 1, \ldots, i_0$. We also have μ_2 is a p.m. satisfying $\int x d\mu_2(x) = x_{i_0}$ and $\mu_2[x_i, \infty) \leqq c_0^{-1} a_i \leqq \mu_2^*[x_i, \infty)$ for $i = i_0 + 1, \ldots, N+1$; and may apply the induction hypothesis to obtain p.m. ν_2 and martingale $Z = (Z_t)_{0 \leqq t \leqq 1}$ satisfying $Z_0 \equiv x_{i_0}$, $M(Z) \overset{\mathscr{D}}{=} \nu_2$, $Z_1 \overset{\mathscr{D}}{=} \mu_2$ and $\nu_2[x_i, \infty) = c_0^{-1} a_i$ for $i = i_0 +$

$1, \ldots, N+1$. Define martingale $X = (X_t)_{0 \leqq t \leqq 1}$ as in Case 1; then one verifies as in Case 1 that $M(X) \stackrel{\mathscr{D}}{=} \nu$, $X_1 \stackrel{\mathscr{D}}{=} \mu$, and $\nu[x_i, \infty) = a_i$ for $i = 1, \ldots, N+1$. □

PROOF OF THEOREM 2.1. We prove (2.6); Theorem 2.1 is then established. Let μ and ν be p.m.'s on $\mathbf{R}$ with $\mu < \nu < \mu^*$. From (2.2) we have that $\mu[x, \infty) \leqq \nu[x, \infty) \leqq \mu^*[x, \infty)$ for every $x \in \mathbf{R}$. Let C_n be a set of numbers $c_{0,n} < c_{1,n} < \cdots < c_{k(n),n} < \infty$ for each $n > 1$, for which $C_n \subset C_{n+1}$ for all n and $C := \bigcup_{n>1} C_n$ is dense in $\mathbf{R}$. For each $n > 1$, $a_i = \nu[c_{i,n}, \infty)$, $i = 0, \ldots, k(n)$, are numbers satisfying $\mu[c_{i,n}, \infty) \leqq a_i \leqq \mu^*[c_{i,n}, \infty)$; and hence from Proposition 2.3 there is a p.m. $\nu^n \in \mathscr{M}(\mu)$ satisfying $\nu^n[c_{i,n}, \infty) = a_i$ for $i = 0, \ldots, k(n)$. We have that $\lim_n \nu^n[c, \infty) = \nu[c, \infty)$ for each $c \in C$; and $\nu^n[c, \infty) = \nu[c, \infty)$ from some n onwards, for each $c \in C$. This implies $\nu^n \Rightarrow \nu$. From Proposition 1.6, it follows that $\nu \in \mathscr{M}(\mu)$, and (2.6) holds.

3. Martingales with given initial, maxima, and terminal distributions

The goal of this section is stochastic ordering characterization of the collection of p.m.'s

$$\mathscr{M}(\mu_0, \mu_1) = \{\nu \in \mathscr{P}(\mathbf{R}) : \text{there is a martingale } X = (X_t)_{0 \leqq t \leqq 1} \text{ with } X_0 \stackrel{\mathscr{D}}{=} \mu_0, M \stackrel{\mathscr{D}}{=} \nu, \text{ and } X_1 \stackrel{\mathscr{D}}{=} \mu_1\}, \tag{3.1}$$

for given p.m.'s μ_0 and μ_1 on $\mathbf{R}$ with $\int |x| \, d\mu_1(x) < \infty$. This is Theorem 3.4. Two preliminary characterizations are given of collections related to (3.1).

For the first result (stated in (0.3)), let μ be any p.m. with $\int x d\mu(x) = 0$, and denote $\mathscr{M}_0(\mu) = \mathscr{M}(\varepsilon_0, \mu)$. Observe that $\mathscr{M}_0(\mu)$ is nonempty. Indeed, if Y is an r.v. associated with p.m. μ, define μ_+ to be the p.m. on $\mathbf{R}$ associated with $Y_+ = Y \vee 0$. Then by taking $X_t \equiv 0$ if $0 \leqq t < \frac{1}{2}$, and $= Y$ if $\frac{1}{2} \leqq t \leqq 1$, we have that $\mu_+ \in \mathscr{M}_0(\mu)$. The martingale $(Z_t)_{0 \leqq t \leqq 1}$ of Section 2 shows again that $\mu^* \in \mathscr{M}_0(\mu)$.

THEOREM 3.1. *For any p.m. on* $\mathbf{R}$ *with* $\int x d\mu(x) = 0$,

$$\mathscr{M}_0(\mu) = \{\nu \in \mathscr{P}(\mathbf{R}) : \mu_+ < \nu < \mu^*\}. \tag{3.2}$$

PROOF. The containment '$\subset$' is clear from (2.2). For a proof of the containment '$\supset$' one can use Theorem 2.1 to obtain a martingale $Y = (Y_t)_{0 \leqq t \leqq 1}$ for which $M(Y) \stackrel{\mathscr{D}}{=} \nu$ and $Y_1 \stackrel{\mathscr{D}}{=} \mu$; and then define martingale $X = (X_t)_{0 \leqq t \leqq 1}$ by

$$X_t = 0 \text{ if } 0 \leqq t < \tfrac{1}{2}, \quad \text{and} \quad = Y_{2t-1} \text{ if } \tfrac{1}{2} \leqq t \leqq 1$$

to show that $\nu \in \mathscr{M}_0(\mu)$.

A second proof of the containment '$\supset$', independent of Theorem 2.1, could be carried out as follows. Let $\mathscr{S}$ be the set of all functions s: $[0, 1] \times [0, 1] \to [0, 1]$ with the properties $s(\cdot, v)\uparrow$ for each v fixed; $s(u, \cdot)\uparrow$ for each u fixed; $s(u, v) \leqq u$ for all u, v; $s(\cdot, v)$ is right continuous for each v; $s(u, v)$ equals either u or $\lim_{t\uparrow u} s(t, v)$; and $s(0, \cdot) = 0$, $s(1, \cdot) = 1$. Let μ be a p.m. on $\mathbf{R}$ with $\int x d\mu(x) = 0$ and $(Z_t)_{0 \leqq t \leqq 1}$ be the Dubins and Gilat martingale of (2.4) associated with μ. Let the probability space be $\Omega = [0, 1] \times [0, 1]$, $\mathscr{F} = \mathscr{B}(\Omega)$, and $P =$ Lebesgue measure on Ω. For each $s \in \mathscr{S}$, define $(X_t)_{0 \leqq t \leqq 1}$ on $(\Omega, \mathscr{F}, P)$ by $X_t(\omega) = Z_{s(t, \omega_2)}(\omega_1)$ for $\omega = (\omega_1, \omega_2) \in \Omega$. Then $(X_t)_{0 \leqq t \leqq 1}$ is a martingale with respect to filtration $\{\mathscr{F}_{s(t)}\}$ where $\mathscr{F}_t = \sigma(\mathscr{B}([0, t]), (t, 1]) \times \mathscr{B}([0, 1])$ for $t \in [0, 1]$. One shows that for each p.m. ν with $\mu_+ < \nu < \mu^*$, there is a function s in $\mathscr{S}$ with associated martingale $(X_t)_{0 \leqq t \leqq 1}$ satisfying $X_0 \equiv 0$, $M(X) \overset{\mathscr{D}}{=} \nu$, and $X_1 \overset{\mathscr{D}}{=} \mu$ by using techniques similar to those used in this paper. This involves showing that $\{\nu \in \mathscr{P}([0, \infty))$: there is a martingale $Z^s = (Z_{s(t, \cdot)})_{0 \leqq t \leqq 1}$ with $M(Z^s) \overset{\mathscr{D}}{=} \nu$ for some $s \in \mathscr{S}\}$ is convex and closed with respect to weak convergence, and contains an appropriate dense subset, obtained by explicit construction.

Since van der Vecht [23] has shown that $(Z_t)_{0 \leqq t \leqq 1} \overset{\mathscr{D}}{=} (B_{T(t)})_{0 \leqq t \leqq 1}$ where $(B_s)_{s \geqq 0}$ is standard Brownian motion and $(T(t))_{0 \leqq t \leqq 1}$ is a standard family of stopping times of the type described by Azema and Yor [1], we have $(X_t)_{0 \leqq t \leqq 1} \overset{\mathscr{D}}{=} (B_{T(s(t))})_{0 \leqq t \leqq 1}$. □

For the second result of this section, let ξ be any p.m. on $(\mathbf{R}^2, \mathscr{B}(\mathbf{R}^2))$, with marginals denoted by μ_0 and μ_1, for which $\int |x| d\mu_1(x) < \infty$ and $\int x_1 \xi(dx_1 \mid x_0) = x_0$ a.e. $[\mu_0]$. (Here and below regular conditional p.m.'s are used.) For characterization of such p.m.'s ξ, see [16] or [22]. Define

$$\begin{aligned} \mathscr{M}^2(\xi) = \{\nu \in \mathscr{P}(\mathbf{R})\colon & \text{ there is a martingale } X = (X_t)_{0 \leqq t \leqq 1} \\ & \text{for which } (X_0, X_1) \overset{\mathscr{D}}{=} \xi \text{ and } M \overset{\mathscr{D}}{=} \nu\}. \end{aligned} \tag{3.3}$$

The following result extends Theorem 3.1.

THEOREM 3.2.

$$\begin{aligned} \mathscr{M}^2(\xi) = \{\nu \in \mathscr{P}(\mathbf{R})\colon & \text{ there is a p.m. } \lambda \text{ on } \mathbf{R}^2, \\ & \text{with marginals } \mu_0 \text{ and } \nu, \text{ satisfying } \xi(\cdot \mid x) < \lambda(\cdot \mid x) < (\xi(\cdot \mid x))^* \\ & \text{and } \varepsilon_x(\cdot) < \lambda(\cdot \mid x) \text{ a.e. } [\mu_0]\}. \end{aligned} \tag{3.4}$$

PROOF. Let ξ be any p.m. on $\mathbf{R}^2$ with marginals μ_0 and μ_1 given as above. For the proof of the containment '$\subset$' in (3.4), let $\nu \in \mathscr{P}(\mathbf{R})$ and $X = (X_t)_{0 \leqq t \leqq 1}$ be a martingale for which $(X_0, X_1) \overset{\mathscr{D}}{=} \xi$ and $M(X) \overset{\mathscr{D}}{=} \nu$. Let λ be the probability measure on $\mathbf{R}^2$ associated with $(X_0, M(X))$, and $\lambda(A \mid x)$ denote a regular conditional p.m. version of $P(M(X) \in A \mid X_0 = x)$. Then λ has marginals μ_0 and ν, and $\xi(\cdot \mid x) < \lambda(\cdot \mid x)$ and $\varepsilon_x(\cdot) < \lambda(\cdot \mid x)$ a.e. $[\mu_0]$. To show that $\lambda(\cdot \mid x) < (\xi(\cdot \mid x))^*$ a.e. $[\mu_0]$, one uses martingales $Y^x = (Y_t^x)_{0 \leqq t \leqq 1}$ satisfying $Y_t^x \equiv x$ if $0 \leqq t \leqq \frac{1}{2}$ and $(Y_{t_1}^x, \ldots, Y_{t_n}^x) \overset{\mathscr{D}}{=} (X_{2t_1 - 1}, \ldots, X_{2t_n - 1}) \mid X_0 = x$, if $\frac{1}{2} \leqq t_1 \leqq \cdots \leqq t_n \leqq 1$ and $n \geqq 1$, so that $Y_0^x \equiv x$, $M(Y^x) \overset{\mathscr{D}}{=} \lambda(\cdot \mid x)$, $Y_1^x \overset{\mathscr{D}}{=} \xi(\cdot \mid x)$ a.e. $[\mu_0]$. From Theorem 3.1, one obtains that $\lambda(\cdot \mid x) < (\xi(\cdot \mid x))^*$ a.e. $[\mu_0]$. From Theorem 3.1, one obtains that $\lambda(\cdot \mid x) < (\xi(\cdot \mid x))^*$ a.e. $[\mu_0]$.

To show the containment '$\supset$' in (3.4), let λ be a p.m. on $\mathbf{R}^2$ with marginals μ_0 and ν satisfying

$$\xi(\cdot \mid x) < \lambda(\cdot \mid x) < (\xi(\cdot \mid x))^* \quad \text{and} \quad \varepsilon_x(\cdot) < \lambda(\cdot \mid x) \text{ a.e. } [\mu_0]. \tag{3.5}$$

We ensure appropriate measurability conditions are satisfied in our martingale construction by taking the following approach. In the following, the filtrations for the martingales are taken to be the natural filtrations. Let $\mathbf{R} \times \mathscr{P}(D[0, 1])$ be given the product Borel σ-algebra, denote by Γ the set in this space given by

$$\begin{aligned} \Gamma = \{(x, P) :\ & P \text{ is the p.m. } P_Y \text{ on } D[0,1] \text{ induced} \\ & \text{by some martingale } Y = (Y_t)_{0 \leqq t \leqq 1} \\ & \text{for which } Y_0 \equiv x,\ M(Y) \overset{\mathscr{D}}{=} \lambda(\cdot \mid x), \text{ and } Y_1 \overset{\mathscr{D}}{=} \xi(\cdot \mid x)\}. \end{aligned}$$

Now Γ is a Borel set. This follows, for example, from results in Chapter 7 of [3]; uniform integrability and right continuity of the processes; and the representation

$$\begin{aligned} \Gamma = \{(x, P) : & P(\pi_1^{-1}(E)) = \xi(E \mid x),\ P(T^{-1}(E)) = \lambda(E \mid x), \quad \text{and} \\ & P(\pi_0^{-1}(E)) = \varepsilon_x(E) \qquad \text{for all } E \in \mathscr{E}; \quad \text{and} \\ & \int \pi_s(y) \prod_{j=1}^{k} f_j(\pi_{u(j)}(y))\,dP = \int \pi_t(y) \prod_{j=1}^{k} f_j(\pi_{u(j)}(y))\,dP \\ & \text{for all } f_1, \ldots, f_k \in \mathscr{C}, \text{ and all } u(1), \ldots, u(k), s, t \\ & \text{in } I \text{ with } u(j) \leqq s < t, \text{ for } k = 1, 2, \ldots \}, \end{aligned} \tag{3.6}$$

where $\pi_s(y) = y_s$ and $Ty = \sup_{0 \leqq t \leqq 1} y_t$; $\mathscr{E}$ is a countable collection of open subsets of $\mathbf{R}$ generating $\mathscr{B}(\mathbf{R})$; $\mathscr{C}$ is a countable dense subset of the unifomly continuous functions on $\mathbf{R}$; and I is a dense subset of $[0, 1]$ including number 1. Also, from (3.5) and Theorem 3.1, $\Gamma(x)$, the section of Γ at x, is nonempty for almost all x $[\mu_0]$. From the Jankov–von Neumann Selection Theorem (see e.g., [3]: Proposition 7.49), one obtains a Borel measurable mapping $\bar{P}$ from $\mathbf{R}$ into $\mathscr{P}(D[0, 1])$ for which $(x, \bar{P}(x)) \in \Gamma$ a.e. $[\mu_0]$ (for another selection result along these lines, see Proposition 3.3 of [18]). Finally, define p.m. P on $D[0, 1]$ by $P(C) = \int (\bar{P}(x))(C) d\mu_0(x)$ for $C \in \mathscr{B}(D[0, 1])$, and let $Y = (Y_t)_{0 \leqq t \leqq 1}$ be the process associated with P. Then Y is a martingale satisfying for $B \in \mathscr{B}(\mathbf{R}^2)$ and $A \in \mathscr{B}(\mathbf{R})$,

$$P((Y_0, Y_1) \in B) = \int (\bar{P}(x))(\{y \in D : (\pi_0(y), \pi_1(y)) \in B\}) d\mu_0(x)$$
$$= \int I_B(x, x_1) \xi(dx_1 \mid x) \mu_0(dx) = \xi(B),$$

and

$$P(M(Y) \in A) = \int (\bar{P}(x))(\{y \in D : Ty \in A\}) d\mu_0(x) = \int \lambda(A \mid x) d\mu_0(x) = \nu(A).$$

This proves that $\nu \in \mathscr{M}^2(\xi)$. □

REMARK 3.3. For p.m. ξ given as above with marginals μ_0 and μ_1, let $\mu_0 \vee \mu_1$ be the p.m. on $\mathbf{R}$ defined by $(\mu_0 \vee \mu_1)(A) = \xi(\{(x_0, x_1) : x_0 \vee x_1 \in A\})$ for $A \in \mathscr{B}(\mathbf{R})$. We claim that

$$\mathscr{M}^2(\xi) \subset \{\nu \in \mathscr{P}(\mathbf{R}) : \mu_0 \vee \mu_1 < \nu < \mu_1^*\}. \tag{3.7}$$

This follows from (3.3), (3.4), and the result that

$$\int (\xi(\cdot \mid x))^* d\mu_0(x) < \left(\int \xi(\cdot \mid x) d\mu_0(x) \right)^* = \mu_1^*. \tag{3.8}$$

The inequality of (3.8) follows from

$$\left(\sum_{i=1}^{n} p_i \mu_i^* \right) [x, \infty) \leqq \left(\sum_{i=1}^{n} p_i \mu_i \right)^* [x, \infty) \qquad \text{for all } x \in \mathbf{R}, \tag{3.9}$$

for any p.m.'s $\mu_1, \ldots, \mu_n$ and nonnegative numbers $p_1, \ldots, p_n$ with $\sum_{i=1}^n p_i = 1$. This is proved using the maximal function of Section 2 and observing that the p.m. on the left-hand side of (3.9) involves the appropriate rearrangement and averaging first, and then a mixture, and the p.m. on the right-hand side of (3.9) involves a mixture first, and then the appropriate rearrangement and averaging. To see that the containment in (3.7) can be strict, let $\xi = (1-x)\cdot\varepsilon_{(0,0)} + x\cdot\varepsilon_{(1,1)}$, and let $\nu = \mu_1^*$ with d.f. $H(u) = 0$ if $u \leqq x$, $= 1 - xu^{-1}$ if $x \leqq u < 1$, and $= 1$ if $1 \leqq u < \infty$. In this case, $\mu_0 \vee \mu_1 = \mu_1 < \nu < \mu_1^*$, but $\nu \notin \mathscr{M}(\xi) = \{\mu_0\}$ (use e.g., (3.4) and $\xi(\cdot \mid x) = (\xi(\cdot \mid x))^*$ to see this last equality).

As an immediate corollary to Theorem 3.2, we have the following result.

THEOREM 3.4. *The set of p.m.'s $\mathscr{M}(\mu_0, \mu_1)$ equals the set of p.m.'s $\nu \in \mathscr{P}(\mathbf{R})$ for which*

(i) *there is a p.m. ξ on $\mathbf{R}^2$ with marginals μ_0 and μ_1 and $\int x_1 \xi(dx_1 \mid x_0) = x_0$ a.e. $[\mu_0]$, and*

(ii) *there is a p.m. λ on $\mathbf{R}^2$ with marginals μ_0 and ν and $\xi(\cdot \mid x) < \lambda(\cdot \mid x) < (\xi(\cdot \mid x))^*$ and $\varepsilon_x(\cdot) < \lambda(\cdot \mid x)$ a.e. $[\mu_0]$.*

4. Sharp martingale inequalities and prophet problems

We first give a result which illustrates the use of these stochastic order comparisons of the previous sections to prove sharp martingale inequalities.

THEOREM 4.1. *Let μ be any p.m. on* $[0, 1]$, *with associated d.f. F. Then for any martingale $(X_t)_{0 \leqq t \leqq 1}$ with $X_1 \stackrel{\mathscr{D}}{=} \mu$ and $M = \sup_{0 \leqq t \leqq 1} X_t$,*

$$E(M) \leqq \int_{[0,1]} (1 - F(z)) - (1 - F(z))\ln(1 - F(z))dz \tag{4.1}$$

(*the integrand is taken to be zero if $F(z) = 1$*). *Inequality* (4.1) *is attained.*

PROOF. Let $x = \int z d\mu(z)$, and let $Z = (Z_t)_{0 \leqq t \leqq 1}$ be a martingale constructed as in the beginning of Section 2 for which $Z_0 \equiv x$, $M(Z) \stackrel{\mathscr{D}}{=} \mu^*$, and $Z_1 \stackrel{\mathscr{D}}{=} \mu$. From Theorem 2.1, $M(X) < M(Z)$ in the stochastic ordering and so $E(M(X)) \leqq E(M(Z))$. Letting $s(u) = (1-u)\ln(1-u) - (1-u)$ if $0 \leqq u < 1$ and $= 0$ if $u = 1$, inequality (4.1) follows from the calculation

$$\begin{aligned} E(M(Z)) &= \int_{(0,1)} z d\mu^*(z) = \int_{(0,1)} (1-w)^{-1} \int_{(w,1)} F^{-1}(u) du dw \\ &= \int_{(0,1)} F^{-1}(u)(-\ln(1-u))du = \int_{(0,1)} F^{-1}(u) ds(u) \\ &= \lim_{a \downarrow 0, b \uparrow 1} \int_{[a,b]} -s(u)dF^{-1}(u) + F^{-1}(b)s(b) - F^{-1}(a)s(a) \\ &= \int_{(0,1)} -s(F(x))dx. \end{aligned}$$

□

COROLLARY 4.2. ([7]: Inequality (9)). *For any martingale* $(X_t)_{0 \leqq t \leqq 1}$ *taking values in* $[0, 1]$ *with* $EX_0 = x$,

$$E(M) \leqq x - x \ln x. \tag{4.2}$$

The inequality (4.2) *is attained within this class of martingales.*

PROOF. The function $x - x \ln x$ is concave on $[0, 1]$. Inequality (4.2) thus follows from (4.1), Jensen's inequality, and $\int_{(0,1)} (1 - F(x))dx = \int_{[0,1]} z dF(z) = x$. For attainment of inequality (4.2), let $\mu = (1 - x) \cdot \varepsilon_x + x \cdot \varepsilon_1$ and let $(Z_t)_{0 \leqq t \leqq 1}$ be a martingale of the type constructed at the beginning of Section 2 so that $Z_0 \equiv x$, $M(Z) \overset{\mathscr{D}}{=} \mu^*$, and $Z_1 \overset{\mathscr{D}}{=} \mu$. Then p.m. μ^* has associated d.f. $H(u) = 0$ if $-\infty < u \leqq x$, $= 1 - xu^{-1}$ if $x \leqq u < 1$, and $= 1$ if $1 \leqq u < \infty$; and $E(M(Z)) = \int z d\mu^*(z) = x - x \ln x$. □

We give an interpretation of our results of prophet vs. gambler type. Consider a subclass $\mathscr{C}$ of processes $X = (X_t)_{0 \leqq t \leqq 1}$ in $D[0, 1]$ which satisfy the property that the gambler can achieve the value $\sup\{EX_\tau : \tau$ is a stop rule for $(X_t)_{0 \leqq t \leqq 1}\}$ as EX_{τ^*} with some stopping time τ^*. Now, let X vary within $\mathscr{C}$ under the constraint that X_{τ^*} has fixed p.m. μ, and find the possible distributions of $M = M(X)$ and associated values of EM, the possible rewards of the prophet. If we assume that $\mathscr{C}$ is the collection of uniformly integrable martingales in $D[0, 1]$ and that μ is a p.m. on $\mathbf{R}$ with $\int |x| d\mu(x) < \infty$, then a uniformly applicable optimal stopping time in this setting is $\tau^* \equiv 1$. Theorem 2.1 says that if we let X vary within $\mathscr{C}$ under the constraint that $X_1 \overset{\mathscr{D}}{=} \mu$, so that the gambler receives $\int x d\mu(x)$, then the possible distributions of M are the p.m.'s ν satisfying $\mu < \nu < \mu^*$, and the possible associated reward for the prophet are the values $\int x d\nu(x)$ between $\int x d\mu(x)$ and $\int x d\mu^*(x)$.

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