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A NOTE ON THE UNSTEADY FLOW OF DUSTY VISCOUS FLUID BETWEEN TWO PARALLEL PLATES

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ABSTRACT. We study the isothermal flow of a dusty viscous incompressible conducting fluid between two types of boundary motions- oscillatory and non-oscillatory, under the influence of gravitational force. Within the frame work of some physically realistic approximations and suitable boundary conditions, closed form solutions were obtained for the velocity profiles and the skin friction of the particulate flow. These results show that for a constant pressure gradient, only the velocity profile of the fluid and the skin friction are unaffected by gravity, while magnetic field is seen to affect both the fluid, particle velocities and the skin friction. Thus, our results are extension of previous results in literature, and graphical demonstration of some these solutions have been presented.

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1. Introduction

The subject of heat and mass transfer in particulate flows came into scientific prominence since the 1950's, and has been motivated by a number of scientific and technological applications, such as purification of drinking water, sedimentation, petroleum industries, and use of dust gas cooling to enhance heat transfer process, e.t.c. These numerous areas of applications have thus foster a continous attention on these areas of research.

Saffman [9] pioneered the study of the fliud - particle system. They derived the equations describing the motion of a gas carrying small dust particles and the equations satisfied by small disturbances of a steady laminar flow. They examined the convective stability of the particulate poiseuille flow on the assumption that the solid phase is distributed homogeneously.

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Ganguly and Lahiri [4] considered the motion of an isothermal dusty viscous incompressible fluid between two infinite parallel plates where both plates are assumed to be oscillating harmonically with different amplitudes and frequency in their own planes. Exact solutions for the velocity of the fluid and dusty particles were obtained.

Uwanta [10] studied the oscillating free convection flow of incompressible rigid conducting fluid which contains suspended inert rigid spherical particles between two infinite plates. They showed that temperature has significant effect on the fluid velocity but has no effect on the particle velocity.

Erdogan [3] considered the flow of a viscous fluid produced by a plane boundary moving in its own plane with a sinusoidal variation of velocity. An analytical solution describing the flow at small and large times after the start of the boundary is obtained by the Laplace Transform method. The solution gives not only the steady solution but also the transient solution. The time required to attain steady flows for the cosine oscillation of the boundary is one-half cycle and it is a full cycle for the sine oscillation of the boundary.

Chandran et al. [2] investigated the hydro-magnetic flow of electrically conducting liquids whose Prandtl number are different from unity when the flow takes place near an infinite vertical plate, subject to uniform heat flux and accelerated motion. A unified exact solution has been derived for the boundary layer velocity and skin friction for the cases of magnetic field being fixed relative to the fluid or to the vertical plate.

In another similar application, Haldar [6] investigated the flow of blood through a constricted artery in the presence of an external transverse magnetic field using Adomian's decomposition method. The expressions for the two-term approximation to the solution of stream function, axial velocity component and wall shear stress are obtained in this analysis.

Lyubimov et al. [8] considered the non-isothermal two-phase flow in a closed cavity, where one of the phases is gas (or liquid) and another phase consists of solid particles. The particles are subjected to the gravitational precipitation, high frequency vibrations and the drag exerted by the flow. They examined the linear stability of the plane parallel flow between differently heated vertical plates for the static gravity field. In addition, the transient processes of the thermal buoyancy convection of a fluid with solid inclussions is determined numerically.

Soundalgekar and Bhat [12] obtained an exact solution to fully developed flow of a viscous, incompressible fluid in a porous medium between two vertical parallel plates. The temperature and velocity profiles are shown graphically.
Bratsun and Teplov [1] also studied the two-phase flow in a vertical slot differently heated from the side walls where one of the phase is fluid and another phase consisits of small solid particles. The particles are subjected to downward drag exerted by gravity and the drag exerted by finite frequency horizontal vibrations along the layer. In the framework of generalized Boussinesq approximations,

they derived a set of governing equations, wherein a linear stability analysis of the pulsed base flow is carried out.

Since not all partial differential equations are amenable to exact solutions, Gachpazan et al. [5] introduced a new method for finding the approximate solution of a second order nonlinear partial differential equation. In this method the problem is transformed to an equivalent optimization problem. Then, by considering it as a distributed parameter control system, the theory of measure is used for obtaining the approximate solution of the original problem.

As it is known, when the flow field comprises an electrically conducting fluid under the influence of an externally applied magnetic field and gravitational force, the combined effects of these influence render the system of Navier Stokes equations highly nonlinear and coupled. However, this may be difficult to handle analytically and the degree of difficulty depends on the physical situation and the assumption inherent to the problem. Analytical solutions, though mostly representing idealized situations, are also important partly because of their wider applicability in understanding the basic physics of the problem, and partly because of their possible applications in industrial and technological field. It is known that nonlinear problem can be reduced to a linear system, which admits closed form solution, if the quadratic convection terms in the governing equations are neglected and a suitable chioce of boundary condition is proposed.

Hence, the focus of this work is to consider the motion of an isothermal conducting dusty viscous incompressible fluid between two infinite parallel plates under the influence of gravitational precipitation, using an analytical framework. This is to enable us appreciate the contributions of gravity and the magnetic field through analytical solutions in order to validate previous works in which numerical methods or qualitative analysis have been carried out (see [1], [8] and [12]).

2. Problem formulation and basic equations

Consider two dimensional incompressible plane($\frac{\partial}{\partial z} = 0$)viscous flow between two parallel plates with distance d apart. We assume that the plates are very wide and very long, so that the flow is essentially axial. We take the x-axis along the lower plate in the direction of the flow and the y- axis perpendicular to the plates drawn into the region of the fluid. For an isothermal flow, the equations of motion of an unsteady dusty viscous incompressible fluid under gravity are (see Lyubimov et al. [8], Saffman [7], Ganguly and Lahiri [4] and Uwanta [11] and White [12]),

$$\frac{\partial U}{\partial t} + (U.\nabla)U = -\frac{1}{\rho}\nabla p - \frac{\sigma B_0^2 U}{\rho} + \nu\nabla^2 U + \frac{KN}{\rho}(V-U) + g, \quad (1)$$

$$\nabla.U = 0, \quad (2)$$

$$\frac{\partial}{\partial t}V + (V.\nabla)V = g + \frac{K}{m}(U - V), \quad (3)$$

$$\frac{\partial N}{\partial t} + \nabla.(NV) = 0 \quad (4)$$

where U and V denote the local velocity vectors of fluid and dust particles,
ρ is the density,
p is the static fluid pressure,
ν is the kinetic viscosity,
N is the number of dust particles per unit volume and
K is a resistance coefficient (e.g. by stokes formula, spherical partcle has $K = 6\pi\mu a$),
B_0 is a constant external magnetic field parameter,
g is the acceleration due to gravity,
m is the mass of the particles.

From the continuity equations,

$$\nabla.U = 0, \text{ i.e., } \frac{\partial u}{\partial x} = 0 \ \Rightarrow u = u(y)$$

and

$$\nabla.V = 0, \text{ i.e., } \frac{\partial v}{\partial x} = 0 \ \Rightarrow v = v(y).$$

Thus, equations (1) - (4) reduce to

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\sigma B_0^2 u}{\rho}u + \nu\frac{\partial^2 u}{\partial y^2} + \frac{KN}{\rho}(v - u) + g, \quad (5)$$

$$\frac{\partial}{\partial t}v = g + \frac{K}{m}(u - v) \quad (6)$$

where u and v are the velocities of fluid and dust particles respectively along the field of flow(x-direction).

2.1. Oscillating parallel plates

We suppose that the lower plate oscillates at an amplitude a_1 and frequency λ_1 while the upper plate oscillates at an amplitude a_2 and frequency λ_2. We shall further assume that flow has a negligible convection, the pressure gradient is constant and N is fixed. The boundary conditions may be taken

$$u = a_1 e^{-i\lambda_1 t}, \ y = 0 \text{ and } u = a_2 e^{-i\lambda_2 t}, \ y = d. \quad (7)$$

We eliminate v between (5) and (6) by substituting for v_t and v in (5) to give

$$u_{tt} = \nu u_{yyt} + \frac{K\nu}{m}u_{yy} - \left(\frac{K}{m} + \frac{KN}{\rho} + \frac{\sigma B_0^2}{\rho}\right)u_t$$
$$- \frac{K\sigma B_0^2 u}{m\rho} + \left(\frac{Kg}{m} + \frac{KNg}{\rho} - \frac{K}{m\rho}\frac{dp}{dx}\right). \quad (8)$$

A special case of conservative force is gravity and for fluid in equilibrum or fluid with constant pressure gradient(see [12] and [10]), we may assume that

$$\frac{dp}{dx} = (\rho + mN)\, g. \tag{9}$$

Thus, (8) reduces to

$$u_{tt} \;=\; \nu u_{yyt} \;+\; \frac{K\nu}{m} u_{yy} - \left(\frac{K}{m} + \frac{KN}{\rho} + \frac{\sigma B_0^2}{\rho}\right) u_t \;-\; \frac{K\sigma B_0^2 u}{m\rho}. \tag{10}$$

From the boundary condition (7) we seek a solution of the form

$$a_1 f(y) e^{-i\lambda_1 t} \;+\; a_2 g(y) e^{-i\lambda_2 t} \tag{11}$$

where

$$A = \frac{K\nu}{m},\; B = \frac{K}{m} + \frac{KN}{\rho} + \frac{\sigma B_0^2}{\rho},\; C = \frac{K\sigma B_0^2 u}{m\rho}$$

and $f(y)$ and $g(y)$ are to be determined. The substitution of equation (11) into (10) gives,

$$\begin{aligned} -\lambda_1^2 a_1 f e^{-i\lambda_1 t} - \lambda_2^2 a_2 g e^{-i\lambda_2 t} \;=\; & \nu\Big(- i\lambda_1 a_1 f'' e^{-i\lambda_1 t} - i\lambda_2 a_2 g'' e^{-i\lambda_2 t}\Big) \\ & + A\Big(a_1 f'' e^{-i\lambda_1 t} + a_2 g'' e^{-i\lambda_2 t}\Big) \\ & + B\Big(i\lambda_1 a_1 f e^{-i\lambda_1 t} + i\lambda_2 a_2 g e^{-i\lambda_2 t}\Big) \\ & - C\Big(a_1 f e^{-i\lambda_1 t} + a_2 g e^{-i\lambda_2 t}\Big). \end{aligned}$$

Collecting terms, we have

$$-\lambda_1^2 f = -\nu\lambda_1 f'' + Af'' + Bi\lambda_1 f - Cf \tag{12}$$

and

$$-\lambda_2^2 g = -\nu\lambda_2 g'' + Ag'' + Bi\lambda_1 g - Cg. \tag{13}$$

We re-write (12) and (13) as

$$f'' + Pf = 0 \tag{14}$$

and

$$g'' + Qg = 0, \tag{15}$$

where

$$P \;=\; \frac{\lambda_1^2 + iB\lambda_1 - C}{A - i\nu\lambda_1} \text{ and } Q \;=\; \frac{\lambda_2^2 + iB\lambda_2 - C}{A - i\nu\lambda_2}.$$

The solution to (14) is

$$f(y) \;=\; C_0 \cos Py + C_1 \sin Py$$

with boundary conditions

$$f(0) \;=\; 1 \text{ and } f(d) = 0\ .$$

Hence,

$$f(y) = \frac{\sin P(d-y)}{\sin Pd}.$$

Similarly, the solution to (15) is

$$g(y) \;=\; C_2 \cos Py + C_3 \sin Py$$

with boundary conditions

$$g(0) \;=\; 0 \text{ and } g(d) = 0 \;,$$

where C_0, C_1, C_2 and C_4 are constants of integration. Using these boundary conditions, we obtain,

$$g(y) = \frac{\sin Qy}{\sin Qd}.$$

Hence,

$$u \;=\; a_1 \frac{\sin P(d-y)}{\sin Pd} e^{-i\lambda_1 t} \;+\; a_2 \frac{\sin Qy}{\sin Qd} e^{-i\lambda_2 t}. \tag{16}$$

Substituting (16) into (5), we obtain

$$v = \frac{mg}{K} + \left(1 - \frac{i\rho\lambda_1}{KN} + \frac{\nu P^2 \rho}{KN} + \frac{\sigma B_0^2}{KN}\right) a_1 \frac{\sin P(d-y)}{\sin Pd} e^{-i\lambda_1 t}$$
$$+ \left(1 - \frac{i\rho\lambda_2}{KN} + \frac{\nu Q^2 \rho}{KN} + \frac{\sigma B_0^2}{KN}\right) a_2 \frac{\sin Qy}{\sin Qd} e^{-i\lambda_2 t}. \tag{17}$$

We observe that the special case of $g \;=\; B_0 = 0$, solutions (16) and (17) are in full agreement with the results of Ganguly and Lahiri[4]. It is obvious that gravity(g) and magnetic parameter(B_0) have the effect of increasing the velocity of the dust particles. However, the magnetic field parameter(B_0) has effect on both the velocity of the fluid as well as the particles.

Consider the special case of a fixed upper plate and the lower executes simple harmonic motion in its own plane. Then we have $a_2 = 0$, $\lambda_2 = 0$ and the velocity of the fluid is given by

$$u \;=\; a_1 \frac{\sin P(d-y)}{\sin Pd} e^{-i\lambda_1 t} \tag{18}$$

and the velocity of the dust particle is given by

$$v \;=\; \frac{mg}{K} + \left(1 - \frac{i\rho\lambda_1}{KN} + \frac{\nu P^2 \rho}{KN} + \frac{\sigma B_0^2}{KN}\right) a_1 \frac{\sin P(d-y)}{\sin Pd} e^{-i\lambda_1 t}. \tag{19}$$

In another special case, both plates oscillate with the same amplitude and frequency, we have

$$a_1 = a_2 = a,\;\; \lambda_1 = \lambda_2 = \lambda \text{ and } P = Q = R,$$

and then the velocities reduce to

$$u \;=\; a \frac{\sin R(d-y)}{\sin Rd} e^{-i\lambda t} \;+\; a \frac{\sin Ry}{\sin Rd} e^{-i\lambda t} \tag{20}$$
$$= a \frac{\cos R(\frac{d}{2} - y)}{\cos \frac{Rd}{2}} e^{-i\lambda t},$$

and

$$v = \frac{mg}{K} + \left(1 - \frac{i\rho\lambda}{KN} + \frac{\nu R^2\rho}{KN} + \frac{\sigma B_0^2}{KN}\right) a \frac{\cos(\frac{d}{2} - y)}{\cos R\frac{Rd}{2}} e^{-i\lambda t}. \tag{21}$$

2.2. Skin friction

In order to evaluate the shear stress at the boundary, we consider the skin friction $\tau = (\partial u/\partial y)_{y=0}$ and $\tau = (\partial u/\partial y)_{y=d}$. Hence the skin friction is expressed as

$$\begin{aligned}\tau &= \left(\frac{\partial u}{\partial y}\right)_{y=d} \\ &= \frac{-pa_1 \cos P(d-y)}{\sin Pd} \exp(-i\lambda_1 t) + Qa_2 \frac{\cos Qy}{\sin Qd} \exp(-i\lambda_2 t) \\ &= -Pa_1 \csc Pd \exp(-i\lambda_1 t) + a_2 Q \cot Qd \exp(-i\lambda_2 t).\end{aligned} \tag{22}$$

At the lower plate,

$$\tau = \left(\frac{\partial u}{\partial y}\right)_{y=0} = -Pa_1 \cot Pd \exp(-i\lambda_1 t) + a_2 Q \csc Qd \exp(-i\lambda_2 t). \tag{23}$$

The expressions, (22) and (23) are in full agreement with the results obtained previously(see Ganguly and Lahiri [4]).

2.3. Fixed parallel plates

Suppose we consider the case in which both plates are fixed(i.e., $a_1 = a_2 = 0$). Then the solution (11) is no longer appropriate. Hence we seek a new approach. When the upper and lower plates are fixed, we subject the wall to the no-slip conditions

$$u = 0 \text{ at } y = d \text{ and } y = 0. \tag{24}$$

If we take ansatz, a separable solution which satisfies the boundary conditions (24) may be expressed in the form,

$$u(y,t) = h(t) \sin\left(\frac{(d-y)}{d}\pi\right), \tag{25}$$

where $h(t)$, satisfying the condition $h(\infty) \to 0$, is to be determined. Substituting equation (25) into equation (10), we obtain

$$h'' + Rh' + Sh = 0, \tag{26}$$

where

$$R = \left(\frac{\nu\pi^2}{d^2} + \frac{K}{m} + \frac{KN}{\rho} + \frac{\sigma B_0^2}{\rho}\right) \text{ and } S = \left(\frac{K\nu\pi^2}{md^2} + \frac{K\sigma B_0^2}{m\rho}\right).$$

By substituting $h(t) = e^{qt}$, we obtain the quadratic equation

$$q^2 + Rq + S = 0, \text{ whose solution is } q_{1,2} = \frac{-R \pm \sqrt{R^2 - 4S}}{2}.$$

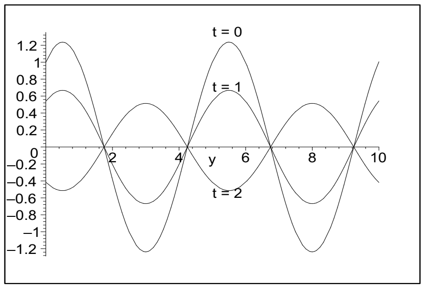

Figure 1: u vs y, $B_0 = 0$.

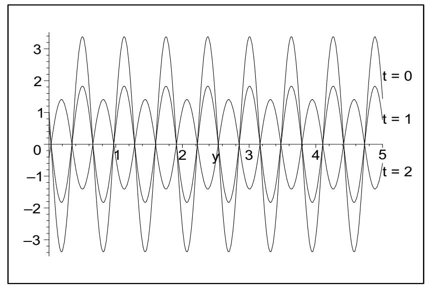

Figure 2: u vs y, $B_0 = 10$

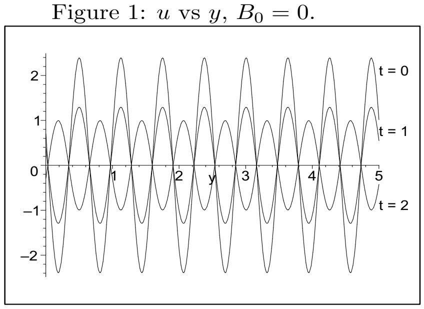

Figure 3: v vs y,$B_0 = 10, g = 0$.

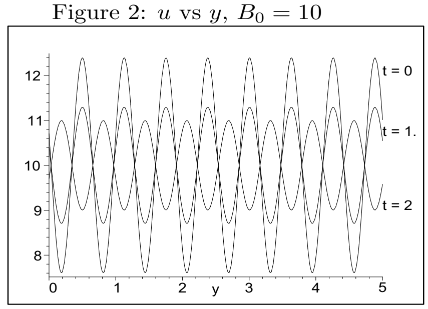

Figure 4: v vs y,$B_0 = g = 10$.

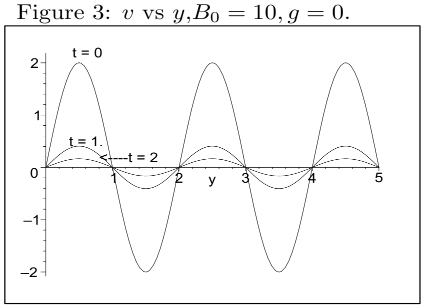

Figure 5: u vs y, $B_0 = 0$.

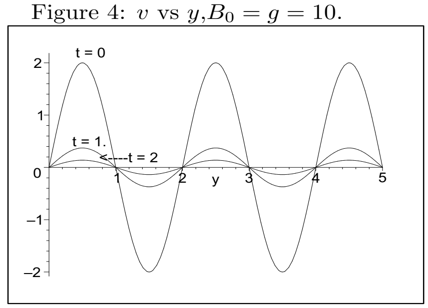

Figure 6: u vs y, $B_0 = 10$

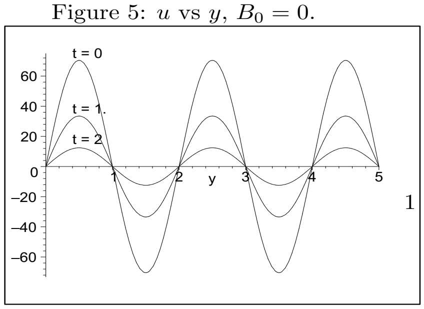

Figure 7: v vs y,$B_0 = 10, g = 0$.

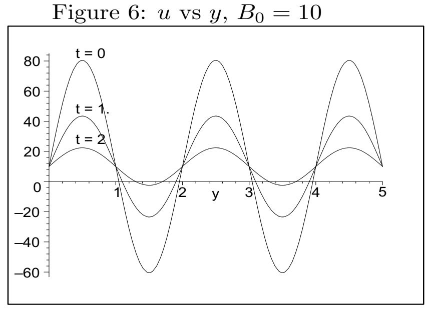

Figure 8: v vs y,$B_0 = g = 10$.

Thus

$$h(t) = C_4 e^{q_1 t} \ + \ C_5 e^{q_2 t}$$

and equation (25) becomes

$$u(y,t) = \left(C_4 e^{q_1 t} \ + \ C_5 e^{q_2 t}\right) \sin\left(\frac{(d-y)}{d}\pi\right), \tag{27}$$

where R and S are carefully chosen such that $q_{1,2} < 0$. We also substitute (27) to obtain

$$v = \left(C_6 e^{q_1 t} + C_7 e^{q_2 t}\right) \sin\left(\frac{(d-y)\pi}{d}\right) + \frac{mg}{K}, \tag{28}$$

$$C_6 = \left(q_1 + 1 - \frac{\rho\pi^2\nu}{KNd^2} + \frac{\sigma B_0^2}{KN}\right) \text{ and } C_7 = \left(q_2 + 1 - \frac{\rho\pi^2\nu}{KNd^2} + \frac{\sigma B_0^2}{KN}\right).$$

2.4. Skin friction

We may also obtain the skin friction τ for the velocity profiles (27) and (28) as

$$\tau \ = \ \left(\frac{\partial u}{\partial y}\right)_{y=d} = \frac{-\pi}{d}\left(e^{q_1 t} + e^{q_2 t}\right). \tag{29}$$

And at the lower plate,

$$\tau \ = \ \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\pi}{d}\left(e^{q_1 t} + e^{q_2 t}\right). \tag{30}$$

It is also worthy of note that the magnetic field parameter(B_0) also affects the skin friction in a similar pattern as the previous solutions.

Figures 1-2 Plot (u)(eqn. (16)) and Figures 2-4(v)(eqn. (17)) against(y), for some values of B_0, g and $t = \{0, 1, 2\}$ Figures 5-6 Plot (u)(eqn. (27)) and Figures 7-8(v)(eqn. (28)) against(y), for some values of B_0, g and $t = \{0, 1, 2\}$.

The figures are plots of velocity of the fluid $(u)$and the particle velocity (v) against y for $t = \{0, 1, 2\}$, where other parameters are fixed.

From figures 1 and 2 it is shown that (B_0) has the effect of increasing the velocity of the fluid u. While in figures 3 and 4, as expected, gravity (g) is seen to increase the velocity of the particles in addition to the magnetic field.

For the fixed parallel plates, figures 5 and 7 show that varying B_0, does not have a considerable effect on the velocity of fluid while figures 6 and 8 show the contributions of gravity, in addition to magnetic field, on the velocity of the particles. These observations have shown that the particles tend to respond faster to changes in the magnetic field and gravity than the fluid. Furthermore, the velocities are monotonically decreasing functions of time t.

3. Conclusion and discussion

We have considered an isothermal flow of a dusty viscous incompressible conducting fluid between two cases- oscillating and non-oscillating parallel plates under gravitational force. Within the limit of a physically reasonable approximations and boundary conditions, closed form solutions were obtained for the velocity profiles of the fluid as well as the dusty particles. In addition to these, the skin frictions at the walls are evaluated. These results when compared with previous results, show that the velocity profile of the fluid is unaffected by gravity, whereas it increases the velocity of the dusty particle considerably. Furthermore, the magnetic field parameter affects the velocities of the fluid, particles and the skin friction. However, these analytical solutions may be seen as representing idealized situations, they are considered important partly because of their wider applicability in understanding some basic physics of the problem, and partly because it could be of some interest to researchers in the subject area.

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