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OPTIMAL CONTROL OF THE HEAT EQUATION IN AN INHOMOGENEOUS BODY

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ABSTRACT. In this paper we consider a heat flow in an inhomogeneous body without internal source. There exists special initial and boundary conditions in this system and we intend to find a convenient coefficient of heat conduction for this body so that body cool off as much as possible after definite time. We consider this problem in a general form as an optimal control problem which coefficient of heat conduction is optimal function. Then we replace this problem by another in which we seek to minimize a linear form over a subset of the product of two measures space defined by linear equalities. Then we construct an approximately optimal control.

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1. Introduction

The optimal control problem for partial differential equation with an optimal control function as heat conduction is investigated in [4], [5]. In these papers, the methods are based on "linearization" and have used many of conditions for finding optimal control. We intend to consider this system by giving a method based on measure theory used in optimal control problems on a system of diffusion equations and a control function, (see [6], [7], [8], [9], [12], [13]). Then we will find an approximate optimal control for this problem.

Let D be a bounded domain in R^n with smooth boundary ∂D. We consider the bounded cylindrical region $Q_T = D \times (0, T)$ in R^{n+1}, here T is a positive

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real number, define

$$\begin{aligned}\Gamma_T &= \{x \in \partial D, 0 < t < T\} = \partial D \times (0, T),\\ D_0 &= \{x \in D, t = 0\} = D \times \{0\},\\ D_T &= \{x \in D, t = T\} = D \times \{T\}.\end{aligned}$$

We now consider the diffusion equation by operator L as follows:

$$Lu = u_t - \operatorname{div}(k(x)\nabla u) \equiv 0, \tag{1}$$

where $k(x) \in C^1(\overline{Q}_T)$ and $k(x) \geq c > 0$, (c is a positive constant), with the initial the condition

$$u|_{D_0} = \rho(x), \tag{2}$$

and the boundary condition

$$u|_{\Gamma_T} = 0. \tag{3}$$

The function $k(\cdot)$ is the control function and assume it gets its values in a bounded set as $K \subset R$. We consider the control function $v(\cdot)$ in terminal condition such that

$$u|_{D_T} = v, \tag{4}$$

where the control function $v(\cdot) \in V \subset R$ is Lebesgue measurable.

Definition 1. A triple (u, k, v) of trajectory function u and two control functions k and v is said to be admissible if:

i) The trajectory function

$$u(x,t) \in C^{2,1}(Q_T) \bigcap C(Q_T \bigcup \Gamma_T \bigcup D_0)$$

satisfies the problem (1)-(4).

ii) The control functions $k(\cdot)$ and $v(\cdot)$ are in $C^1(\overline{Q}_T)$ and $C(D_T)$, respectively.

Let Υ be the nonempty set of admissible triples. We intend to find a triple in Υ, such that minimizes the functional

$$J(u, k, v) = \int_{Q_T} f_\circ(t, x, u, \nabla u, k)dxdt + \int_{D_T} g_\circ(x, v)dx, \tag{5}$$

where $f_\circ$ and $g_\circ$ are nonnegative, continuous and real functions respectively on R^{2n+3} and R^{n+1} and suppose that there exists a constant $h > 0$, such that

$$f_\circ(t, x, u, \nabla u, k) \leq h|u|, \ \forall\ (x, t) \in Q_T.$$

From (3) we can find the bounded sets $A \subset R$ and $B \subset R^n$ such that (see [10]),

$$u(x,t) \in A, \quad \nabla u(x,t) \in B. \tag{6}$$

We mention that A and B are intersection of such sets, that satisfy in (6), and furthermore these sets must be locally compact.

2. Change of the space

In the given classical control problem, in general it is not possible to find a triple in Υ such that to minimize the functional (5). So we may extend the problem to measure space which is larger than the classic space of controls, then we obtain a solution in the new space for the problem and finally we obtain an approximate solution for the original problem in the classic space.

For this purpose in first we consider the following theorem.

Theorem 1. *Let $u(x,t) \in C^{2,1}(Q_T) \bigcap C(Q_T \bigcup \Gamma_T \bigcup D_0)$ be a classic solution of (1)-(4). Then this solution satisfies the following equation*

$$\int_0^T \int_D (-u\varphi_t + k\nabla\varphi\nabla u)dxdt + \int_{D_T} \varphi u dx = \int_{D_0} \varphi u dx, \tag{7}$$

where φ's are in $C^1(\overline{Q}_T)$ and satisfy

$$\varphi|_{\Gamma_T} = 0. \tag{8}$$

Proof. Assume Φ be the set of all φ's in $C^1(\overline{Q}_T)$ that satisfy in (8). Multiplying (1) by a member $\varphi \in \Phi$, we have

$$\varphi u_t = \varphi \mathrm{div}(k(x)\nabla u).$$

Besides, $\mathrm{div}(k(x)\nabla u) = k\Delta u + \nabla k \nabla u$, since by the following definitions (see chapter VI of [10])

$$\nabla u(x,t) = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right), \ \mathrm{div}(w_1(x,t), \cdots, w_n(x,t)) = \frac{\partial w}{\partial x_1} + \cdots + \frac{\partial w}{\partial x_n},$$

and

$$\triangle u = \mathrm{div}\nabla u(x,t) = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2},$$

thus

$$\begin{aligned}\operatorname{div}(k(x)\nabla u) &= \operatorname{div}\left(k(x)\frac{\partial u}{\partial x_1},\cdots,k(x)\frac{\partial u}{\partial x_n}\right)\\ &= \frac{\partial}{\partial x_1}\left(\frac{k(x)\partial u}{\partial x_1}\right)+\cdots+\frac{\partial}{\partial x_n}\left(\frac{k(x)\partial u}{\partial x_n}\right)\end{aligned}$$

$$\begin{aligned}&= k(x)\frac{\partial^2 u}{\partial x_1^2}+\cdots+k(x)\frac{\partial^2 u}{\partial x_n^2}+\frac{\partial k(x)}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right)\\ &\quad+\cdots+\frac{\partial k(x)}{\partial x_n}\left(\frac{\partial u}{\partial x_n}\right)\\ &= k(x)\triangle u+\nabla k\nabla u.\end{aligned}$$

Now by multiplying this equality by φ we have

$$\varphi\operatorname{div}(k(x)\nabla u)=\varphi k\Delta u+\varphi\nabla k\nabla u,$$

then by integrating over D,

$$\int_D \varphi\operatorname{div}(k(x)\nabla u)dx-\int_D \varphi k\Delta u dx-\int_D \varphi\nabla k\nabla u dx=0.$$

By Green's theorem [10] for any two differentiable functions z, y on space Q we have

$$\int_Q y\Delta z dx=\int_{\partial Q} y\frac{\partial z}{\partial n}ds-\int_Q \nabla z\nabla y dx.$$

Now by substituting of φk and u respectively for y and z the second term of the above formula changes as follows:

$$\begin{aligned}\int_D \varphi k\Delta u dx &= \int_{\partial D}\varphi k\frac{\partial u}{\partial n}ds-\int_D \nabla\varphi k\nabla u dx\\ &= \int_{\partial D}\varphi k\frac{\partial u}{\partial n}ds-\int_D k\nabla\varphi\nabla u dx-\int_D \varphi\nabla k\nabla u dx.\end{aligned}$$

By using this equality, since $\varphi u_t=\varphi\operatorname{div}(k(x)\nabla u)$ so we have

$$\int_D \varphi u_t dx-\int_{\partial D}\varphi k\frac{\partial u}{\partial n}ds+\int_D k\nabla\varphi\nabla u dx=0.$$

Now by integrating of the above equality on $[0,T]$ we have

$$\int_0^T\int_D \varphi u_t dxdt-\int_0^T\int_{\partial D}\varphi k\frac{\partial u}{\partial n}dsdt+\int_0^T\int_D k\nabla\varphi\nabla u dxdt=0,$$

and by using part-part integration on the first term we obtain

$$\int_{D_T}\varphi u dx-\int_{D_0}\varphi u dx-\int_0^T\int_D \varphi_t u dxdt=\int_0^T\int_D \varphi u_t dxdt.$$

Since φu_t is integrable, thus by Fubini's theorem [15] and from two above equality we have

$$-\int_0^T\int_{\partial D}\varphi k\frac{\partial u}{\partial n}dsdt+\int_0^T\int_D(-u\varphi_t+k\nabla\varphi\nabla u)dxdt+\int_{D_T}\varphi udx=\int_{D_0}\varphi udx.$$

If u be a solution of (1)-(4), then by (8) we have

$$\int_0^T\int_D(-u\varphi_t+k\nabla\varphi\nabla u)dxdt+\int_{D_T}\varphi udx=\int_{D_0}\varphi udx,$$

for all φ's in $C^1(\overline{Q}_T)$. □

In the following we define the weak solution of (1)-(4).

Definition 2. The function $u(x,t)$ in $H^{2,1}(Q_T)$ is called a weak solution of (1)-(4), if this function satisfies in (3) and the equation (7) for all φ's in $H^1(Q_T)$ that satisfy in (8).

In general there exists a weak solution for the system (1)-(4) and if the weak solution of this system is in $C^{2,1}(Q_T)\bigcap C(Q_T\bigcup\Gamma_T\bigcup D_0)$, then this solution is a classic solution, [10].

Let $F\in C(\Omega)$ and $G\in C(\Omega)$, where $\Omega=Q_T\times A\times B\times K$ and $\omega=D_T\times V$. Now consider the following mappings,

$$\Lambda\ :\ F\ \longrightarrow\int_{Q_T}F(t,x,u,w,k)dxdt,$$

and

$$\Pi\ :\ G\ \longrightarrow\int_{D_T}G(x,v)dx,$$

where Λ and Π are positive, continuous and bounded respectively on $C(\Omega)$ and $C(\omega)$. By Riesz's theorem there exist the measures μ and λ such that

$$\Lambda(F)=\int_\Omega F(t,x,u,w,k)d\mu,$$

and

$$\Pi(G)=\int_\omega G(x,v)d\lambda.$$

Of course we can use the Riesz's theorem because Q_T and D_T are locally compact sets. In fact to each pair (u,k), we correspond a measure μ and to each control v correspond a measure λ. Now (7) changes to the following form:

$$\int_\Omega F_\varphi(t,x,u,w,k)d\mu+\int_\omega G_\varphi(x,v)d\lambda=c_\varphi, \tag{9}$$

where

$$F_{\varphi} = -u\varphi_t + k\nabla u\nabla\varphi, \quad (10)$$

$$G_{\varphi} = u|_{D_T}\varphi = v\varphi, \quad (11)$$

from (2) we have

$$c_{\varphi} = \int_{D_0} \varphi u dx,$$

where $F_{\varphi} \in C(\Omega)$ and $G_{\varphi} \in C(\omega)$. Using these concepts we can put our non-classical problem (9) with functional (5) in its definitive form. Thus, we seek measures μ and λ which minimizes the functional

$$I(\mu, \lambda) = \mu(f_{\circ}) + \lambda(g_{\circ}),\ (f_{\circ} \in C(\Omega), g_{\circ} \in C(\omega)), \quad (12)$$

subject to (by 9):

$$\mu(F_{\varphi}) + \lambda(G_{\varphi}) = c_{\varphi},\ \forall\varphi \in \Phi, \quad (13)$$

where from (10) and (11)

$$\mu(F_{\varphi}) = \int_{\Omega} F_{\varphi} d\mu,$$

$$\lambda(G_{\varphi}) = \int_{\omega} G_{\varphi} d\lambda.$$

So the problem of minimizing the functional (5) on Υ will convert to the problem of minimizing (12) by the pairs (μ, λ) such that these pairs satisfy in (13). We call the set of all positive Radon measures on Ω and ω by $\mathcal{M}^+(\Omega)$ and $\mathcal{M}^+(\omega)$ respectively. We choose (μ, λ) from $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$. Now consider the functions $\xi : \Omega \to R$ such that these functions are depend only on $(x, t) \in Q_T$. We have

$$\mu(\xi) = \int_{\Omega} \xi d\mu = a_{\xi}. \quad (14)$$

Similarly we consider the functions $\eta : \omega \to R$ depend only on $x \in D_T$ and so we have

$$\lambda(\eta) = \int_{\omega} \eta d\lambda = b_{\eta}. \quad (15)$$

Note that a_{ξ} and b_{η} are the Lebesgue integral of the functions ξ and η on Ω and ω respectively. Thus if 1_{Ω} and 1_{ω} are characteristic functions of Ω and ω and L_{Ω} and L_{ω} be the Lebesgue measures of D and D_T respectively then

$$\mu(1_{\Omega}) = TL_{\Omega}, \quad (16)$$

$$\lambda(1_{\omega}) = L_{\omega}. \quad (17)$$

3. The existence of approximate optimal measure

Let P be the subset of measures in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ satisfy the equalities (13)-(17). We intend to show there exists an optimal measure pair (μ, λ) in P such that this pair minimizes the functional (12). We assume P is nonempty.

To find an optimal measure pair we must use a convenient topological space for $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ such that P be a compact subset of this space. If we topologize the space $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ by the weak *-topology, we can say P is compact and much as that Theorem II.1 in [11] any continuous function gets its minimum on a compact subset of a Hausdorff space.

Theorem 2. *The set P, that is the set of measure pairs in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ that satisfy in*

$$\begin{aligned} \mu(F_\varphi) + \lambda(G_\varphi) = c_\varphi, \ \forall \varphi \in \Phi, \\ \mu(\xi) = \int_\Omega \xi d\mu = a_\xi, \\ \lambda(\eta) = \int_\omega \eta d\lambda = b_\eta \end{aligned} \tag{18}$$

for all ξ's and η's that satisfy in (14) and (15) respectively, is compact respect to weak *-topology on $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$.

Proof. The proof is similar to the proof of Proposition 4.4 in [1]. □

The functional $(\mu, \lambda) \to \mu(f_\circ) + \lambda(g_\circ)$ is continuous (see [1]) and thus we have the following theorem:

Theorem 3. *There exists an optimal measure pair, (μ^*, λ^*) in P such that for any pair, (μ, λ) in P*

$$I(\mu^*, \lambda^*) = \mu^*(f_\circ) + \lambda^*(g_\circ) \leq \mu(f_\circ) + \lambda(g_\circ) = I(\mu, \lambda),$$

thus the functional I achieves a minimum on P.

The problem (12)-(13) is an infinite dimensional linear form, the underlying space $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ is not finite dimensional and the number of equations in (13) is not finite. In following we intend to find a way for converting this problem to a finite dimensional problem.

Proposition 1. *Let $\mathcal{P} \subset P$ be the set of measure pairs (μ, λ) in P correspond to triples (u, k, v) of piecewise constant functions on Q_T and D_T that satisfy in (18) then $\mathcal{P}$ is dense in P respect to weak *-topology.*

Proof. The proof is like as the Proposition in appendix of [7]. □

Definition 3. We call the functions $\varphi_i \in C^1(\overline{Q}_T),\ i = 1, 2, \cdots,$ *total* if for each $\varphi \in C^1(\overline{Q}_T)$ and for given $\epsilon \geq 0$, there exists a positive integer N and real numbers as $\gamma_i,\ i = 1, 2, \cdots, N$ such that

$$\max_{Q_T} |\varphi - \sum_{i=1}^{N} \gamma_i \varphi_i| < \epsilon,$$

$$\max_{Q_T} |\varphi_t - \sum_{i=1}^{N} \gamma_i \varphi_{it}| < \epsilon,$$

$$\max_{Q_T} \|\nabla\varphi - \sum_{i=1}^{N} \gamma_i \nabla\varphi_i\| < \epsilon.$$

Now by (10) and (11) we define

$$\forall\ i\ F_{\varphi_i} = F_i,\ G_{\varphi_i} = G_i,\ c_{\varphi_i} = c_i.$$

Furthermore we consider a different form of total functions in $C(Q_T)$ and $C(D_T)$ respectively corresponding to the functions in (14) and (15) as follows

$$\{\xi_i,\ i = 1, 2, \cdots\},\ \{\eta_i,\ i = 1, 2, \cdots\},$$

respectively, such that Lebesgue integral of them on Ω and ω are a_i and b_i for a_{ξ_i} and b_{η_i}. Now consider the following theorem that its proof is like as the Theorem 3 of [13].

Theorem 4. *Let M_1, M_2 and M_3 are positive integers. Now we consider the problem of minimizing the functional*

$$I\ :\ (\mu, \lambda)\ \to\ \mu(f_\circ) + \lambda(g_\circ), \tag{19}$$

on the set $P(M_1, M_2, M_3) \subset P$ of measures in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ that satisfy in

$$\begin{aligned} \mu(F_i) + \lambda(G_i) = c_i,\ i = 1, 2, \cdots, M_1, \\ \mu(\xi_j) = a_j,\ j = 1, 2, \cdots, M_2, \\ \lambda(\eta_k) = b_k,\ k = 1, 2, \cdots, M_3; \\ \mu(1_\Omega) = TL_\Omega \textit{ and } \lambda(1_\omega) = L_\omega, \end{aligned} \tag{20}$$

then as $M_1,\ M_2,\ M_3 \to \infty$

$$inf_{P(M_1, M_2, M_3)} [\mu(f_\circ) + \lambda(g_\circ)]\ \to\ inf_{P} [\mu(f_\circ) + \lambda(g_\circ)].$$

One can show that (see [11]),

$$\inf_P I\ \leq\ \inf_\Upsilon J.$$

We can proceed now the construction of suboptimal triples of trajectory and controls for functional (5). In the first step we obtain the optimal pairs (μ^*, λ^*) in P that its existence is shown in Theorem 3. For this purpose, we consider the pairs (μ, λ) in P correspond to triples (u, k, v) of piecewise constant functions on Q_T and D_T that satisfy in (18) which we called the set of all these pairs $\mathcal{P}$. By Proposition 1, $\mathcal{P}$ is dense in P, thus we apply Theorem 4 for members of $\mathcal{P} \cap P(M_1, M_2, M_3)$. Now by optimal measure obtained from (19)-(20), we find a triple (u, k, v) of piecewise constant functions. The obtained function v belongs to $L^2(D_T)$, because D_T is bounded and v is piecewise constant. By a similar reason the piecewise constant function k belongs to $L^2(Q_T)$. We call the function u corresponding to k, v in any triple by u_v^k. Now by the weak solution of (1)-(4) and Definition 2, since the function u_v^k belongs to $H^1(Q_T)$, so is a weak solution for (1)-(4) as well. In [10] is shown this weak solution exists. Someone can see a same framework in [2] and [13], by borrowing a term from [14], we call the triple (u_v^k, k, v) of trajectory and control functions *asymptotically admissible* if:

i) the control functions

$$v(\cdot) \in L^2(D_T),\ v(x) \in V$$

and

$$k(\cdot) \in L^2(Q_T),\ k(x) \in K.$$

ii) trajectory function u_v^k is the weak solution of (1)-(4) corresponding to the control functions $k(\cdot)$ and $v(\cdot)$ and satisfies in (7).

Finally in a theorem we will show if the numbers M_1, M_2 and M_3, that are introduced in Theorem 4, are sufficiently large and the approximate optimal measure pair, that is obtained by above manner, be sufficiently good then the value of $J(u_v^k, k, v)$, the value of functional J in (5) by (u_v^k, k, v), is close to $\inf_P I$.

Note that we do not need to obtain the trajectory function which is made by the control functions $k(\cdot)$ and $v(\cdot)$.

Theorem 5. *Let (u_v^k, k, v) be the triple of controls and trajectory that is obtained in above discussion then*

i) *The triple is asymptotically admissible*

ii) *As $M_1,\ M_2,\ M_3 \to \infty$ then*

$$J(u_v^k, k, v) \to inf_P I.$$

Proof. Firstly we call the pairs which is considered in P corresponding to the triples (u, k, v) of piecewise constant functions by (μ_u^k, λ_v). In fact $\mathcal{P}$ is the set of all these pairs. By Proposition 1, $\mathcal{P}$ is compact in P. Now if we assume that the pair (μ^*, λ^*) be the solution of problem (19)-(20) then in each neighborhoods of (μ^*, λ^*) there exists a member (μ_u^k, λ_v) such that

$$|I(\mu_u^k, \lambda_v) - I(\mu^*, \lambda^*)| < \frac{\epsilon}{2}.$$

Now by the definition I in (12) we have

$$|\{\mu_u^k(f_\circ) + \lambda_v(g_\circ)\} - \{\mu^*(f_\circ) + \lambda^*(g_\circ)\}| < \frac{\epsilon}{2}, \tag{21}$$

and

$$|\{\mu_u^k(F_i) + \lambda_v(G_i)\} - \{\mu^*(F_i) + \lambda^*(G_i)\}| < \frac{\epsilon}{2},\ i = 1, 2, \cdots, M_1.$$

Thus by (20) we can write

$$|\{\mu_u^k(F_i) + \lambda_v(G_i)\} - c_i| < \frac{\epsilon}{2},\ i = 1, 2, \cdots, M_1. \tag{22}$$

We do not need to prove (i), since by previous discussion the triple of (u_v^k, k, v) is asymptotically admissible.

To prove (ii), for given $\epsilon = \frac{1}{M_1}$, we need to show by choosing M_1, M_2 and M_3, sufficiently large, then

$$|J(u_v^k, k, v) - \{\mu^*(f_\circ) + \lambda^*(g_\circ)\}| < \epsilon, \tag{23}$$

where (μ^*, λ^*) is the optimal solution of problem (19)-(20). Now by above inequality we have

$$\begin{aligned} |J(u_v^k, k, v) - \{\mu^*(f_\circ) + \lambda^*(g_\circ)\}| \ &< \ |J(u_v^k, k, v) - (\mu_u^k(f_\circ) + \lambda_v(g_\circ))| \\ &\quad + |(\mu_u^k(f_\circ) + \lambda_v(g_\circ)) - (\mu^*(f_\circ) + \lambda^*(g_\circ))|. \end{aligned} \tag{24}$$

Now by (21) is enough to show

$$|J(u_v^k, k, v) - (\mu_u^k(f_\circ) + \lambda_v(g_\circ))| < \frac{\epsilon}{2}, \tag{25}$$

where from (5),

$$J(u_v^k, k, v) = \int_{Q_T} f_\circ(t, x, u_v^k, \nabla u_v^k, k)dxdt + \int_{D_T} g_\circ(x, v)dx.$$

Now by definition of functionals, Λ and Π and the measures μ and λ we suppose that $(\mu^k_{u_v^k}, \lambda_v)$ is corresponding to triple of (u_v^k, k, v), then since,

$$\lambda_v(g_\circ) = \int_{D_T} g_\circ(x, v)dx,$$

we have

$$J(u_v^k, k, v) - (\mu_u^k(f_\circ) + \lambda_v(g_\circ)) = \mu^k_{u_v^k}(f_\circ) - \mu_u^k(f_\circ),$$

thus is enough to show

$$|\mu^k_{u_v^k}(f_\circ) - \mu_u^k(f_\circ)| < \frac{\epsilon}{2},$$

but the function $f_\circ \in C(Q_T)$ satisfies in $f_\circ(t,x,u,\nabla u,k) \leq h|u|$, thus

$$\left|(\mu^k_{u^k_v} - \mu^k_u)(f_\circ)\right| \leq h\left|\mu^k_{u^k_v} - \mu^k_u\right|(\vartheta). \tag{26}$$

We suppose that $\vartheta = u$ and without loss of the generality assume $u > 0$. Now we consider a set of total functions in $C^1(Q_T)$ as $\varphi_i,\ i = 1,2,\cdots,N$ such that for $\epsilon' > 0$ we have

$$\left\|\nabla\varphi - \sum_{i=1}^{N}\gamma_i\nabla\varphi_i\right\| < \epsilon', \tag{27}$$

$$\left|\varphi_t - \sum_{i=1}^{N}\gamma_i\varphi_{it}\right| < \epsilon'. \tag{28}$$

By multiplying relation (27) in $\|\nabla u\|$ we have

$$-\epsilon'\|\nabla u\| \leq \|\nabla u\|\left\|\nabla\varphi - \sum_{i=1}^{N}\gamma_i\nabla\varphi_i\right\| \leq \epsilon'\|\nabla u\|.$$

For each two vectors a and b we have

$$-\|a\|\|b\| \leq a.b \leq \|a\|\|b\|,$$

thus

$$-\|\nabla u\|\|\nabla\varphi - \sum_{i=1}^{N}\gamma_i\nabla\varphi_i\| \leq \nabla u.(\nabla\varphi - \sum_{i=1}^{N}\gamma_i\nabla\varphi_i) \leq \|\nabla u\|\left\|\nabla\varphi - \sum_{i=1}^{N}\gamma_i\nabla\varphi_i\right\|,$$

and by above inequalities we can write

$$-\epsilon'\|\nabla u\| \leq \nabla u.\left(\nabla\varphi - \sum_{i=1}^{N}\gamma_i\nabla\varphi_i\right) \leq \epsilon'\|\nabla u\|.$$

Multiplying above inequality by $k(\cdot)$ and add $\nabla u.\sum_{i=1}^{N}\gamma_i\nabla\varphi_i$, we have

$$-\epsilon' k\|\nabla u\| + k\nabla u.\sum_{i=1}^{N}\gamma_i\nabla\varphi_i \leq k\nabla u.\nabla\varphi \leq \epsilon' k\|\nabla u\| + k\nabla u.\sum_{i=1}^{N}\gamma_i\nabla\varphi_i. \tag{29}$$

From (28)

$$-\epsilon' + \sum_{i=1}^{N}\gamma_i\varphi_{it} \leq \varphi_t \leq \sum_{i=1}^{N}\gamma_i\varphi_{it} + \epsilon',$$

and multiply it by $-u(\cdot)$ we find ,

$$-u\epsilon' - \sum_{i=1}^{N}\gamma_i\varphi_{it} \leq -u\varphi_t \leq -\sum_{i=1}^{N}\gamma_i\varphi_{it} + \epsilon'. \tag{30}$$

Now from (29) and (30) we have

$$\begin{aligned}&-\epsilon' k\|\nabla u\| - u\epsilon' + \sum_{i=1}^{N} \gamma_i(-u\varphi_{it} + k\nabla u \nabla \varphi_i)\\ &\leq -u\varphi_t + k\nabla u\nabla\varphi\\ &\leq \epsilon' k\|\nabla u\| + u\epsilon' + \sum_{i=1}^{N} \gamma_i(-u\varphi_{it} + k\nabla u \nabla \varphi_i).\end{aligned}$$

Assume $F_\varphi = -u\varphi_t + k\nabla u\nabla\varphi$ and

$$F_i = F_{\varphi_i} = -u\varphi_{it} + k\nabla u\nabla\varphi_i,$$

so we can write

$$-\epsilon' k\|\nabla u\| - u\epsilon' + \sum_{i=1}^{N} \gamma_i F_i \leq F_\varphi \leq \epsilon' k\|\nabla u\| + u\epsilon' + \sum_{i=1}^{N} \gamma_i F_i. \tag{31}$$

But

$$\mu_u^k(F_{\varphi_i}) + \lambda_v(G_{\varphi_i}) = \int_{D_0} \rho\varphi_i dx,$$

and

$$\mu_{u_v^k}^k(F_{\varphi_i}) + \lambda_v(G_{\varphi_i}) = \int_{D_0} \rho\varphi_i dx,$$

thus

$$\mu_{u_v^k}^k(F_{\varphi_i}) - \mu_u^k(F_{\varphi_i}) = 0.$$

By linearity of $\mu_{u_v^k}^k$ and μ_u^k we obtain

$$\begin{aligned}-\mu_u^k(\epsilon' k\|\nabla u\| + u\epsilon') + \sum_{i=1}^{N}\gamma_i\mu_u^k(F_i) \;&\leq\; \mu_u^k(F_\varphi)\\ &\leq\; \mu_u^k(\epsilon' k\|\nabla u\| + u\epsilon') + \sum_{i=1}^{N}\gamma_i\mu_u^k(F_i),\end{aligned}$$

and

$$\begin{aligned}-\mu_{u_v^k}^k(\epsilon' k\|\nabla u\| + u\epsilon') + \sum_{i=1}^{N}\gamma_i\mu_{u_v^k}^k(F_i) \;&\leq\; \mu_{u_v^k}^k(F_\varphi)\\ &\leq\; \mu_{u_v^k}^k(\epsilon' k\|\nabla u\| + u\epsilon') + \sum_{i=1}^{N}\gamma_i\mu_{u_v^k}^k(F_i),\end{aligned}$$

From above two inequality,

$$-\mu^k_{u^k_v}(\epsilon' k\|\nabla u\| + u\epsilon') - \mu^k_u(\epsilon' k\|\nabla u\| + u\epsilon') + \sum_{i=1}^{N}\gamma_i(\mu^k_{u^k_v}(F_i) - \mu^k_u(F_i)) \leq \mu^k_{u^k_v}(F_\varphi) - \mu^k_u(F_\varphi),$$

and

$$\mu^k_{u^k_v}(F_\varphi) - \mu^k_u(F_\varphi) \leq \sum_{i=1}^{N}\gamma_i(\mu^k_{u^k_v}(F_i) - \mu^k_u(F_i)) + \mu^k_{u^k_v}(\epsilon' k\|\nabla u\| + u\epsilon') + \mu^k_u(\epsilon' k\|\nabla u\| + u\epsilon'),$$

since $(\mu^k_{u^k_v} - \mu^k_u)(F_i) = 0$, so

$$\begin{aligned}-(\mu^k_{u^k_v} + \mu^k_u)(\epsilon' k\|\nabla u\| + u\epsilon') &\leq \mu^k_{u^k_v}(F_\varphi) - \mu^k_u(F_\varphi)\\ &\leq (\mu^k_{u^k_v} + \mu^k_u)(\epsilon' k\|\nabla u\| + u\epsilon'),\end{aligned}$$

thus

$$\begin{aligned}|(\mu^k_{u^k_v} - \mu^k_u)(F_\varphi)| &\leq (\mu^k_{u^k_v} + \mu^k_u)(\epsilon' k\|\nabla u\| + u\epsilon')\\ &\leq (\mu^k_{u^k_v} + \mu^k_u)(\epsilon' k\text{meas}B + \epsilon'\text{meas}A)\\ &\leq 4L_{Q_T}(k\text{meas}B + \text{meas}A)\epsilon',\end{aligned}$$

where A and B are the bounded subset of R and R^n that are contain of $u(x,t)$ and $\nabla u(x,t)$ respectively and L_{Q_T} is the Lebesgue measure on the space of Ω. Now we choose the φ's functions such that $|\vartheta(1-\varphi_t)| < \epsilon'$ and $|k\nabla u\nabla\varphi| < \epsilon'$ then by linearity of measures and

$$\vartheta = \vartheta(1-\varphi_t) + k\nabla u\nabla\varphi - (-\vartheta\varphi_t + k\nabla u\nabla\varphi),$$

we have

$$\begin{aligned}|\mu^k_{u^k_v} - \mu^k_u|(\vartheta) &\leq |\mu^k_{u^k_v} - \mu^k_u|(|\vartheta(1-\varphi_t) + k\nabla u\nabla\varphi|)\\ &\quad + |\mu^k_{u^k_v} - \mu^k_u|(| - \vartheta\varphi_t + k\nabla u\nabla\varphi|).\end{aligned} \tag{32}$$

On the other hand it is easily to show that

$$|\mu^k_{u^k_v} - \mu^k_u|(|\vartheta(1-\varphi_t)|) \leq (\mu^k_{u^k_v} + \mu^k_u)\epsilon',$$

and

$$|\mu^k_{u^k_v} - \mu^k_u|(|k\nabla u\nabla\varphi|) \leq (\mu^k_{u^k_v} + \mu^k_u)\epsilon',$$

thus

$$|\mu^k_{u^k_v} - \mu^k_u|(|\vartheta(1-\varphi_t) + k\nabla u\nabla\varphi|) \leq 2(\mu^k_{u^k_v} + \mu^k_u)\epsilon' \leq 4L_{Q_T}\epsilon'.$$

Now by (32) and above mentioned inequalities we have

$$|\mu^k_{u^k_v} - \mu^k_u|(\vartheta) \leq 4L_{Q_T}\epsilon' + 4L_{Q_T}(k\text{meas}B + \text{meas}A)\epsilon',$$

where by choosing

$$\epsilon' = \min\left\{\frac{\epsilon}{16hL_{Q_T}}, \frac{\epsilon}{16hL_{Q_T}(k\text{meas}B + \text{meas}A)}\right\},$$

we have

$$|\mu^k_{u^k_v} - \mu^k_u|(\vartheta) \le \frac{\epsilon}{2h},$$

and by (26)

$$|(\mu^k_{u^k_v} - \mu^k_u)(f_\circ)| \le \frac{\epsilon}{2},$$

and the proof is completed. □

4. The approximate optimal pair measures

Let (μ^*, λ^*) be the optimal pair measures that is obtained by solving the linear programming problem (19)-(20). Now by unitary atomic measures we can write μ^* and λ^* as a finite linear combination of unitary atomic measures as follows

$$\mu^* = \sum_{m=1}^{M} \alpha^*_m \delta(Z^*_m),$$

$$\lambda^* = \sum_{n=1}^{N} \beta^*_n \delta(z^*_n),$$

where $\alpha^*_m \ge 0,\ m = 1, 2, \cdots, M$ and $\beta^*_n \ge 0,\ n = 1, 2, \cdots, N$, $Z^*_m \in \Omega$, $z^*_n \in \omega$ for any m, n and $\delta(Z)$, $\delta(z)$ are unitary atomic measures respectively supported by Z and z.

Now by using Proposition III.3 of [11], by considering dense sets as $\Omega_1 \subset \Omega$ and $\omega_1 \subset \omega$ and by choosing $Z^*_m \in \Omega_1,\ m = 1, 2, \cdots, M$, and $z^*_n \in \omega,\ n = 1, 2, \cdots, N$, the optimal pair measures (μ^*, λ^*) that is obtained from problem of (19)-(20) can be approximate by pair measures (μ, λ) where

$$\mu = \sum_{m=1}^{M} \alpha_m \delta(Z_M),\ \lambda = \sum_{n=1}^{N} \beta_n \delta(z_n), \tag{33}$$

and $\alpha_m \ge 0,\ m = 1, 2, \cdots, M$, and $\beta_n \ge 0,\ n = 1, 2, \cdots, N$, will obtain by solving a linear programming problem as follows

$$\text{Minimize} \sum_{m=1}^{M} \alpha_m f_\circ(Z_m) + \sum_{n=1}^{N} \beta_n g_\circ(z_n), \tag{34}$$

subject to

$$\sum_{m=1}^{M} \alpha_m F_i(Z_m) + \sum_{n=1}^{N} \beta_n G_i(z_n) = c_i,\ i = 1, 2, \cdots, M_1,$$

$$\sum_{m=1}^{M} \alpha_m \xi_j(Z_m) = a_j,\ j = 1, 2, \cdots, M_2,$$

$$\sum_{n=1}^{N} \beta_n \eta_k(z_n) = b_k,\ k = 1, 2, \cdots, M_3; \tag{35}$$

$$\sum_{m=1}^{M} \alpha_m = 1_\Omega$$

$$\sum_{n=1}^{N} \beta_n = 1_\omega$$

$$\alpha_m \geq 0,\ m = 1, 2, \cdots, M,\ \beta_n \geq 0,\ n = 1, 2, \cdots, N,$$

where $1_\Omega = TL_\Omega$ and $1_\omega = L_\omega$, (L_Ω and L_ω are defined already). For obtaining Z_m's and z_n's that are dense in Ω and ω we divided the sets of K, B, A, D and $(0, T)$ respectively to m_1, m_2, m_3, m_4 and m_5 subrectangulars and so we have $M = m_1 m_2 m_3 m_4 m_5$ subrectangulars of Ω as $\Omega_m,\ m = 1, 2, \cdots, M$. We choose from each Ω_m a member as $Z_m = (t_m, x_m, u_m, w_m, k_m)$. In a similar framework we obtain z_n's by dividing D_T and V to n_1 and n_2 subrectangulars and we will have $N = n_1 n_2$ subrectangular as ω_n $n = 1, 2, \cdots, N$, and we choose $z_n = (x_n, v_n)$ from each ω_n.

5. Numerical example

In this section we apply the mentioned method for finding a control function $k(\cdot)$ for an optimal control problem that is to minimize a certain given functional of $u(x, t)$ at time $t = T$ as

$$\int_0^1 u^2(x, T)dx, \tag{36}$$

on a system governed by following parabolic equation

$$u_t = (ku_x)_x, \tag{37}$$

where suitable initial and boundary conditions are as follows

$$u(x, 0) = \rho(x), \tag{38}$$

$$u_x(0,t) = 0, \tag{39a}$$

$$u(1,t) = 0, \tag{39b}$$

this problem is considered in [5] as well and has a physical motivation as below.

Suppose an inhomogeneous rod of length 1 is to be constructed with some specification for the conductivity coefficient $k(x)$, $(0 \leq x \leq 1)$. We wish to design the rod in such a way that with prescribed initial temperature (38) and boundary conditions (39) the rod cools off as much as possible after T units of time, The cooling off is measured by (36). Furthermore $k \in \mathcal{K}$ where

$$\mathcal{K} = \left\{ k \ \mid \int_0^1 k(x)dx = \rho,\ \sigma \leq k(\cdot) \leq \tau, k(\cdot) \text{ is measurable} \right\}.$$

We convert this problem as in Sections 1-4. For this means, observe that the parabolic system (36)-(39) is equivalent to the problem

$$\begin{aligned} u_t &= (k(x)u_x)_x \ (x,t) \in (-1,1) \times (0,T), \\ u(x,0) &= \rho(x),\ -1 < x < 1, \\ u(\pm 1, t) &= 0,\ 0 < t < T, \end{aligned}$$

provided $\rho(x) = \rho(-x)$ and $k(-x) = k(x)$. We set $u(x,T) = v(x)$ and suppose that $v(x)$ is a control function and thus our control problem is to minimize

$$\int_{-1}^{1} v^2(x)dx,$$

by the following constraints

$$\begin{aligned} u_t &= (k(x)u_x)_x \ (x,t) \in (-1,1) \times (0,T), \\ u(x,0) &= \rho(x),\ -1 < x < 1, \\ u(x,T) &= v(x),\ -1 < x < 1, \\ u(\pm 1, t) &= 0,\ 0 < t < T, \end{aligned}$$

this form of problem is same as problem (1)-(4) by minimization (5) that is defined in Section.1, where we have

$$D = (-1,1),\ Q_T = (-1,1) \times (0,T),\ f_\circ(t,x,u,w,k) = 0$$

and $g_\circ(x,v) = v^2$. For our numerical example we considered $T = 1$, $\rho(x) = \cos(\frac{\pi x}{2})$, $A = [0,1]$, $B = [0,1]$ and $V = [0,1]$. By assuming $\sigma = 0.05$ and $\tau = 0.85$ and by even property of $k(\cdot)$ we have

$$\mathcal{K} = \left\{ k \ \mid \int_0^1 k(x)dx = 2\rho,\ 2\sigma \leq k(\cdot) \leq 2\tau,\ k(\cdot) \text{ is measurable} \right\},$$

thus we have $K = [0.1, 1.7]$. We divide the intervals $(0, T) = (0, 1)$, $D = (-1, 1)$, A, K and B to 5 equal subintervals, thus we have $M = 3125$. As above we divide $D_T = (-1, 1)$ and V to 25 equal subintervals, thus $N = 625$. We define the functions φ that are defined in Section.2 as

$$\varphi(x, t) = (t + 0.1)^p \sin(l\pi x).$$

Note that φ's satisfy in (8), i.e. $\varphi(\pm 1, t) = 0$. We consider four various form of this functions for $p = 1, 2$ and $l = 1, 2$ thus in the problem (35) we have $M_1 = 4$ and by (12) and (13)

$$G_i = G_{\varphi_i} = v\varphi_i,$$
$$F_i = F_{\varphi_i} = -u\varphi_{it} + k\nabla u\nabla\varphi_i = -u\varphi_{it} + kw\nabla\varphi_i,$$

and by definition φ in above when $p = l = 1$ and by $i = 1$ we have

$$F_1(Z_j) = F_1((t_j, x_j, u_j, w_j, k_j) = -u_j \sin(\pi x_j) + \pi k_j w_j (t_j + 0.1)\cos(\pi x_j),$$
$$G_1(z_j) = G_1(x_j, v_j) = 1.1 v_j \sin(\pi x_j),$$

as above we define $i = 2$ with $p = 2,\ l = 1$, $i = 3$ with $p = 1,\ l = 2$, $i = 4$ with $p = 2,\ l = 2$. We consider the functions ξ's and η's as functions that are depend only to $(x, t) \in Q_T$ and $x \in D_T$ respectively. Note that where $k(\cdot) \in K$ we have

$$\int_{-1}^{1} k(x)dx = 2\rho.$$

Thus by choosing $\rho = \frac{1}{2}$ we can write

$$\int_0^1 \int_{-1}^{1} k(x)dxdt = 1,$$

thus

$$\int_{Q_T} k(x)dxdt = 1$$

and we shall consider an additional constraint for problem by setting

$$H(t, x, u, w, k) = k(x),$$

where $H \in C(Q_T)$. Thus we have the following linear programming problem

$$\text{Minimize} \sum_{n=1}^{625} \beta_n v_n^2,$$

subject to

$$\sum_{m=1}^{3125} \alpha_m F_i(Z_m) + \sum_{n=1}^{625} \beta_n G_i(z_n) = c_i,\ i = 1, 2, 3, \cdots, M_1,$$
$$\sum_{m=1}^{3125} \alpha_m \xi_j(Z_m) = a_j,\ j = 1, 2, \cdots, M_2,$$

$$\sum_{n=1}^{625} \beta_n \eta_k(z_n) = b_k,\ k = 1, 2, \cdots, M_3,$$
$$\sum_{m=1}^{3125} \alpha_m = 2, \qquad \sum_{n=1}^{625} \beta_n = 1,$$
$$\sum_{m=1}^{3125} \alpha_m k_m = 1,$$
$$\alpha_m \geq 0,\ m = 1, 2, \cdots, 3125,\ \beta_n \geq 0,\ n = 1, 2, \cdots, 625,$$

by solving this linear programming problem we construct the optimal control problem by the method that is proposed in [9, Sec.5]. The value of cost function is 0.0032 and the optimal control function that is a manner for designing rod is shown in the Fig.1.
Experimentally, the best design is to take the conductivity coefficient "as large as possible" near the end points and "as small as possible" near the center.

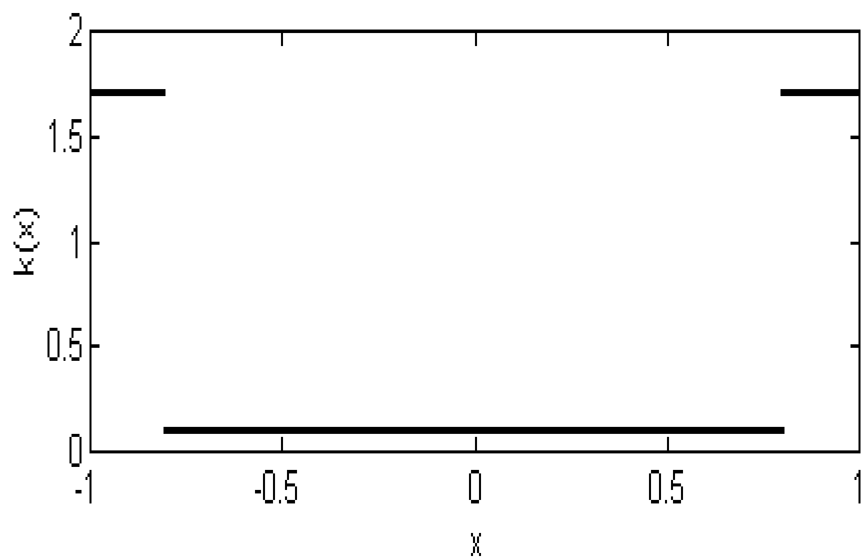

Figure 1. The approximate control function $k(\cdot)$

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