J. Appl. Math. & Computing Vol. 15(2004), No. 1 - 2, pp. 91 - 107

TWO NEW OPERATORS ON FUZZY MATRICES

AMIYA K. SHYAMAL AND MADHUMANGAL PAL

Abstract. The fuzzy matrices are successfully used when fuzzy uncertainty occurs in a problem. Fuzzy matrices become popular for last two decades. In this paper, two new binary fuzzy operators $\oplus$ and $\odot$ are introduced for fuzzy matrices. Several properties on $\oplus$ and $\odot$ are presented here. Also, some results on existing operators along with these new operators are presented.

AMS Mathematics Subject Classification : 03E73, 04A72.
Key words and phrases : Fuzzy set, fuzzy matrix, fuzzy operators.

1. Introduction

It is well known, that matrices play major role in various areas such as mathematics, physics, statistics, engineering, social sciences and many others. Several works on classical matrices are available in different journals even in books also. But in daily life situations, the problems in economics, engineering, environment, social science, medical science etc. do not always involve crisp data. Consequently, we can not successfully use traditional classical matrices because of various types of uncertainties present in daily life problems. Now a days probability, fuzzy sets, intuitionistic fuzzy sets, vague sets, rough sets are used as mathematical tools for dealing uncertainties. Fuzzy matrices arise in many applications, one of which is as adjacency matrices of fuzzy relations and fuzzy relational equations have important applications in pattern classification and in handing fuzziness in knowledge based systems [11]. Several authors presented a number of results on fuzzy matrices. Hashimoto [1] introduced the concept of fuzzy matrices and studied the canonical form of a transitive matrix. Kim et. al. [4] studied the canonical form of an idempotent matrix. Kolodziejczyk [5]

Received April 5, 2003. Revised December 10, 2003.

presented the canonical form of a strongly transitive matrix. Xin [14,15] studied controllable fuzzy matrices. Hemasinha et. al. [2] investigated iterations of fuzzy circulants matrices. Ragab et. al. [10] presented some properties of the min-max composition of fuzzy matrices. Thomason [12] and Kim [6] defined the adjoint of square fuzzy matrix. Tian et. al. [13] studied power sequence of a fuzzy matrices. Kim et. al. [7] studied determinant of square fuzzy matrices. Ragab et. al. [9] presented some properties on determinant and adjoint of a square fuzzy matrix. Pal [8] defined intuitionistic fuzzy determinant. Khan, Shyamal and Pal [3] introduced intuitionistic fuzzy matrices.

In this paper, we introduce two new binary operators on fuzzy matrices which are denoted by the symbol $\oplus$ and $\odot$. Also, some properties of the fuzzy matrices over these new operators and some pre-defined operators are presented.

2. Definitions

We define some operators on fuzzy matrices whose elements are confined in the closed interval [0,1]. For all $x, y, \alpha \in [0,1]$ the following operators are defined.

(i) $x \vee y = \max\{x, y\}$
(ii) $x \wedge y = \min\{x, y\}$
(iii) $x \ominus y = \begin{cases} x, & \text{if } x > y \\ 0, & \text{if } x \leq y \end{cases}$
(iv) $x^{(\alpha)}$(upper α -cut)$=\begin{cases} 1, & \text{if } x \geq \alpha \\ 0, & \text{if } x < \alpha \end{cases}$
(v) $x_{(\alpha)}$(lower α -cut)$=\begin{cases} x, & \text{if } x \geq \alpha \\ 0, & \text{if } x < \alpha \end{cases}$
(vi) x^c (complement)$=1 - x$.

Now, we define two new operators $\oplus$ and $\odot$ as follows:

(vii) $x \oplus y = x + y - x.y$ and
(viii) $x \odot y = x.y$,

where the operations '+', '–' and '.' are ordinary addition, subtraction and multiplication respectively.

It may be noted that the values of $x \vee y$, $x \wedge y$, $x \ominus y$, $x \oplus y$, $x \odot y$, $x^{(\alpha)}$, $x_{(\alpha)}$ and x^c belong to the closed interval [0, 1].

It is obvious that (i) $1 \oplus x = 1$, (ii) $1 \odot x = x$, (iii) $0 \oplus x = x$ and (iv) $0 \odot x = 0$.

Now, we define some operations on fuzzy matrices.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two fuzzy matrices of order $m \times n$. Then

(i) $A \oplus B = [a_{ij} + b_{ij} - a_{ij}.b_{ij}]$
(ii) $A \odot B = [a_{ij}.b_{ij}]$
(iii) $A \vee B = [a_{ij} \vee b_{ij}]$

(iv) $A \wedge B = [a_{ij} \wedge b_{ij}]$
(v) $A \ominus B = [a_{ij} \ominus b_{ij}]$
(vi) $A^{[k+1]} = A^{[k]} \odot A$, $A^{[1]} = A$, $k = 1, 2, \dots$
(vii) $[k+1]A = [k]A \oplus A, [1]A = A$, $k = 1, 2, \dots$
(viii) $A^{(\alpha)} = [a_{ij}^{(\alpha)}]$(upper α -cut fuzzy matrix)
(ix) $A_{(\alpha)} = [a_{ij(\alpha)}]$ (lower α -cut fuzzy matrix)
(x) $A' = [a_{ji}]$ (the transpose of A)
(xi) $A^c = [1 - a_{ij}]$ (the complement of A)
(xii) $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all i, j.
(xiii) For any two matrices A and B, $\min\{A, B\} = A \wedge B$.

Every fuzzy matrix can be visualized as a three dimensional figure. But, this representation is not possible for classical matrix without any proper scaling. To illustrate this fact, we consider two fuzzy matrices A and B as follows:

$$A = \begin{bmatrix} 0.5 & 0.1 & 0.7 & 0.5 \\ 0.3 & 0.8 & 0.1 & 0.6 \\ 0.6 & 0.4 & 0.9 & 0.8 \\ 0.2 & 0.7 & 0.3 & 0.4 \end{bmatrix}$$

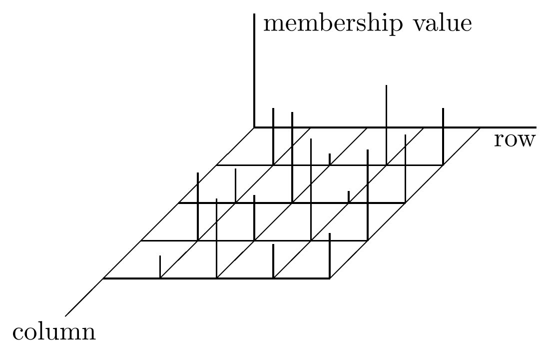

Figure 1. Geometrical representation of the matrix A

$$B = \begin{bmatrix} 0.3 & 0.8 & 0.6 & 0.4 \\ 0.2 & 0.7 & 0.5 & 0.3 \\ 0.7 & 0.5 & 0.3 & 0.9 \\ 0.9 & 0.4 & 0.1 & 0.6 \end{bmatrix}$$

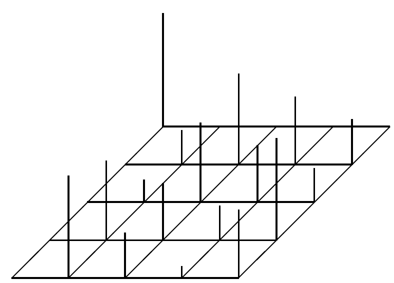

Figure 2. Geometrical representation of the matrix B

The 3D representation of the matrices A and B are shown in Figures 1 and 2. The 3D representation of the matrices $A \oplus B$, $A \ominus B$, $A^{(\alpha)}$, A^c and $A \vee B$ are shown in Figures 3-7.

$$A \oplus B = \begin{bmatrix} 0.65 & 0.82 & 0.88 & 0.70 \\ 0.44 & 0.94 & 0.55 & 0.72 \\ 0.88 & 0.70 & 0.93 & 0.98 \\ 0.92 & 0.82 & 0.37 & 0.76 \end{bmatrix}$$

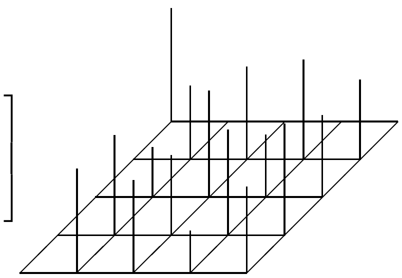

Figure 3. Representation of the matrix $A \oplus B$

$$A \ominus B = \begin{bmatrix} 0.5 & 0.0 & 0.7 & 0.5 \\ 0.3 & 0.8 & 0.0 & 0.6 \\ 0.0 & 0.0 & 0.9 & 0.0 \\ 0.0 & 0.7 & 0.3 & 0.0 \end{bmatrix}$$

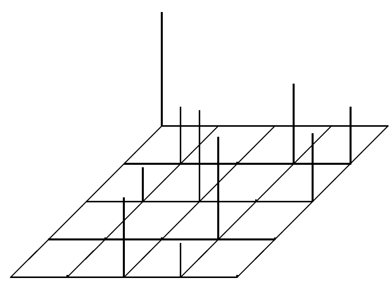

Figure 4. Representation of the matrix $A \ominus B$

$$A^{(0.4)} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

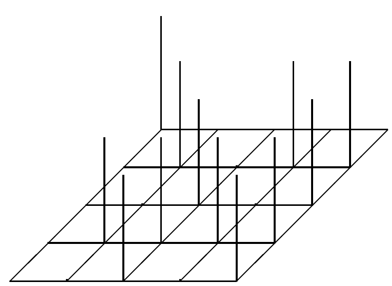

Figure 5. Representation of the matrix $A^{(0.4)}$

$$A^c = \begin{bmatrix} 0.5 & 0.9 & 0.3 & 0.5 \\ 0.7 & 0.2 & 0.9 & 0.4 \\ 0.4 & 0.6 & 0.1 & 0.2 \\ 0.8 & 0.3 & 0.7 & 0.6 \end{bmatrix}$$

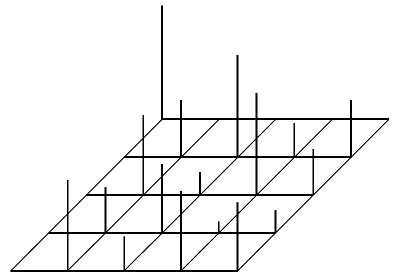

Figure 6. Representation of the matrix A^c

$$A \vee B = \begin{bmatrix} 0.5 & 0.8 & 0.7 & 0.5 \\ 0.3 & 0.8 & 0.5 & 0.6 \\ 0.7 & 0.5 & 0.9 & 0.9 \\ 0.9 & 0.7 & 0.3 & 0.6 \end{bmatrix}$$

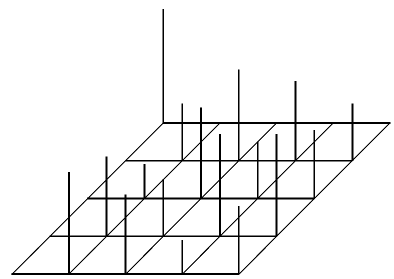

Figure 7. Representation of the matrix $A \vee B$

In the following, we define some special types of matrices. Let $R = [r_{ij}]$ be an $n \times n$ fuzzy matrix. Then

(i) R is reflexive if and only if $r_{ii} = 1$ for all $i = 1, 2, \ldots, n$.
(ii) R is irreflexive if and only if $r_{ii} = 0$ for all $i = 1, 2, \ldots, n$.
(iii) R is nearly irreflexive if and only if $r_{ii} \leq r_{ij}$ for all $i, j = 1, 2, \ldots, n$.
(iv) R is symmetric if and only if $R^{'} = R$.
(v) R is constant if and only if $r_{ij} = r_{kj}$ for all $i, j, k = 1, 2, \ldots, n$.
(vi) R is identity if and only if $r_{ii} = 1$ and $r_{ij} = 0$ $(i \neq j)$ for all i, j.

The identity matrix of order $n \times n$ is generally denoted by I_n.

(vii) R is weakly reflexive if $r_{ii} \geq r_{ij}$ for all i, j.
(viii) R is diagonal if $r_{ii} \geq 0$ and $r_{ij} = 0$ $(i \neq j)$ for all i, j.

If all the entries of a matrix are 0 (respectively 1) then we denote it by O (respectively U).

Throughout the paper we assume that $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}]$ and $D = [d_{ij}]$.

3. Some results

The matrices $A^{[k]}$ and $[k]A$ converge to the matrix O and the matrix U respectively. These results proved in the following property.

Property 1. *Let $A = [a_{ij}]$ be a fuzzy matrix of order $n \times n$.*

(i) *If $a_{ij} < 1$ for all i, j, then $\lim_{k\to\infty} A^{[k]} = O$,*

(ii) *If $a_{ij} > 0$ for all i, j, then $\lim_{k\to\infty} [k]A = U$.*

Proof. For the fuzzy matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ we have, $A \odot B = [a_{ij}.b_{ij}]$. Therefore, $A \odot A = A^{[2]} = [a_{ij}^2]$, $A^{[3]} = A^{[2]} \odot A = [a_{ij}^3]$. In general, for any positive integer k, $A^{[k]} = [a_{ij}^k]$.
Hence, $\lim_{k\to\infty} A^{[k]} = O$.

Again, $[2]A = A \oplus A = [2a_{ij} - a_{ij}^2] = [1 - (1 - a_{ij})^2]$. Also, $1 - a_{ij} \leq 1$. Therefore, for positive integer k, $[k]A = [1 - (1 - a_{ij})^k]$ and hence $\lim_{k\to\infty} [k]A = U$. □

For $0 \leq a, b \leq 1$, $a.b \leq a$ and $a.b \leq b$ imply $a.b \leq \min\{a, b\}$.
Therefore, $A \odot B \leq \min\{A, B\}$.

Property 2. *Let A and B be two fuzzy matrices, then*

(i) $A \oplus B \geq A \odot B$,

(ii) *If A and B are symmetric, then $A \oplus B$ and $A \odot B$ are symmetric,*

(iii) *If A and B are nearly irreflexive, then $A \oplus B$ and $A \odot B$ are nearly irreflexive.*

Proof. (i) The ijth element of $A \oplus B$ is $a_{ij} + b_{ij} - a_{ij}.b_{ij}$ and that of $A \odot B$ is $a_{ij}.b_{ij}$. Assume that $a_{ij} + b_{ij} - a_{ij}.b_{ij} \geq a_{ij}.b_{ij}$. i.e., $a_{ij}.(1-b_{ij}) + b_{ij}.(1-a_{ij}) \geq 0$, which is true as $0 \leq a_{ij} \leq 1$ and $0 \leq b_{ij} \leq 1$. Hence, $A \oplus B \geq A \odot B$.

(ii) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two symmetric fuzzy matrices such that $A \oplus B$ and $A \odot B$ are defined. Therefore, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. Let c_{ij} be the ijth element of $A \oplus B$. Then

$$c_{ij} = a_{ij} + b_{ij} - a_{ij}.b_{ij} = a_{ji} + b_{ji} - a_{ji}.b_{ji} = c_{ji}.$$

Hence, $A \oplus B$ is symmetric.

Again, let d_{ij} be the ijth element of $A \odot B$. Then, $d_{ij} = a_{ij}.b_{ij} = a_{ji}.b_{ji} = d_{ji}$. Hence, $A \odot B$ is symmetric.

(iii) Since A and B are nearly irreflexive, $a_{ii} \leq a_{ij}$ and $b_{ii} \leq b_{ij}$ for all i, j. Let c_{ij} and d_{ij} be the ijth elements of $A \oplus B$ and $A \odot B$ respectively.
Then

$$c_{ij} - c_{ii} = (a_{ij} + b_{ij} - a_{ij}.b_{ij}) - (a_{ii} + b_{ii} - a_{ii}.b_{ii})$$

$$= (1 - a_{ii}).(1 - b_{ii}) - (1 - a_{ij}).(1 - b_{ij}) \geq 0$$

as $1 - a_{ii} \geq 1 - a_{ij}$ and $1 - b_{ii} \geq 1 - b_{ij}$, i.e., $c_{ij} \geq c_{ii}$. Hence, $A \oplus B$ is irreflexive.

Now $d_{ij} - d_{ii} = a_{ij}.b_{ij} - a_{ii}.b_{ii} \geq 0$ i.e., $d_{ij} \geq d_{ii}$. Hence, $A \odot B$ is irreflexive. □

The operator $\oplus$ is expanding operator while the operator $\odot$ is contracting operator. That is, if the operator $\oplus$ is used between two same matrices then the resultant matrix is larger than the original matrix. The fact is opposite for the operator $\odot$. This is shown in the following property.

Property 3. *For any fuzzy matrix A,*

(i) $A \oplus A \geq A$, *and*
(ii) $A \odot A \leq A$.

Proof. (i) The ijth element of $A \oplus A$ is $2a_{ij} - a_{ij}^2 = a_{ij} + a_{ij}.(1 - a_{ij}) \geq a_{ij}$. Therefore, $A \oplus A \geq A$.

(ii) The ijth element a_{ij}^2 of $A \odot A$ is less than a_{ij}. Therefore, $A \odot A \leq A$. □

The following results are obvious. The operators $\oplus$ and $\odot$ are commutative as well as associative.

Property 4. *Let A, B and C be any three fuzzy matrices. Then*

(i) $A \oplus B = B \oplus A$,
(ii) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,
(iii) $A \odot B = B \odot A$,
(iv) $(A \odot B) \odot C = A \odot (B \odot C)$.

De Morgan's laws (over transpose) for the operators $*$ and $\circ$ are

(i) $(A * B)' = A' \circ B'$ and
(ii) $(A \circ B)' = A' * B'$,

where A' is the transpose of A. The operators $\oplus$ and $\odot$ do not obey the De Morgan's laws over transpose.

Property 5. *Let A, B and C be three fuzzy matrices.*

(i) $(A \oplus B)' = A' \oplus B'$,
(ii) $(A \odot B)' = A' \odot B'$,
(iii) *If $A \leq B$, then $A \oplus C \leq B \oplus C$ and $A \odot C \leq B \odot C$.*

Proof. (i) Let c_{ij} and d_{ij} be the ijth elements of $A \oplus B$ and $A' \oplus B'$ respectively. Therefore, $e_{ij} = c_{ji}$ is the ijth element of $(A \oplus B)'$. Then

$$\begin{aligned} c_{ij} &= a_{ij} + b_{ij} - a_{ij}.b_{ij}, \\ d_{ij} &= a_{ji} + b_{ji} - a_{ji}.b_{ji} \\ e_{ij} &= a_{ji} + b_{ji} - a_{ji}.b_{ji} = d_{ij}. \end{aligned}$$

Therefore, $e_{ij} = d_{ij}$ for all i, j. Hence, $(A \oplus B)' = A' \oplus B'$.

(ii) Let c_{ij} and d_{ij} be the ijth elements of $A \odot B$ and $A' \odot B'$ respectively. Obviously, $e_{ij} = c_{ji}$ is the ijth element of $(A \odot B)'$. Then $c_{ij} = a_{ij}.b_{ij}$. Thus $e_{ij} = a_{ji}.b_{ji}$ and $d_{ij} = a_{ji}.b_{ji}$. Therefore, $d_{ij} = e_{ij}$, for all i, j. Hence, $(A \odot B)' = A' \odot B'$.

(iii) Let d_{ij}, e_{ij}, f_{ij} and g_{ij} be the ijth elements of $A \oplus C$, $B \oplus C$, $A \odot C$ and $B \odot C$ respectively. Then

$$\begin{aligned} d_{ij} &= a_{ij} + c_{ij} - a_{ij}.c_{ij}, \;\; e_{ij} = b_{ij} + c_{ij} - b_{ij}.c_{ij} \\ f_{ij} &= a_{ij}.c_{ij}, \;\; g_{ij} = b_{ij}.c_{ij}. \end{aligned}$$

Since $A \leq B$, $a_{ij} \leq b_{ij}$. Then $a_{ij}.(1 - c_{ij}) \leq b_{ij}.(1 - c_{ij})$ or, $a_{ij} + c_{ij} - a_{ij}.c_{ij} \leq b_{ij} + c_{ij} - b_{ij}.c_{ij}$. That is, $d_{ij} \leq e_{ij}$ for all i, j. Hence, $A \oplus C \leq B \oplus C$.

Again, $a_{ij}.c_{ij} \leq b_{ij}.c_{ij}$, i.e., $f_{ij} \leq g_{ij}$ for all i, j. Hence $A \odot C \leq B \odot C$. □

Property 6. *For any $n \times n$ fuzzy matrix A,*

(i) $I_n \oplus (A \oplus A')$ *is reflexive and symmetric,*
(ii) $A \ominus I_n$ *is irreflexive,*
(iii) $A \oplus A'$ *is nearly irreflexive and symmetric,*
(iv) $I_n \oplus (A \oplus A') = I_n \vee (A \oplus A')$.

Proof. (i) $A \oplus A' = [a_{ij} + a_{ji} - a_{ij}.a_{ji}]$ and $I_n \oplus (A \oplus A') = [r_{ij}]$, where $r_{ii} = 1$ and $r_{ij} = a_{ij} + a_{ji} - a_{ij}.a_{ji}$, for $i \neq j$.

Now, $r_{ji} = a_{ji} + a_{ij} - a_{ij}.a_{ji} = r_{ij}$. That is, each diagonal element of $I_n \oplus (A \oplus A^{'})$ is 1 and all non–diagonal elements are $a_{ij} + a_{ji} - a_{ij}.a_{ji}$.
Therefore, $I_n \oplus (A \oplus A')$ is reflexive and also symmetric.

(ii) The diagonal elements of $A \ominus I_n$ are 0 because $a_{ii} \leq 1$. Hence, $A \ominus I_n$ is irreflexive.

(iii) Let $R = A \oplus A'$, i.e., $r_{ij} = a_{ij} + a_{ji} - a_{ij}.a_{ji} = r_{ji}$. Therefore, R is symmetric. Again, $r_{ii} = 2a_{ii} - a_{ii}^2$. Since A is nearly irreflexive, $a_{ii} \leq a_{ij}$. Therefore, $1 - a_{ii} \geq 1 - a_{ij}$.

Now,

$$\begin{aligned} r_{ij} - r_{ii} &= \{1 - (1 - a_{ij}).(1 - a_{ji})\} - \{1 - (1 - a_{ii}).(1 - a_{ii})\} \\ &= (1 - a_{ii}).(1 - a_{ii}) - (1 - a_{ij}).(1 - a_{ji}) \geq 0. \end{aligned}$$

Therefore, $A \oplus A'$ is nearly irreflexive and symmetric.

(iv) Since $\max\{1, a_{ij} + a_{ji} - a_{ij}.a_{ji}\} = 1$ and $\max\{0, a_{ij} + a_{ji} - a_{ij}.a_{ji}\} = a_{ij} + a_{ji} - a_{ij}.a_{ji}$, $I_n \vee (A \oplus A') = [r_{ij}]$ where $r_{ii} = 1$ and $r_{ij} = a_{ij} + a_{ji} - a_{ij}.a_{ji}$, for $i \neq j$. Again $I_n \oplus (A \oplus A') = [r_{ij}]$, where $r_{ii} = 1$ and $r_{ij} = a_{ij} + a_{ji} - a_{ij}.a_{ji}$, $i \neq j$. Hence the result hold. □

Property 7. *Let A and B be two fuzzy matrices. Then*

$$A \oplus B \geq A \vee B \geq A \ominus B.$$

Proof. Let c_{ij}, d_{ij} and e_{ij} be the ijth elements of the fuzzy matrices $A \oplus B$, $A \vee B$ and $A \ominus B$ respectively.
Now,

$$c_{ij} = a_{ij} + b_{ij} - a_{ij}.b_{ij} = \begin{cases} a_{ij} + b_{ij}.(1 - a_{ij}) \geq a_{ij} \\ b_{ij} + a_{ij}.(1 - b_{ij}) \geq b_{ij} \end{cases}$$

$$d_{ij} = \max\{a_{ij}, b_{ij}\} \leq a_{ij} + b_{ij} - a_{ij}.b_{ij} = c_{ij}.$$

Thus, $c_{ij} \geq d_{ij}$ for all i, j. Hence, $A \oplus B \geq A \vee B$.
Again,

$$e_{ij} = a_{ij} \ominus b_{ij} = \begin{cases} a_{ij}, & a_{ij} > b_{ij} \\ 0, & a_{ij} \leq b_{ij}. \end{cases}$$

That is,

$$e_{ij} \leq a_{ij} \leq \max\{a_{ij}, b_{ij}\} \text{ for all } i, j.$$

Thus, $A \ominus B \leq A \vee B$. Finally, $A \oplus B \geq A \vee B \geq A \ominus B$. □

Property 8. *For any two fuzzy matrices A and B,*

(i) $(A \vee B) \vee (A \ominus B) = A \vee B$,
(ii) $(A \vee B) \ominus (A \ominus B) \leq B$,
(iii) $A \oplus B \geq (A \vee B) \vee (A \ominus B)$,
(iv) $A \oplus B \geq (A \vee B) \ominus (A \ominus B)$.

Proof. (i) Let c_{ij}, d_{ij} and e_{ij} be the ijth elements of $A \vee B$, $A \ominus B$ and $(A \vee B) \vee (A \ominus B)$ respectively.
Then

$$c_{ij} = \max\{a_{ij}, b_{ij}\}, \;\; d_{ij} = \begin{cases} a_{ij}, & \text{if } a_{ij} > b_{ij} \\ 0, & \text{if } a_{ij} \leq b_{ij}. \end{cases}$$

$$e_{ij} = \begin{cases} \max\{\max\{a_{ij}, b_{ij}\}, a_{ij}\}, & \text{if } a_{ij} > b_{ij} \\ \max\{\max\{a_{ij}, b_{ij}\}, 0\}, & \text{if } a_{ij} \leq b_{ij} \end{cases}$$
$$= \begin{cases} \max\{a_{ij}, a_{ij}\}, & \text{if } a_{ij} > b_{ij} \\ \max\{b_{ij}, 0\}, & \text{if } a_{ij} \leq b_{ij} \end{cases}$$
$$= \begin{cases} a_{ij}, & \text{if } a_{ij} > b_{ij} \\ b_{ij}, & \text{if } a_{ij} \leq b_{ij}. \end{cases}$$

That is, ijth element e_{ij} of $(A \vee B) \vee (A \ominus B)$ is either a_{ij} or b_{ij} according as $a_{ij} >$ or $\leq b_{ij}$. Also, the ijth element of c_{ij} is either a_{ij} or b_{ij} according as $a_{ij} >$ or $\leq b_{ij}$. Therefore, $c_{ij} = e_{ij}$ for all i, j. Hence, $(A \vee B) \vee (A \ominus B) = A \vee B$.

(ii) Let f_{ij} be the ijth element of $(A \vee B) \ominus (A \ominus B)$. Then ijth element c_{ij} of $A \vee B$ is

$$c_{ij} = \max\{a_{ij}, b_{ij}\} = \begin{cases} a_{ij}, & \text{if } a_{ij} > b_{ij} \\ b_{ij}, & \text{if } a_{ij} \leq b_{ij}, \end{cases}$$

and the ijth element of $A \ominus B$ is $d_{ij} = \begin{cases} a_{ij}, & \text{if } a_{ij} > b_{ij} \\ 0, & \text{if } a_{ij} \leq b_{ij}. \end{cases}$

Therefore,

$$f_{ij} = \begin{cases} 0, & \text{if } a_{ij} > b_{ij} \\ b_{ij}, & \text{if } a_{ij} \leq b_{ij}. \end{cases}$$

That is, the elements of $(A \vee B) \ominus (A \ominus B)$ are either 0 or b_{ij}. Hence, $(A \vee B) \ominus (A \ominus B) \leq B$.

(iii) $(A \vee B) \vee (A \ominus B) = A \vee B \leq A \oplus B$ (by property 7).

(iv) It is obvious that $B \leq A \vee B$. Hence $(A \vee B) \ominus (A \ominus B) \leq B \leq A \vee B \leq A \oplus B$. □

Property 9. *Let A, B and C be three fuzzy matrices. Then*

(i) $A \oplus (B \vee C) = (A \oplus B) \vee (A \oplus C)$,
(ii) $A \oplus (B \ominus C) \geq (A \oplus B) \ominus (A \oplus C)$,
(iii) $A \ominus (B \oplus C) \leq (A \ominus B) \oplus (A \ominus C)$,
(iv) $A \ominus (B \vee C) \leq (A \ominus B) \vee (A \ominus C)$,
(v) $A \vee (B \oplus C) \leq (A \vee B) \oplus (A \vee C)$,
(vi) $A \vee (B \ominus C) \geq (A \vee B) \ominus (A \vee C)$.

Proof. (i) Let $d_{ij}, e_{ij}, f_{ij}, g_{ij}$ and h_{ij} be the ijth elements of $B \vee C, A \oplus B, A \oplus C, A \oplus (B \vee C)$ and $(A \oplus B) \vee (A \oplus C)$ respectively. Then

$$\begin{aligned} d_{ij} &= \max\{b_{ij}, c_{ij}\}, \;\; e_{ij} = a_{ij} + b_{ij} - a_{ij}.b_{ij} \\ f_{ij} &= a_{ij} + c_{ij} - a_{ij}.c_{ij}. \\ g_{ij} &= a_{ij} \oplus \max\{b_{ij}, c_{ij}\} = \begin{cases} a_{ij} + b_{ij} - a_{ij}.b_{ij}, & \text{if } b_{ij} > c_{ij} \\ a_{ij} + c_{ij} - a_{ij}.c_{ij}, & \text{if } b_{ij} \leq c_{ij}. \end{cases} \\ h_{ij} &= \max\{e_{ij}, f_{ij}\} = \begin{cases} e_{ij}, & \text{if } e_{ij} > f_{ij} \\ f_{ij}, & \text{if } e_{ij} \leq f_{ij}. \end{cases} \end{aligned}$$

If $b_{ij} > c_{ij}$, then $b_{ij}.(1-a_{ij}) > c_{ij}.(1-a_{ij})$. i.e., $a_{ij}+b_{ij}-a_{ij}.b_{ij} > a_{ij}+c_{ij}-a_{ij}.c_{ij}$ or, $e_{ij} > f_{ij}$.

Again, if $b_{ij} \leq c_{ij}$, then $a_{ij} + b_{ij} - a_{ij}.b_{ij} \leq a_{ij} + c_{ij} - a_{ij}.c_{ij}$ or, $e_{ij} \leq f_{ij}$. That is,

$$h_{ij} = \begin{cases} a_{ij} + b_{ij} - a_{ij}.b_{ij}, & \text{if } b_{ij} > c_{ij} \\ a_{ij} + c_{ij} - a_{ij}.c_{ij}, & \text{if } b_{ij} \leq c_{ij}. \end{cases}$$

Therefore, $g_{ij} = h_{ij}$ for all i, j. Hence, $A \oplus (B \vee C) = (A \oplus B) \vee (A \oplus C)$.

(ii) Let d_{ij}, e_{ij}, f_{ij}, s_{ij} and t_{ij} be the ijth elements of $B \ominus C$, $A \oplus B$, $A \oplus C$, $A \oplus (B \ominus C)$ and $(A \oplus B) \ominus (A \oplus C)$ respectively. Then

$$d_{ij} = \begin{cases} b_{ij}, & \text{if } b_{ij} > c_{ij} \\ 0, & \text{if } b_{ij} \leq c_{ij}. \end{cases}$$

$$e_{ij} = a_{ij} + b_{ij} - a_{ij}.b_{ij} \;\text{ and }\; f_{ij} = a_{ij} + c_{ij} - a_{ij}.c_{ij}.$$

$$s_{ij} = a_{ij} \oplus d_{ij} = \begin{cases} a_{ij} + b_{ij} - a_{ij}.b_{ij}, & \text{if } b_{ij} > c_{ij} \\ a_{ij}, & \text{if } b_{ij} \leq c_{ij}. \end{cases}$$

$$t_{ij} = e_{ij} \ominus f_{ij} = \begin{cases} e_{ij}, & \text{if } e_{ij} > f_{ij} \\ 0, & \text{if } e_{ij} \leq f_{ij}. \end{cases}$$

If $b_{ij} > c_{ij}$, then $b_{ij}.(1-a_{ij}) \geq c_{ij}.(1-a_{ij})$ as $0 \leq a_{ij} \leq 1$. i.e., $a_{ij}+b_{ij}-a_{ij}.b_{ij} \geq a_{ij} + c_{ij} - a_{ij}.c_{ij}$ i.e., $e_{ij} > f_{ij}$.

But, when $b_{ij} \leq c_{ij}$ then $a_{ij} + b_{ij} - a_{ij}.b_{ij} \leq a_{ij} + c_{ij} - a_{ij}.c_{ij}$. i.e., $e_{ij} \leq f_{ij}$. Thus $e_{ij} > f_{ij}$ or $e_{ij} \leq f_{ij}$ according as $b_{ij} > c_{ij}$ or $b_{ij} \leq c_{ij}$. Therefore,

$$t_{ij} = \begin{cases} a_{ij} + b_{ij} - a_{ij}.b_{ij}, & \text{if } b_{ij} > c_{ij} \\ 0, & \text{if } b_{ij} \leq c_{ij}. \end{cases}$$

Hence, $s_{ij} \geq t_{ij}$, whatever may be the values of a_{ij}, b_{ij} and c_{ij}, for all i, j. Thus $A \oplus (B \ominus C) \geq (A \oplus B) \ominus (A \oplus C)$.

(iii) Let d_{ij}, e_{ij}, f_{ij}, g_{ij} and h_{ij} be the ijth elements of $B \oplus C$, $A \ominus B$, $A \ominus C$, $A \ominus (B \oplus C)$ and $(A \ominus B) \oplus (A \ominus C)$ respectively. Then

$$\begin{aligned} d_{ij} &= b_{ij} + c_{ij} - b_{ij}.c_{ij}, \\ e_{ij} &= \begin{cases} a_{ij}, & \text{if } a_{ij} > b_{ij} \\ 0, & \text{if } a_{ij} \leq b_{ij} \end{cases} \\ f_{ij} &= \begin{cases} a_{ij}, & \text{if } a_{ij} > c_{ij} \\ 0, & \text{if } a_{ij} \leq c_{ij}. \end{cases} \end{aligned}$$

Now

$$\begin{aligned} g_{ij} &= a_{ij} \oplus d_{ij} = \begin{cases} a_{ij}, & \text{if } a_{ij} > d_{ij} \\ 0, & \text{if } a_{ij} \leq d_{ij} \end{cases} \\ h_{ij} &= e_{ij} \oplus f_{ij}. \end{aligned}$$

Case 1. If $a_{ij} > b_{ij}$ and $a_{ij} > c_{ij}$, then $e_{ij} = a_{ij}$ and $f_{ij} = a_{ij}$. Therefore

$$g_{ij} = \begin{cases} a_{ij}, & \text{if } a_{ij} > b_{ij} + c_{ij} - b_{ij}.c_{ij} \\ 0, & \text{if } a_{ij} \leq b_{ij} + c_{ij} - b_{ij}.c_{ij} \end{cases}$$

and $h_{ij} = 2a_{ij} - a_{ij}^2$. Thus $g_{ij} \leq h_{ij}$.

Case 2. If $a_{ij} < b_{ij}$ and $a_{ij} < c_{ij}$, then $g_{ij} = 0$ and $h_{ij} = 0$. i.e., $g_{ij} = h_{ij}$.

Case 3. If $c_{ij} < a_{ij} < b_{ij}$, then $g_{ij} = 0$ and $h_{ij} = a_{ij}$. i.e., $g_{ij} \leq h_{ij}$.

Case 4. if $b_{ij} < a_{ij} < c_{ij}$, then $g_{ij} = 0$ and $h_{ij} = a_{ij}$. i.e., $g_{ij} \leq h_{ij}$. Therefore for all the cases $g_{ij} \leq h_{ij}$, for whatever may be the values of a_{ij}, b_{ij} and c_{ij}. That is, $g_{ij} \leq h_{ij}$ for all $i.j$. Hence, $A \ominus (B \oplus C) \leq (A \ominus B) \oplus (A \ominus C)$. Proofs of (iv), (v) and (vi) are similar to (iii). □

4. Results on α-cut of fuzzy matrix

The upper α-cut fuzzy matrix is basically a boolean fuzzy matrix. It represents only two states 0 and 1. But the lower α-cut fuzzy matrix is a multi-graded fuzzy matrix. When the elements of this matrix are less than α then lower α-cut fuzzy matrix represents same state 0 and all other cases it represent the actual states.

To illustrate the behavior of α-cut matrices, we consider a network N consisting of n nodes and m edges. We assume that the nodes represent the cities of a country and edges represent the roads connecting them. The weight of each road is taken as the amount of crowdness at a particular time interval. It is well known, that "crowdness" of a road is highly fuzzy quantity. If two cities are not connected by a road directly then we assume that the road is fully crowd, i.e., the gradation of the crowdness of the road is taken as 1. The network $N = [a_{ij}]$ can be defined as a fuzzy matrix in the following way.

The ijth element of the matrix N is given by

$$a_{ij} = \begin{cases} 0, & \text{if the cities } i \text{ and } j \text{ are same },\\ 1, & \text{if there is no direct road between the cities } i \text{ and } j\\ & \text{ or the road is fully crowd,}\\ w_i, & w_i \text{ is the gradation of crowdness of the road}\\ & \text{connecting the cities } i \text{ and } j. \end{cases}$$

Let the fuzzy matrix A represents the crowdness status of the network N at any time period. If we consider the crowdness as two states i.e., if we consider the road is fully crowd when the crowdness gradation is greater than some value, say, α, $0 < \alpha < 1$, and the road is free from crowd when the gradation is less than α, then the fuzzy matrix A becomes $A^{(\alpha)}$, the upper α-cut fuzzy matrix.

Now, if we consider another case. If the gradation of crowdness is less than α (a specified value) then we assume that the road is crowd free and if crowdness is greater than α then it remains unchanged. In this case, the fuzzy matrix A becomes $A_{(\alpha)}$.

Some results on α-cuts fuzzy matrices are presented in the following:

Property 10. *For any two fuzzy matrices A and B,*

(i) $(A \ominus B)^{(\alpha)} \geq A^{(\alpha)} \ominus B^{(\alpha)}$,
(ii) $(A \vee B)^{(\alpha)} = A^{(\alpha)} \vee B^{(\alpha)}$,
(iii) $(A \oplus B)^{(\alpha)} \geq A^{(\alpha)} \oplus B^{(\alpha)}$.

Proof. (i) Let $C = A \ominus B, E = A^{(\alpha)} \ominus B^{(\alpha)}$ and $D = C^{(\alpha)}$.

Case 1. $a_{ij} \geq b_{ij}$. Therefore, $c_{ij} = a_{ij}$ and $d_{ij} = a_{ij}^{(\alpha)}$.

Subcase 1.1. $a_{ij} \geq b_{ij} \geq \alpha$. Then $d_{ij} = 1$ and $a_{ij}^{(\alpha)} = 1, b_{ij}^{(\alpha)} = 1$ so $e_{ij} = 0$. That is, $d_{ij} > e_{ij}$.

Subcase 1.2. $a_{ij} \geq \alpha \geq b_{ij}$. Also, in this case $d_{ij} = 1, a_{ij}^{(\alpha)} = 1, b_{ij}^{(\alpha)} = 0$ so $e_{ij} = 1$. Therefore, $d_{ij} = e_{ij}$.

Subcase 1.3. $\alpha \geq a_{ij} \geq b_{ij}$. Here, $d_{ij} = 0, a_{ij}^{(\alpha)} = b_{ij}^{(\alpha)} = 0$ and hence $e_{ij} = 0$. That is, $d_{ij} = e_{ij}$. Hence, $d_{ij} \geq e_{ij}$ for all i, j. Thus, $(A \oplus B)^{(\alpha)} \geq A^{(\alpha)} \ominus B^{(\alpha)}$.

Case 2. $b_{ij} > a_{ij}$. In this case, $c_{ij} = 0$ and $d_{ij} = 0$.

Subcase 2.1. $b_{ij} \geq a_{ij} \geq \alpha$. In this case, $b_{ij}^{(\alpha)} = 1$ and $a_{ij}^{(\alpha)} = 1$ and so $e_{ij} = 0$.

Subcase 2.2. $b_{ij} \geq \alpha \geq a_{ij}$. Then, $b_{ij}^{(\alpha)} = 1$ and $a_{ij}^{(\alpha)} = 0$. Therefore, $e_{ij} = 0$.

Subcase 2.3. $\alpha \geq b_{ij} \geq a_{ij}$. Here, $a_{ij}^{(\alpha)} = b_{ij}^{(\alpha)} = 0$. $e_{ij} = 0$. Therefore, $d_{ij} = e_{ij}$ when $b_{ij} \geq a_{ij}$ and hence $(A \ominus B)^{(\alpha)} = A^{(\alpha)} \ominus B^{(\alpha)}$. Finally, $(A \ominus B)^{(\alpha)} \geq A^{(\alpha)} \ominus B^{(\alpha)}$.

(ii) Let $a_{ij} \geq b_{ij}$. Then $a_{ij} \vee b_{ij} = \max\{a_{ij}, b_{ij}\} = a_{ij}$. Therefore, $(a_{ij} \vee b_{ij})^{(\alpha)} = a_{ij}^{(\alpha)}$.

Again, $a_{ij}^{(\alpha)} \vee b_{ij}^{(\alpha)} = a_{ij}^{(\alpha)}$ since $a_{ij} \geq b_{ij}$. Hence, $(a_{ij} \vee b_{ij})^{(\alpha)} = a_{ij}^{(\alpha)} \vee b_{ij}^{(\alpha)}$. That is, $(A \vee B)^{(\alpha)} = A^{(\alpha)} \vee B^{(\alpha)}$. Proof is similar when $b_{ij} > a_{ij}$. Hence $(A \vee B)^{(\alpha)} = A^{(\alpha)} \vee B^{(\alpha)}$.

(iii)

$$\begin{aligned}
(a_{ij} \oplus b_{ij})^{(\alpha)} &= (a_{ij} + b_{ij} - a_{ij}.b_{ij})^{(\alpha)} \\
&= \begin{cases} 1, & if\ a_{ij} + b_{ij} - a_{ij}.b_{ij} \geq \alpha \\ 0, & if\ a_{ij} + b_{ij} - a_{ij}.b_{ij} < \alpha. \end{cases} \\
a_{ij}^{(\alpha)} \oplus b_{ij}^{(\alpha)} &= a_{ij}^{(\alpha)} + b_{ij}^{(\alpha)} - a_{ij}^{(\alpha)}.b_{ij}^{(\alpha)}
\end{aligned}$$

Case 1. $a_{ij}, b_{ij} \geq \alpha$.

$$a_{ij} + b_{ij} - a_{ij}.b_{ij} = a_{ij} + b_{ij}.(1 - a_{ij}) \geq a_{ij} \geq \alpha$$

Therefore, $(a_{ij} \oplus b_{ij})^{(\alpha)} = 1$ and $a_{ij}^{(\alpha)} \oplus b_{ij}^{(\alpha)} = 1+1-1 = 1$. That is, $(a_{ij} \oplus b_{ij})^{(\alpha)} = a_{ij}^{(\alpha)} \oplus b_{ij}^{(\alpha)}$. Hence, $(A \oplus B)^{(\alpha)} = A^{(\alpha)} \oplus B^{(\alpha)}$.

Case 2. $a_{ij}, b_{ij} < \alpha$. $a_{ij} \oplus b_{ij} = a_{ij} + b_{ij} - a_{ij}.b_{ij}$. In this case $a_{ij} \oplus b_{ij}$ may or may not be greater than α. Therefore, $(a_{ij} \oplus b_{ij})^{(\alpha)} = 1$ or 0. But, $a_{ij}^{(\alpha)} \oplus b_{ij}^{(\alpha)} = 0 \oplus 0 = 0$. Therefore, $(a_{ij} \oplus b_{ij})^{(\alpha)} \geq a_{ij}^{(\alpha)} \oplus b_{ij}^{(\alpha)}$. Hence, $(A \oplus B)^{(\alpha)} \geq A^{(\alpha)} \oplus B^{(\alpha)}$.

Case 3. $a_{ij} < \alpha$ and $b_{ij} > \alpha$. Then

$$\begin{aligned}
b_{ij} > a_{ij} &\quad \text{and} \\
a_{ij} \oplus b_{ij} &= a_{ij} + b_{ij} - a_{ij}.b_{ij} = b_{ij} + a_{ij}.(1 - b_{ij}) \geq b_{ij} > \alpha.
\end{aligned}$$

Therefore $(a_{ij} \oplus b_{ij})^{(\alpha)} = 1$. Again, $a_{ij}^{(\alpha)} = 0$, $b_{ij}^{(\alpha)} = 1$ and hence $a_{ij}^{(\alpha)} \oplus b_{ij}^{(\alpha)} = 1$. Thus, $(a_{ij} \oplus b_{ij})^{(\alpha)} = a_{ij}^{(\alpha)} \oplus b_{ij}^{(\alpha)}$. Hence, $(A \oplus B)^{(\alpha)} \geq A^{(\alpha)} \oplus B^{(\alpha)}$. □

Property 11. *For the fuzzy matrices A and B,*

(i) $(A \oplus B)_{(\alpha)} \geq A_{(\alpha)} \oplus B_{(\alpha)}$,
(ii) $(A \ominus B)_{(\alpha)} = A_{(\alpha)} \ominus B_{(\alpha)}$,
(iii) $(A \vee B)_{(\alpha)} = A_{(\alpha)} \vee B_{(\alpha)}$.

Proof. (i) $(A \oplus B)_{(\alpha)} = [(a_{ij} + b_{ij} - a_{ij}.b_{ij})_{(\alpha)}]$ and $A_{(\alpha)} \oplus B_{(\alpha)} = [a_{ij(\alpha)} + b_{ij(\alpha)} - a_{ij(\alpha)}.b_{ij(\alpha)}]$.

Case 1. $a_{ij}, b_{ij} \geq \alpha$. Therefore

$$(a_{ij} \oplus b_{ij})_{(\alpha)} = (a_{ij} + b_{ij} - a_{ij}.b_{ij})_{(\alpha)} = a_{ij} + b_{ij} - a_{ij}.b_{ij}$$

as $a_{ij} + b_{ij} - a_{ij}.b_{ij} = a_{ij} + b_{ij}.(1 - a_{ij}) \geq a_{ij} \geq \alpha$. Again, $a_{ij(\alpha)} \oplus b_{ij(\alpha)} = a_{ij(\alpha)} + b_{ij(\alpha)} - a_{ij(\alpha)}.b_{ij(\alpha)} = a_{ij} + b_{ij} - a_{ij}.b_{ij}$ as $a_{ij}, b_{ij} \geq \alpha$. Thus, in this case $(A \oplus B)_{(\alpha)} = A_{(\alpha)} \oplus B_{(\alpha)}$.

Case 2. $a_{ij}, b_{ij} < \alpha$. Here, $(a_{ij} \oplus b_{ij})_{(\alpha)} = (a_{ij} + b_{ij} - a_{ij}.b_{ij})_{(\alpha)} = a_{ij} + b_{ij} - a_{ij}.b_{ij}$ or 0 according as $a_{ij} + b_{ij} - a_{ij}.b_{ij} \geq \alpha$ or $< \alpha$. But, $a_{ij(\alpha)} \oplus b_{ij(\alpha)} = 0 \oplus 0 = 0$. Thus, $(A \oplus B)_{(\alpha)} \geq A_{(\alpha)} \oplus B_{(\alpha)}$.

Case 3. $a_{ij} < \alpha$ and $b_{ij} \geq \alpha$. Then

$$(a_{ij} \oplus b_{ij})_{(\alpha)} = (a_{ij} + b_{ij} - a_{ij}.b_{ij})_{(\alpha)} = a_{ij} + b_{ij} - a_{ij}.b_{ij}$$

because $a_{ij} + b_{ij} - a_{ij}.b_{ij} = b_{ij} + a_{ij}.(1 - b_{ij}) \geq b_{ij} \geq \alpha$. Again, $a_{ij(\alpha)} \oplus b_{ij(\alpha)} = a_{ij(\alpha)} + b_{ij(\alpha)} - a_{ij(\alpha)}.b_{ij(\alpha)} = 0 + b_{ij} - 0 = b_{ij}$.

Therefore, $(A \oplus B)_{(\alpha)} \geq A_{(\alpha)} \oplus B_{(\alpha)}$. Hence finally, $(A \oplus B)_{(\alpha)} \geq A_{(\alpha)} \oplus B_{(\alpha)}$.

(ii) Let $C = A \ominus B$, $E = A_{(\alpha)} \ominus B_{(\alpha)}$ and $D = C_{(\alpha)}$.

Case 1. $a_{ij} \geq b_{ij}$ Then $c_{ij} = a_{ij}$ and $d_{ij} = a_{ij(\alpha)}$.

Subcase 1.1. $a_{ij} \geq b_{ij} \geq \alpha$ then $d_{ij} = a_{ij}$ and $e_{ij} = a_{ij} \ominus b_{ij} = a_{ij}$. Thus, $d_{ij} = e_{ij}$.

Subcase 1.2. $a_{ij} \geq \alpha > b_{ij}$ then $d_{ij} = a_{ij}$ and $e_{ij} = a_{ij} \ominus 0 = a_{ij}$. Hence, $d_{ij} = e_{ij}$.

Subcase 1.3. $\alpha > a_{ij} \geq b_{ij}$. Then, $d_{ij} = 0$ and also $e_{ij} = 0$ Thus, $d_{ij} = e_{ij}$. Therefore in this case, $(A \ominus B)_{(\alpha)} = A_{(\alpha)} \ominus B_{(\alpha)}$.

Case 2. $b_{ij} \geq a_{ij}$. Then $c_{ij} = 0$ and hence $d_{ij} = 0$.

Subcase 2.1. $b_{ij} \geq a_{ij} \geq \alpha$. Here, $a_{ij(\alpha)} = a_{ij}$ and $b_{ij(\alpha)} = b_{ij}$ and hence $e_{ij} = a_{ij} \ominus b_{ij} = 0$ as $b_{ij} \geq a_{ij}$.

Subcase 2.2. $b_{ij} \geq \alpha > a_{ij}$. $b_{ij(\alpha)} = b_{ij}$ and $a_{ij(\alpha)} = 0$, then $e_{ij} = 0$.

Subcase 2.3. $\alpha > b_{ij} \geq a_{ij}$. Then $e_{ij} = 0$ as $a_{ij(\alpha)} = 0$ and $b_{ij(\alpha)} = 0$. All the cases, $d_{ij} = e_{ij}$ for all i, j. Hence, $(A \ominus B)_{(\alpha)} = A_{(\alpha)} \ominus B_{(\alpha)}$.

(iii) Let $a_{ij} \geq b_{ij}$. Then $a_{ij} \vee b_{ij} = \max\{a_{ij}, b_{ij}\} = a_{ij}$. Therefore, $(a_{ij} \vee b_{ij})_{(\alpha)} = a_{ij(\alpha)}$. Again, $a_{ij(\alpha)} \vee b_{ij(\alpha)} = a_{ij(\alpha)}$ as $a_{ij} \geq b_{ij}$. Therefore, $(a_{ij} \vee b_{ij})_{(\alpha)} = a_{ij(\alpha)} \vee b_{ij(\alpha)}$. The proof is similar when $a_{ij} \leq b_{ij}$. Hence, $(A \vee B)_{(\alpha)} = A_{(\alpha)} \vee B_{(\alpha)}$. □

6. Results on complement of fuzzy matrix

The complement of a fuzzy matrix is used to analysis the complement nature of any system. For example, if A represents the crowdness of a network at a particular time period then its complement A^c represents the clearness at the same time period. Using the following results we can study the complement nature of a system with the help of original fuzzy matrix.

The operator complement obey the De Morgan's laws for the operator $\oplus$ and $\odot$. This is established in the following property.

Property 12. *For the fuzzy matrices A and B,*

(i) $(A \oplus B)^c = A^c \odot B^c$,
(ii) $(A \odot B)^c = A^c \oplus B^c$,
(iii) $(A \oplus B)^c \leq A^c \oplus B^c$,
(iv) $(A \odot B)^c \geq A^c \odot B^c$.

Proof. (i) $A^c \oplus B^c = [1 - a_{ij}.b_{ij}]$ and $A \odot B = [a_{ij}.b_{ij}]$. Then $(A \odot B)^c = [1 - a_{ij}.b_{ij}]$. Therefore, $(A \odot B)^c = A^c \oplus B^c$.

(ii) $(A \oplus B)^c = [1 - (a_{ij} + b_{ij} - a_{ij}.b_{ij})]$, $A^c = [1 - a_{ij}]$ and $B^c = [1 - b_{ij}]$. Now, $A^c \odot B^c = [1 - (a_{ij} + b_{ij} - a_{ij}.b_{ij})]$. Hence, $(A \oplus B)^c = A^c \odot B^c$.

(iii) From property 2(i) $A \oplus B \geq A \odot B$. Then $(A \oplus B)^c \leq (A \odot B)^c = A^c \oplus B^c$.

(iv) $(A \odot B)^c \geq (A \oplus B)^c = A^c \odot B^c$. □

Acknowledgement

The authors are grateful to referees for their suggestions to improve the quality and presentation of the paper.

References

1. H. Hashimoto, *Canonical form of a transitive matrix*, Fuzzy Sets and Systems, **11** (1983) 157-162.
2. R.Hemasinha, N.R.Pal and J.C.Bezdek, *Iterates of fuzzy circulant matrices*, Fuzzy Sets and Systems, **60** (1993) 199-206.
3. S. K. Khan, A. K. Shyamal and M. Pal, *Intuitionistic fuzzy matrices*, Notes on Intuitionistic Fuzzy Sets, **8**(2) (2002) 51-62.
4. K. H. Kim and F. W. Roush, *Generalised fuzzy matrices*, Fuzzy Sets and Systems, **4** (1980) 293-315.
5. W. Kolodziejczyk, *Canonical form of a strongly transitive fuzzy matrix*, Fuzzy Sets and Systems, **22** (1987) 292-302.
6. J. B. Kim, *Determinant theory for fuzzy and boolean matrices*, Congressus Numerantium, (1988) 273-276.
7. J. B. Kim, A.Baartmans and S.Sahadin, *Determinant theory for fuzzy matrices*, Fuzzy Sets and Systems, **29** (1989) 349-356.
8. M. Pal, *Intuitionistic fuzzy determinant*, V.U.J.Physical Sciences, **7** (2001) 87-93.
9. M. Z. Ragab and E. G. Emam, *The determinant and adjoint of a square fuzzy matrix*, Fuzzy Sets and Systems, **61** (1994) 297-307.
10. M. G. Ragab and E. G. Emam, *On the min-max composition of fuzzy matrices*, Fuzzy Sets and Systems, **75** (1995) 83-92.
11. S. Tamura, S. Higuchi and K. Tanaka, *Pattern classification based on fuzzy relations*, IEEE Trans. Syst. Man Cybern, SMC, **1(1)** (1971) 61-66.

12. M. G. Thomason, *Convergence of powers of a fuzzy matrix*, J.Math Anal. Appl., **57** (1977) 476-480.
13. Zhou-Tian and De-Fu Liu, *On the power sequence of a fuzzy matrix convergent power sequence*, J. Appli. Math. & Computing(old:KJCAM)**4** (1997) 147–166.
14. L. J. Xin, *Controllable fuzzy matrices*, Fuzzy Sets and Systems, **45** (1992) 313-319.
15. L. J. Xin, *Convergence of powers of controllable fuzzy matrices*, Fuzzy Sets and Systems, **62** (1994) 83-88.

Amiya K. Shyamal has started his Ph. D. work at Vidyasagar University on 2000. His research interests include fuzzy and intuitionistic fuzzy sets.

Madhumangal Pal received his M. Sc from Vidyasagar University, India and Ph. D from Indian Institute of Technology, Kharagpur, India in 1990 and 1996 respectively. He is engaged in research since 1991. In 1996, he received Computer Division Award from Institution of Engineers (India), for best research work. During 1997 to 1999 he was a faculty member of Midnapore College and since 1999 he has been at the Vidyasagar University, India. His research interest include computational graph theory, parallel algorithms, data structure, combinatorial algorithms, genetic algorithms, fuzzy sets and intuitionistic fuzzy sets.

Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore - 721 102, West Bengal, India,
`e-mail: mmpal@vidyasagar.ac.in`