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A Direct Proof of a Theorem of Blaschke and Lebesgue

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ABSTRACT. The Blaschke–Lebesgue Theorem states that among all planar convex domains of given constant width B the Reuleaux triangle has minimal area. It is the purpose of this article to give a direct proof of this theorem by analyzing the underlying variational problem. The advantages of the proof are that it shows uniqueness (modulo rigid deformations such as rotation and translation) and leads analytically to the shape of the area-minimizing domain. Most previous proofs have relied on foreknowledge of the minimizing domain. Key parts of the analysis extend to the higher-dimensional situation, where the convex body of given constant width and minimal volume is unknown.

1. Introduction

A convex body in $\mathbb{R}^d$ is said to have constant width B if any two distinct parallel planes tangent to its boundary are separated by a distance B. For $d = 2$ such bodies are often called *orbiforms,* and for $d = 3$ they are called *spheroforms.* A well-known example is the *Reuleaux triangle,* whose boundary consists of three equally long circular arcs with curvature $1/B$. The arcs meet at the corners of an equilateral triangle. Reuleaux polygons with any odd number of sides likewise enjoy the property of constant width.

It has long been known that among all two-dimensional convex bodies of constant width, the Reuleaux triangle has the smallest area. W. Blaschke [2] and H. Lebesgue [14, 15] were the first to show this, and the succeeding decades have seen several other works on the problem of minimizing the area or volume of an object given a constant width; see [10, 4, 7, 1, 6, 19, 11], and [5]. Objects of constant width have several practical uses, and have been entertainingly discussed in [8, 13]. For instance, coins are sometimes designed with such shapes, because constant width allows their use in vending machines.

The disadvantage of the arguments of Blaschke and Lebesgue and most subsequent proofs of the Blaschke–Lebesgue Theorem is that they are not sufficiently analytic to derive the minimality of the Reuleaux triangle without prior knowledge of the minimizer. No doubt this is one of the reasons that the higher-dimensional analog of the problem has remained open: What body (or bodies) of constant width in three or more dimensions has the smallest volume?

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Another reason may be the rigidity of the condition of constant width. The Reuleaux triangle is only about 10% smaller than the disk of the same width, and the not quite tetrahedrally symmetric Meissner bodies [4, Section 67], which are the best-known conjectured minimizers in the three-dimensional case, are less than 20% smaller than the ball.

It is my purpose here to prove the Blaschke–Lebesgue Theorem in a directly analytic way, and to frame the problem in higher dimensions as a step toward answering the question just posed.

Two previous attempts to give analytical proofs can be cited. Fujiwara [10] expressed the area in terms of $r(\theta)$ and showed through a lengthy calculation that in general the area of an orbiform exceeds that of the Reuleaux triangle of the same width. His proof gives little indication how to find the optimal geometry from first principles. More recently Ghandehari [11] gave an argument via optimal control theory and Pontryagin's maximum principle, which resembles the one of this article in a few respects.

2. On the minimal volume of a convex body of constant width

A body K of constant width is strictly convex, and therefore ∂K may be expressed as a continuous image of the sphere S^{d-1} via the mapping $\Gamma(\boldsymbol{\omega})$ which associates to any unit vector $\boldsymbol{\omega}$ the point of ∂K with supporting plane perpendicular to $\boldsymbol{\omega}$. (At smooth points of ∂K, Γ is the inverse of the Gauss map.)

If $\mathbf{x}$ denotes a point on the boundary, then the *support function* of K will be defined in the usual manner as $h(\boldsymbol{\omega}) := \mathbf{x} \cdot \boldsymbol{\omega}$, where $\mathbf{x} = \Gamma(\boldsymbol{\omega})$. Notice that $h(\boldsymbol{\omega})$ is the distance from the origin of a plane in contact with ∂K, provided that the origin is within K, which may always be assumed. Given the support function $h(\boldsymbol{\omega})$ of a convex set, the set itself can be reconstructed as the envelope of its supporting planes. Choosing the independent variable as $\boldsymbol{\omega}$ will be convenient for several reasons, among them the simple form of the formula for the width of K:

$$h(\boldsymbol{\omega}) + h\left(\boldsymbol{\omega}^{\mathbf{a}}\right) = B\,, \tag{2.1}$$

where $\boldsymbol{\omega}^{\mathbf{a}}$ designates the point on S^1 antipodal to $\boldsymbol{\omega}$: for S^1 one could identify ω as the usual angular variable and write $\omega^{\mathbf{a}} = \omega \pm \pi$, but dimension-independent notation will be preferred as far as possible.

A simple exercise using the divergence theorem shows that the volume can be written in terms of the support function:

$$\mathrm{Vol}(K) = \frac{1}{d}\int_{\partial K} h(\boldsymbol{\omega})\, dS = \frac{1}{d}\int_{S^{d-1}} h(\boldsymbol{\omega}) \frac{d\omega}{\prod_j \kappa_j}\,.$$

In this formula κ_j are the principal curvatures of ∂K. Here and elsewhere, it will be more convenient to express quantities in terms of the radii of curvature $R_j := \frac{1}{\kappa_j}$ (or zero, at non-smooth points of ∂K). Hence

$$\mathrm{Vol}(K) = \frac{1}{d}\left\langle h, \prod_{j=1}^{d-1} R_j \right\rangle_{S^{d-1}}\,. \tag{2.2}$$

The brackets here designate the inner product on $L^2(S^{d-1})$. Because (2.2) is expressed in terms of R_j rather than κ_j, there is no difficulty when ∂K fails to be smooth. The set-up described to this stage is classical; for instance see [4], [3].

The question under consideration is the following.

Problem 2.1. *Determine the minimal volume of a convex body K of fixed width B.*

This problem will be recast with the benefit of several observations, beginning with a useful formula, which results from a direct calculation:

$$\nabla^2_{S^{d-1}} h = \sum_{j=1}^{d-1} R_j - (d-1)h \,. \tag{2.3}$$

Equation (2.3) was known to Weingarten [20] in the nineteenth century (see also [18]). Together with (2.1) it implies that

$$\sum_j R_j(\omega) + \sum_j R_j\left(\omega^{\mathrm{a}}\right) = (d-1)B \,. \tag{2.4}$$

Observe that $d-1$ is the second eigenvalue of $-\nabla^2_{S^{d-1}}$, so the differential equation (2.3) is not uniquely solvable. However, according to the Fredholm alternative theorem, it is uniquely solvable with a reduced resolvent $G : \mathcal{H}_1 \hookleftarrow$, where

$$\mathcal{H}_1 := L^2\left(S^{d-1}\right) \ominus \operatorname{span}\left[Y_1^m\right] \,,$$

and Y_1^m are the spherical harmonics [17] such that

$$-\nabla^2_{S^{d-1}} Y_1^m = (d-1) Y_1^m \,.$$

(If $d = 2$, then $Y_1^m = \sin\omega$ and $\cos\omega$. The notation Y_ℓ^m will be used in any dimension.)

Now, G is a bounded, smoothing operator. That is,

$$\operatorname{Vol}(K) = \frac{1}{d}\left\langle G\left[\sum_j R_j\right], \prod_j R_j \right\rangle_{S^{d-1}} \tag{2.5}$$

is a bounded quadratic form on $L^2(S^{d-1})$, and the operator G maps $L^2(S^{d-1})$ into $W^2(S^{d-1}) \cap \mathcal{H}_1$. Moreover, the condition of orthogonality to the span of the Y_1^m is quite natural geometrically. For the support function, this restriction means that the Steiner point (centroid) has been moved to the center of the coordinate system. Any given set of nonzero coefficients of Y_1^m could be specified, and this would merely correspond to rigidly displacing the body K by a fixed vector with respect to the centroid. On the other hand, the condition that $\sum_{j=1}^{d-1} R_j$ be orthogonal to Y_1^m is necessary for ∂K to be a closed boundary: If $d = 2$, it is the condition that the curve ∂K be closed. If $d = 3$, this condition is necessary and essentially sufficient for the Gauss curvature to determine an immersed closed surface ∂K (uniquely up to rigid motions) [16, p. 130].

When $d = 2$, there is only one curvature defined on the boundary, and V becomes a symmetric quadratic form

$$\operatorname{Vol}(K) = \frac{1}{2}\langle G[R], R\rangle_{S^1} \,. \tag{2.6}$$

When $d = 3$, a theorem of Blaschke [2] [6, p. 66]) states that for objects of constant width B, the volume and surface-area S are related by

$$\operatorname{Vol}(K) = \frac{BS}{2} - \frac{\pi B^3}{3} \,.$$

It follows that the minimizers of the volume functional are identical to the minimizers of the surface-area functional, which for $d = 3$ may be written as a symmetric quadratic form in $R := \sum_j R_j$:

$$\Phi_1[R] := \frac{1}{d-1}\langle G[R], R\rangle_{S^{d-1}} \tag{2.7}$$

[4, p. 63]. Recall that the support function enters through $G[R] = h$. The functional (2.7) will be considered here as the objective in any dimension, although its interpretation involves Quermass integrals and is not easily intuited when $d > 3$.

With this notation, (2.4) is written:

$$R(\omega) + R\left(\omega^{\mathbf{a}}\right) = (d-1)B\,. \tag{2.8}$$

This implies that admissible R must satisfy

$$0 \le R(\omega) \le (d-1)B\,, \tag{2.9}$$

and the averages of R and h are both determined: It follows from (2.8) and (2.1) that

$$R_{\text{ave}} = \frac{(d-1)B}{2},\, h_{\text{ave}} = B/2\,. \tag{2.10}$$

Because of (2.10) and the fact that G maps the set of functions of mean zero to itself, a simplification is achieved by subtracting the averages of R and h, so $\overline{R} := R - \frac{(d-1)B}{2}$ and $\overline{h} := h - \frac{B}{2}$. In these terms, just as $h = G[R], \overline{h} = G[\overline{R}]$. There results an alternative to Problem 2.1.

Problem 2.2. *Minimize the functional*

$$\Phi\left[\overline{R}\right] := \left\langle G\left[\overline{R}\right], \overline{R}\right\rangle_{S^{d-1}} \tag{2.11}$$

for $\overline{R} \in \mathcal{H} := \left\{f \in L^2(S^{d-1}) : f \perp \operatorname{span}\left\{Y_1^m\right\}, f(\omega^{\mathbf{a}}) = -f(\omega), |f(\omega)| \le \frac{(d-1)B}{2}\right\}$.

Remarks.

1. Functions in $\mathcal{H}$ are orthogonal to the lowest two eigenspaces of $-\nabla^2$. It follows that Φ is a negative-definite quadratic form on $\mathcal{H}$. In particular, the function corresponding to the ball, $\overline{R} = 0$, maximizes Φ. Because of the concavity of Φ, the minimizers are extremals of $\mathcal{H}$. This statement is made somewhat more precise in Theorem 2.3 below.

2. When $d = 2$, minimizing Φ on $\mathcal{H}$ is equivalent to finding the convex region of smallest area for a given B. When $d = 3$, the theorem of Blaschke alluded to above ensures that minimizing Φ is equivalent to minimizing the volume functional, but some elements of $\mathcal{H}$ may not correspond to embedded convex bodies. Hence Problem 2.2 is fully equivalent to Problem 2.1 only for $d = 2$.

Now, the derivative of Φ with respect to the variation $\overline{R} \to \overline{R} + \delta\zeta$ is simply

$$\left.\frac{d\Phi}{d\delta}\right|_0 = 2\left\langle G\left[\overline{R}\right], \zeta\right\rangle_{S^{d-1}} = 2\left\langle \overline{h}, \zeta\right\rangle\,. \tag{2.12}$$

It is then possible to conclude the following.

Theorem 2.3. *Minimizers of Problem 2.2 exist, and every minimizing* $\overline{R}$ *has the properties that*

(a) $\mu\left\{\omega : \overline{h} > 0, |\overline{R}| < \frac{(d-1)B}{2}\right\} = \mu\left\{\omega : \overline{h} < 0, |\overline{R}| < \frac{(d-1)B}{2}\right\} = 0$

(b) $\overline{h} > 0 \Rightarrow \overline{R} = -\frac{(d-1)B}{2}, \overline{h} < 0 \Rightarrow \overline{R} = \frac{(d-1)B}{2}$ *a.e.*

Proof. The existence of a minimizer follows in a standard way from the compactness of the operator G, considered as an operator on the Hilbert space

$$\left\{f \in L^2\left(S^{d-1}\right) : f \perp \operatorname{span}\left\{Y_1^m, 1\right\}\right\} .$$

(Minimizers are non-unique at least by rotation.)

Consider now admissible variations for Φ, normalizing B temporarily for convenience so that $\frac{(d-1)B}{2} = 1$, and thus $-1 \leq \overline{R} \leq 1$.

Suppose that for some minimizing $\overline{R}$ and some $\epsilon > 0$, the set

$$S_\epsilon := \left\{\omega : \overline{h} > 0, \left|\overline{R}\right| \leq 1 - \epsilon\right\}$$

is of positive measure. Then the antipodal set $S_\epsilon^{\mathbf{a}}$ is also of positive measure, and any variation ζ supported in S_ϵ must be extended to $S_\epsilon^{\mathbf{a}}$ antisymmetrically by $\zeta(\omega^{\mathbf{a}}) = -\zeta(\omega)$. Observe here that it is unnecessary to restrict ζ to be orthogonal to Y_1^m, as any such component is orthogonal to $\overline{h}$ and hence will not contribute to (2.12).

Let ζ run through a basis for $L^2[S_\epsilon] \ominus \chi_{S_\epsilon}$ consisting of bounded functions ζ_n, extended antisymmetrically to $S_\epsilon^{\mathbf{a}}$ as mentioned above. (Boundedness, together with $\epsilon > 0$, ensures admissibility. The case ζ proportional to χ_{S_ϵ} will be considered separately below.) From (2.11), with $\overline{h}(\omega) := G[\overline{R}]$ the first variation (2.12) is proportional to

$$\begin{aligned}
\langle \overline{h}, \zeta \rangle &= \int_{S_\varepsilon} \overline{h}(\omega) \zeta(\omega) \, d\omega + \int_{S_\varepsilon^{\mathbf{a}}} \overline{h}(\omega) \zeta(\omega) \, d\omega \\
&= \int_{S_\varepsilon} \overline{h}(\omega) \zeta(\omega) \, d\omega + \int_{S_\varepsilon} \left(-\overline{h}(\omega)\right) (-\zeta(\omega)) \, d\omega \\
&= 2 \int_{S_\varepsilon} \overline{h}(\omega) \zeta(\omega) \, d\omega .
\end{aligned}$$

Optimality implies that this vanishes and hence that $\overline{h} =$ constant a.e. on S_ϵ.

Next consider (2.11) subjected to the variation $\zeta = -\chi_{S_\epsilon} + \chi_{S_\epsilon^{\mathbf{a}}}$.

If $\mu(S_\epsilon) > 0$, then

$$\frac{d\Phi}{d\delta} = -2 \int_{S_\epsilon} \overline{h} + 2 \int_{S_\epsilon^{\mathbf{a}}} \overline{h} < 0 , \tag{2.13}$$

which contradicts optimality. This concludes the proof of (a).

For (b), observe from (a) that either $\overline{h} = 0$ a.e., which corresponds to the sphere, i. e., the maximizing shape, or else there is a set of positive measure for which $\overline{h} > 0$ and $\overline{R} = -1$ or $+1$. But if $\overline{R} = +1$, then the variation leading to (2.13) is still admissible for $\delta \geq 0$, so (2.13) yields a contradiction. Similarly, $\overline{h} < 0$ if $\overline{R} = -1$. □

Corollary 2.4 (The Blaschke–Lebesgue Theorem). *Among all two–dimensional convex regions of a given constant width B, the Reuleaux triangle has the smallest area.*

Proof. Here ω is treated as the angular variable for S^1, and it will be assumed that $B = 1$. As the circle is not the minimizer, statement (b) of Theorem 2.3 implies that $m := \max \overline{h} > 0$. By performing a rotation, it may be assumed that $\overline{h}(0) = m$, and by continuity there is an interval around 0 such that, when rewritten in terms of $\overline{h}$ and specialized to one variable, (2.3) becomes

$$\overline{h}'' = -\overline{h} - \frac{1}{2}, \tag{2.14}$$

yielding

$$\overline{h} = \left(m + \frac{1}{2}\right)\cos\omega - \frac{1}{2} \tag{2.15}$$

on that interval. The endpoints of the interval correspond to $\overline{h} = 0$, i. e., $\omega = \pm \arccos \frac{1}{2m+1} =: \pm\alpha$. At these points, $\overline{h}' \neq 0$. Since standard regularity theory implies that $\overline{h}$ has an absolutely continuous derivative [12, p. 158], $\pm\alpha$ cannot abut an interval on which $\overline{h} = 0$. The only possibility is that $\overline{h}$ becomes negative and on the next interval the differential equation has a solution antisymmetric about α, i. e.,

$$\overline{h} = -\left(m + \frac{1}{2}\right)\cos(2\alpha - \omega) + \frac{1}{2}. \tag{2.16}$$

The function $\overline{h}$ switches between the two forms (2.15) and (2.16), as shown in the figure.

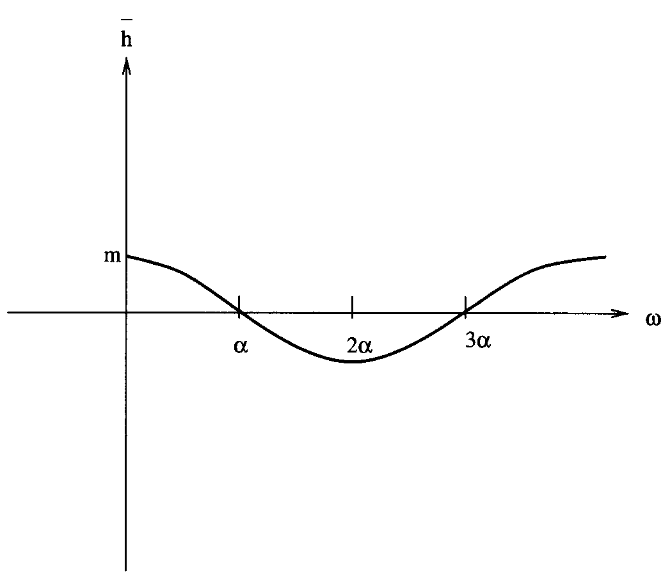

FIGURE 1 The minimizing support function minus 1/2.

The support function is also subject to periodicity ($\omega + 2\pi \cong \omega$) and antisymmetry ($\overline{h}(\omega + \pi) = -\overline{h}(\omega)$). The only candidates for optimality thus correspond to the odd-sided regular Reuleaux polygons with $B = 1$. A calculation [9] shows that the area of any such figure of given width is an increasing function of the number of sides. $\square$

Remarks about three or more dimensions.

There are two barriers to extending the proof of the Blaschke–Lebesgue Theorem to higher dimensions. One of these is connected with the ability to extend solutions of ordinary differential equations uniquely across a boundary; this needs to be replaced by a PDE analysis.

The other, probably more substantial, barrier is the gap between the conditions of Problem 2.2 and Problem 2.1. As remarked already, if the dimension $d > 2$, then the analytic conditions of Problem 2.2 differ from the geometric conditions which would guarantee that the curvature function R defines a convex body as naturally embedded in $\mathbb{R}^d$. Numerical calculations indicate that the simplest generalization of the Reuleaux triangle, viz., the solution of (2.3) with $R = 2\chi_S$, $S = S^2 \cap \{x_1x_2x_3 > 0\}$, is not the support function of an embedded convex body. If, as is plausible, this minimizes Problem 2.2, then additional conditions will have to be imposed for a solution to Problem 2.1.

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